REAL ALGEBRAIC TOOLS IN STOCHASTIC GAMES

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1. Introduction

In game theory and in particular in the theory of stochastic games, we encounter systems of polynomial equalities and inequalities. We start with a few examples.

The first example relates to the minmax value and optimal strategies of a two-person zero-sum game with finitely many strategies. Consider a two-person zero-sum game represented by a $k \times m$ matrix $A = (a_{ij})$, $1 \leq i \leq k$ and $1 \leq j \leq m$. The necessary conditions for the variables v, $x_1, \ldots, x_k$ and $y_1, \ldots, y_m$ to be the minmax value and optimal strategies of player 1 (the maximizer and row player) and player 2, respectively, are given by the following list of polynomial inequalities and equalities in the variables $v, x_1, \ldots, x_k, y_1, \ldots, y_m$:

$$x_i \geq 0 \quad i = 1, \ldots, k, \quad \sum_{i=1}^{k} x_i = 1, \quad y_j \geq 0 \quad j = 1, \ldots, m, \quad \sum_{j=1}^{m} y_j = 1$$

$$\sum_{i=1}^{k} x_i a_{ij} \geq v \quad j = 1, \ldots, m, \quad \text{and} \quad \sum_{j=1}^{m} y_j a_{ij} \leq v \quad i = 1, \ldots, k.$$

The second example concerns the equilibrium strategies and payoffs of an n-person strategic game with finitely many strategies. Consider an n-person game with finite pure strategy sets A^i, $i = 1, \ldots, n$, and payoff functions $g^i : A \to \mathbb{R}$ where $A = \times_{i=1}^{n} A^i$. Let X^i denote the set of mixed strategies of player i. Each element $x^i \in X^i$ is a list of variables $x^i(a^i) \in \mathbb{R}$, $a^i \in A^i$ with $x^i(a^i) \geq 0$ and $\sum_{a^i \in A^i} x^i(a^i) = 1$. The necessary conditions for the variables $x^i \in \mathbb{R}^{A^i}$, $i = 1, \ldots, n$, to be a strategic equilibrium with corresponding payoffs $v^i \in \mathbb{R}$, $i = 1, \ldots, n$, are given by the following list of polynomial inequalities and equalities:

$$x^i(a^i) \geq 0 \quad i = 1, \ldots, n, \quad a^i \in A^i,$$

$$\sum_{a^i \in A^i} x^i(a^i) = 1 \quad i = 1, \ldots, n,$$

$$\sum_{a \in A} (\prod_{j=1}^{n} x^j(a^j)) g^i(a) = v^i \quad i = 1, \ldots, n,$$

$$\sum_{a^{-i} \in A^{-i}} \left[\prod_{j \neq i} x^j(a^j) \right] g^i(a^{-i}, b^i) \leq v^i \quad i = 1, \ldots, n \quad b^i \in A^i,$$

where $A^{-i} = \times_{j \neq i} A^j$ and for $b^i \in A^i$ and $a^{-i} = (a^j)_{j \neq i} \in A^{-i}$, (a^{-i}, b^i) is the element of A whose i-th coordinate is b^i and whose j-th coordinate, $j \neq i$, is a^j.

The present chapter brings together parts of the theory of polynomial equalities and inequalities used in the theory of stochastic games. The theory can be considered as a theory of polynomial equalities and inequalities over the field of real numbers or the field of real algebraic numbers or more generally over an arbitrary real closed field. Real closed fields are defined in the next section. The reader who is interested in the theory over the field of real numbers $\mathbb{R}$ can skip the next section.

2. Real Closed Fields

The content of this section is part of the theory developed by Artin and Schreier for the positive solution of Hilbert's seventeenth problem: Is every polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$ with $P(x_1, \ldots, x_n) \geq 0$ for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$ a sum of squares of rational functions? This material can be found in many books, for example [4].

A *real field* F is a field F such that for every finite list of elements $x_1, \ldots, x_n \in F$ with $\sum_{i=1}^{n} x_i^2 = 0$ we have $x_i = 0$ for every $1 \leq i \leq n$. The characteristic of a real field is 0. The field of real numbers $\mathbb{R}$ is a real field. Every subfield of a real field is a real field and thus the field of real algebraic numbers $\mathbb{R}_{alg}$ and the field of rational numbers $\mathbb{Q}$ are real fields. Another example of a real field is the field of rational functions $\mathbb{R}(X)$ in the variable X.

A *real closed field* is a real field F that has no nontrivial real algebraic extension $F_1 \supset F$, $F_1 \neq F$. Equivalently, a *real closed field* is a field F such that the ring $F[i] = F[X]/(X^2+1)$ is an algebraically closed field. An important property of a real closed field F is that every polynomial of odd degree P in $F[X]$ has a root in F.

An *ordered field* $(F, \leq)$ is a field F together with a total order relation $\leq$ satisfying: (i) $x \leq y \Rightarrow x+z \leq y+z$ and (ii) $0 \leq x$, $0 \leq y \Rightarrow 0 \leq xy$. An element $x \in F$ (where $(F, \leq)$ is an ordered field) is called positive if and only if $0 \leq x$.

The classic examples of ordered fields are the field of rational numbers $\mathbb{Q}$, the field of real numbers $\mathbb{R}$ and the field of real algebraic numbers $\mathbb{R}_{alg}$ with the natural order ($x \leq y \Leftrightarrow 0 \leq y - x$).

We next describe an order on the field $\mathbb{R}(X)$ of rational functions of X. If $P(X) = \sum_{i=k}^{n} a_i X^i$ where $0 \leq k \leq n$ are nonnegative integers and $a_k \neq 0$ then $P(X) > 0$ if and only if $a_k > 0$, and $\frac{P(X)}{Q(X)} > 0$ if and only if $P(X)Q(X) > 0$. An equivalent definition of this ordering is obtained by realizing that each rational function $P(X)/Q(X)$ defines a real-valued function $x \mapsto P(x)/Q(x)$ on any sufficiently small right neighborhood of 0 and then $P(X)/Q(X) > 0$ if and only if for all sufficiently small values of $x > 0$, $P(x)/Q(x) > 0$.

Every sum of squares in an ordered field $(F, \leq)$ is a positive element. Not every positive element in an ordered field $(F, \leq)$ is a sum of squares. However, if F is a real closed field, for every $x \in F$, either x is a square in F or $-x$ is a square in F. Therefore, there is a unique total order $\leq$ on a real closed field F so that $(F, \leq)$ is an ordered field; this unique order is defined by $x \leq y$ if and only if $y - x$ is a square.

3. Puiseux Series

We now turn to describe a field that plays an important role in the theory of stochastic games: the field of real Puiseux series. A Puiseux series (over a field F) is a formal expression f of the form

$$f = \sum_{i=k}^{\infty} a_i X^{i/M}$$

where $a_i \in F$ and M is a positive integer. In other words, a Puiseux series is a formal Laurent series in fractional powers of X. Two Puiseux series $f = \sum_{i=k}^{\infty} a_i X^{i/M}$ and $g = \sum_{j=\ell}^{\infty} b_j X^{j/N}$ are identified if and only if for all $i \geq k$ with $a_i \neq 0$, $j = iN/M$ is an integer $\geq \ell$ and $b_j = a_i$, and for all $j \geq \ell$ with $b_j \neq 0$, $i = jM/N$ is an integer $\geq k$ and $b_j = a_i$. Therefore, given a positive integer N, the Puiseux series $f = \sum_{i=k}^{\infty} a_i X^{i/M}$ is identified with the Puiseux series $f = \sum_{j=kN}^{\infty} \alpha_j X^{j/(MN)}$ where $\alpha_{iN} = a_i$ and $\alpha_j = 0$ whenever j is not a multiple of N. Therefore, given two Puiseux series $f = \sum_{i=k}^{\infty} a_i X^{i/M}$ and $g = \sum_{j=\ell}^{\infty} b_j X^{j/N}$ we can assume without loss of generality that $N = M$ and $k = \ell$, and with that assumption on the representation of f and g, the sum $f + g$ is defined as the formal sum, i.e., $f+g \equiv \sum_{i=k}^{\infty} a_i X^{i/M} + \sum_{i=k}^{\infty} b_i X^{i/M} := \sum_{i=k}^{\infty}(a_i+b_i)X^{i/M}$, and the product of f and g is defined as the formal Abel product of the series, i.e., as the Puiseux series $\sum_{i=2k}^{\infty} c_i X^{i/M}$ where the coefficients c_n, $n \geq 2k$, are defined by $c_n = \sum_{i=k}^{n-2k} a_i b_{n-i}$. The collection of all Puiseux series over a field F is a field $F(X)^{\wedge}$. If F is ordered so is $F(X)^{\wedge}$ by defining $\sum_{i=k}^{\infty} a_i X^{i/M} > 0$ whenever $a_k > 0$. It is known that $\mathbb{C}(X)^{\wedge}$ is algebraically closed ([7], p.98). Of particular importance are the subfields, $\mathbb{C}(X)^{c\wedge}$, $\mathbb{C}_{alg}^{c\wedge}$, $\mathbb{R}(X)^{c\wedge}$ and $\mathbb{R}_{alg}(X)^{c\wedge}$ of $\mathbb{C}(X)^{\wedge}$, consisting of all convergent Puiseux series, i.e., all series $\sum_{i=k}^{\infty} a_i X^{i/M}$ ($a_i \in \mathbb{C}$, $a_i \in \mathbb{C}_{alg}$, $a_i \in \mathbb{R}$ and $a_i \in \mathbb{R}_{alg}$ respectively), such that for all sufficiently small real numbers $x > 0$, the series $\sum_{i=k}^{\infty} |a_i| x^{i/M}$ converges. The fields $\mathbb{C}(X)^{c\wedge}$ and $\mathbb{C}_{alg}(X)^{c\wedge}$ are algebraically closed.

4. Semialgebraic Sets and Functions

Throughout the remainder of this chapter R is a fixed real closed field. The reader that is interested in the theory over the field of real numbers $\mathbb{R}$ can replace in all the definitions and results below the real closed field R with $\mathbb{R}$.

A subset V of R^n is a *semialgebraic set* if V belongs to the smallest Boolean ring of subsets (i.e., closed under complements, finite union and finite intersections) of R^n which contains the sets

$$\{x = (x_1, \ldots, x_n) \in R^n \mid p(x) \geq 0\}, \quad p \in R[X_1, \ldots, X_n].$$

Note that the inequality $\geq$ in the above definition can be replaced with any one of the following inequalities: $\leq$, $>$ or $<$. Indeed, the set $\{x = (x_1, \ldots, x_n) \in R^n \mid p(x) \leq 0\} = \{x = (x_1, \ldots, x_n) \in R^n \mid (-p)(x) \geq 0\}$, the sets $\{x = (x_1, \ldots, x_n) \in R^n \mid p(x) < 0\}$ and $\{x = (x_1, \ldots, x_n) \in R^n \mid p(x) > 0\}$ are the complement (in R^n) of the sets $\{x = (x_1, \ldots, x_n) \in R^n \mid p(x) \geq 0\}$ and $\{x = (x_1, \ldots, x_n) \in R^n \mid p(x) \leq 0\}$ respectively. Similarly, every real algebraic set $\{x = (x_1, \ldots, x_n) \in R^n \mid p(x) = 0\}$, $p \in R[X_1, \ldots, X_n]$, is the intersection of the two semialgebraic sets $\{x = (x_1, \ldots, x_n) \in R^n \mid p(x) \geq 0\}$ and $\{x = (x_1, \ldots, x_n) \in R^n \mid p(x) \leq 0\}$, and thus it is semialgebraic. An equivalent definition of a semialgebraic set is:

Definition 1 *A subset V of R^n is called* semialgebraic *if it is the finite union of sets of the form*

$$(\cap_{i=1}^{k}\{x \in R^n : P_i(x) = 0\}) \cap (\cap_{j=1}^{r}\{x \in R^n : Q_i(x) > 0\}),$$

where for every $i = 1, \ldots, k$ and every $j = 1, \ldots, r$, $P_i \in R[X_1, \ldots, X_n]$ and $Q_j \in R[X_1, \ldots, X_n]$.

Given $P_i \in R[X_1, \ldots, X_n]$, $1 \le i \le k$, then $P_i(x) = 0$ for every $1 \le i \le k$ if and only if $\sum_{i=1}^{k} P_i^2(x) = 0$. Therefore, another equivalent definition of a semialgebraic set is:

Definition 2 *A subset V of R^n is called* semialgebraic *if it is the finite union of sets of the form*

$$\{x \in R^n : P_0(x) = 0 \text{ and } P_i(x) > 0 \ \forall 1 \le i \le r\},$$

where for every $i = 0, \ldots, r$, $P_i \in R[X_1, \ldots, X_n]$.

Definition 3 *A function $\varphi : V \to U \subset R^n$ where $V \subset R^k$ is called* semialgebraic *if its graph, $\{(x, y) \in R^{k+n} \mid x \in V \text{ and } \varphi(x) = y\}$, is semialgebraic.*

We will later see (as a result of the Tarski–Seidenberg Theorem (Theorem 1 below)) that if $\varphi : V \to U \subset R^n$ is semialgebraic so is the set V. Therefore, an equivalent definition of a semialgebraic function is the one that requires in addition that the domain V be semialgebraic. A simple corollary of the above definitions is

Corollary 1 *For every semialgebraic function $\varphi : V \to R^n$, $V \subset R^k$, there is a non-zero polynomial $P \in R[X_1, \ldots, X_k, Y_1, \ldots, Y_n]$, such that $P(x, \varphi(x)) = 0$ for every $x \in V$.*

Proof. We prove the result for the case $R = \mathbb{R}$. The extension of the result to an arbitrary real closed field needs a minor modification to the argument below that uses open sets in $\mathbb{R}^{k+n}$, e.g., by defining properly Euclidean open sets in R^n. The graph of φ is the union of finitely many nonempty sets G_i, $i \in K$ (where K is a finite set), of the form $G_i = \{(x, y) \in \mathbb{R}^{k+n} \mid f_i(x, y) = 0, \text{ and } g_{i,j}(x, y) > 0, \ j = 1, \ldots, k_i\}$ with $f_i, g_{i,j} \in \mathbb{R}[X, Y]$. As φ is a function, its graph does not contain an open set, and therefore each one of the polynomials f_i is not identically zero, and thus the polynomial $P = \prod_i f_i \in \mathbb{R}[X, Y]$ satisfies $P \neq 0$ and $P(x, \varphi(x)) = 0$ for every $x \in V$. ∎

Remarks. The following are immediate corollaries of the definition of semialgebraic sets.

- Every finite subset $V \subset R^n$ is a semialgebraic set.
- If V is a semialgebraic subset of R^n then $V \times R^m$ is a semialgebraic subset of R^{n+m}.
- If V is a semialgebraic subset of R^{n+k} and $x \in R^n$, the set $\{y \in R^k \mid (x, y) \in V\}$ is a semialgebraic subset of R^k.
- Any semialgebraic subset of R is either empty or a finite union of intervals. Equivalently, the semialgebraic subsets of R are exactly the finite unions of points and open intervals (bounded or unbounded).

4.1. EXAMPLES OF SEMIALGEBRAIC SETS

E.1 For each fixed $x \in R^n$ and $r > 0$ the ball of radius r and center x, $\{y \in R^n : \sum_{i=1}^n (y_i - x_i)^2 < r^2\}$ is a semialgebraic subset of R^n, and $\{(x,r,y) \in R^{2n+1} : x \in R^n,\ r \in R,\ y \in R^n, \text{ and } \sum_{i=1}^n (y_i - x_i)^2 < r^2\}$ is a semialgebraic subset of R^{2n+1}.

E.2 If V is a semialgebraic subset of R^n, then $\{(x,r,y) \in R^{2n+1} : x \in R^n,\ r \in R,\ y \in V, \text{ and } \sum_{i=1}^n (y_i - x_i)^2 < r^2\}$ is a semialgebraic subset of R^{2n+1}.

For a fixed finite set K we denote by $\Delta(K)$ the simplex of probabilities on K, i.e., the subset of R^K given by $\{x \in R^K : \forall k \in K, x_k \geq 0, \text{ and } \sum_{k \in K} x_k = 1\}$, and if $K = \{1, \ldots, n\}$ we denote $\Delta(K)$ by Δ_n. We now turn to game-theoretic examples of semialgebraic sets.

4.2. GAME-THEORETIC EXAMPLES OF SEMIALGEBRAIC SETS

E.3 For any fixed list of real numbers a_{ij}, $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the optimal strategies and value of the two-person zero-sum game (a_{ij}), i.e., $\{(x,y,v) \in \mathbb{R}^{n+m+1} : x \in \Delta_n,\ y \in \Delta_m,\ v \in \mathbb{R}, \text{ s.t. } \forall j\ \sum_{i=1}^n x_i a_{ij} \geq v \text{ and } \forall i\ \sum_{j=1}^m y_j a_{ij} \leq v\}$ is a semialgebraic set.

E.4 For any fixed positive integers, n and m, the graph of the correspondence that maps each $n \times m$ two-person zero-sum game to the set of optimal strategies and value, i.e., the set $\{(a,x,y,v) \in \mathbb{R}^{nm+n+m+1} : a \in \mathbb{R}^{n \times m}, x \in \Delta_n,\ y \in \Delta_m,\ v \in \mathbb{R}, \text{ s.t. } \sum_{i=1}^n x_i a_{ij} \geq v\ \forall j \text{ and } \sum_{j=1}^m y_j a_{ij} \leq v\ \forall i\}$ is a semialgebraic subset of $\mathbb{R}^{nm+n+m+1}$.

E.5 Similarly, for any positive integers n, $m_1, \ldots, m_n$, the graph of the equilibrium correspondence that maps each n-person game in which player i has m_i pure strategies to the set of equilibrium strategies and corresponding payoffs is a semialgebraic subset of $\mathbb{R}^{n\prod_{i=1}^n m_i + \sum_{i=1}^n m_i + n}$.

4.3. EXAMPLES OF SEMIALGEBRAIC SETS IN STOCHASTIC GAMES

E.6 The value and optimal strategies correspondence of zero-sum two-person stochastic games. Consider the family of all two-person zero-sum stochastic games with a fixed finite state space S, and fixed finite action sets: for every player $i = 1, 2$ and every state $z \in S$ the set of actions of player i at state $z \in S$ is a finite set $A^i(z)$. We denote by $A(z)$ the cartesian products $A^1(z) \times A^2(z)$. The family of all two-person zero-sum stochastic games with the fixed state space S, and the fixed action sets $A(z)$, $z \in S$, is parameterized by the list of payoffs (to player 1) $r(z,a) \in \mathbb{R}$, ($z \in S$ and $a \in A(z)$), and the list of transition probabilities $p(z,a) \in \Delta(S)$, ($z \in S$ and $a \in A(z)$). A stationary strategy of player i is represented

by a list of vectors $x_z^i \in \Delta(A^i(z))$, $z \in S$. Then the set of all vectors $(\lambda, x_z^i, r(z,a), p(z,a), V_z)$ such that:
1) $0 < \lambda < 1$
2) $\forall z \in S$, $x_z^1 \in \Delta(A^1(z))$ and $x_z^2 \in \Delta(A^2(z))$
3) $\forall z \in S$ and $\forall a \in A(z)$, $r(z,a) \in \mathbb{R}$
4) $\forall z \in S$ and $\forall a \in A(z)$, $p(z,a) \in \Delta(S)$
5) $\forall z \in S$, $V_z \in \mathbb{R}$
6) $\forall z \in S$ and $\forall a^2 \in A^2(z)$,

$$\sum_{a^1 \in A^1(z)} x_z^1(a^1) \left(r(z,a^1,a^2) + \lambda \sum_{z' \in S} p(z,a^1,a^2)(z') V_{z'} \right) \geq V_z$$

and
7) $\forall z \in S$, $\forall a^1 \in A^1(z)$,

$$\sum_{a^2 \in A_z^2} x_z^2(a^2) \left(r(z,a^1,a^2) + \lambda \sum_{z' \in S} p(z,a^1,a^2)(z') V_{z'} \right) \leq V_z,$$

is a semialgebraic subset of

$$\mathbb{R}^{1+\sum_{z\in S}(|A^1(z)|+|A^2(z)|)+\sum_{z\in S}|A(z)|+\sum_{z\in S}|A(z)||S|+|S|}.$$

The above set is the set of all two-person zero-sum stochastic games with the fixed set of states S, discount factors λ, and all corresponding unnormalized value payoffs V_z ($z \in S$) with corresponding stationary optimal strategies $x_z^i \in \Delta(A^i(z))$ (of the discounted stochastic games with payoff functions described by the real numbers $r(z,a)$, $z \in S$ and $a \in A(z)$, and transitions described by the vectors $p(z,a) \in \Delta(S)$) and discount factor λ. Similarly for fixed payoffs $r(z,a)$, $z \in S$ and $a \in A(z)$, and fixed transitions $p(z,a) \in \Delta(S)$, $z \in N$ and $a \in A(z)$, the set of all vectors (λ, x_z^i, V_z) satisfying the polynomial inequalities and equalities 1), 2), 5), 6), and 7), is a semialgebraic set; it is the graph of the correspondence that maps each discount factor λ to the unnormalized value payoffs V_z ($z \in S$) with the corresponding stationary optimal strategies $x_z^i \in \Delta(A^i(z))$ of the discounted stochastic games with discount factor λ with payoff functions described by the real numbers $r(z,a)$, $z \in S$ and $a \in A(z)$, and transitions described by the vectors $p(z,a) \in \Delta(S)$.

E.7 The equilibrium correspondence of n-person stochastic games. Consider the family of all stochastic games with a fixed finite set of players $N = \{1, \ldots, n\}$, a fixed finite state space S, and fixed finite action sets: for every player $i \in N$ and every state $z \in S$ the set of actions of player $i \in N$

at state $z \in S$ is a finite set $A^i(z)$. We denote by $A(z)$ and $A^{-i}(z)$ the cartesian products $\times_{j\in N} A^j(z)$ and $\times_{j\in N, j\neq i} A^j(z)$ respectively. The family of all stochastic games with the fixed set of players N, the fixed state space S, and the fixed action sets $A(z)$, $z \in S$, is parameterized by the list of payoffs $r^i(z,a) \in \mathbb{R}$, ($i \in N$, $z \in S$ and $a \in A(z)$), and the list of transition probabilities $p(z,a) \in \Delta(S)$, ($z \in S$ and $a \in A(z)$). A stationary strategy of player i is represented by a list of vectors $x^i_z \in \Delta(A^i(z))$, $z \in S$. Then the set of all vectors $(\lambda, x^i_z, r^i(z,a), p(z,a), V^i_z)$ such that:
1) $0 < \lambda < 1$
2*) $\forall i \in N$ and $\forall z \in S$, $x^i_z \in \Delta(A^i(z))$
3*) $\forall i \in N$, $\forall z \in S$ and $\forall a \in A(z)$, $r^i(z,a) \in \mathbb{R}$
4) $\forall z \in S$ and $\forall a \in A(z)$, $p(z,a) \in \Delta(S)$
5*) $\forall i \in N$ and $\forall z \in S$, $V^i_z \in \mathbb{R}$
6*) $\forall z \in S$ and $\forall i \in N$,

$$\sum_{a\in A(z)} \Pi_{j\in N} x^j_z(a^j) \left(r^i(z,a) + \lambda \sum_{z'\in S} p(z,a)(z') V^i_{z'} \right) = V^i_z$$

and
7) $\forall z \in S$, $\forall i \in N$ and $\forall b^i \in A^i(z)$,

$$\sum_{a^{-i}\in A^{-i}(z)} \Pi_{j\in N,\, j\neq i} x^j_z(a^j) \left(r^i(z,a^{-i},b^i) + \lambda \sum_{z'\in S} p(z,a^{-i},b^i)(z') V^i_{z'} \right) \leq V^i_z,$$

is a semialgebraic subset of

$$\mathbb{R}^{1+\sum_{i\in N}\sum_{z\in S}|A^i(z)| + \sum_{z\in S}|N||A(z)| + \sum_{z\in S}|A(z)||S| + |N||S|}.$$

The above set is the set of all stochastic games with the fixed players and state sets N and S respectively, discount factors λ, and all corresponding unnormalized equilibrium payoffs V^i_z ($i \in N$ and $z \in S$) with corresponding stationary equilibrium strategies $x^i_z \in \Delta(A^i(z))$ (of the discounted stochastic games with discount factor λ, with payoff functions described by the real numbers $r^i(z,a)$, $z \in S$ and $a \in A(z)$, and transitions described by the vectors $p(z,a) \in \Delta(S)$). Similarly for fixed payoffs $r^i(z,a)$, $i \in N$, $z \in S$ and $a \in A(z)$, and fixed transitions $p(z,a) \in \Delta(S)$, $z \in N$ and $a \in A(z)$, the set of all vectors (λ, x^i_z, V^i_z) satisfying the polynomial inequalities and equalities 1), 2*), 5*), 6*), and 7*), is a semialgebraic set; it is the graph of the correspondence that maps the discount factors λ to all unnormalized equilibrium payoffs V^i_z ($i \in N$ and $z \in S$) with the corresponding stationary equilibrium strategies $x^i_z \in \Delta(A^i(z))$ of the discounted stochastic games with discount factor λ, with payoff functions described by the real numbers $r^i(z,a)$, $z \in S$ and $a \in A(z)$, and transitions described by the vectors $p(z,a) \in \Delta(S)$.

5. The Tarski–Seidenberg Theorem

In this section we state the Tarski–Seidenberg theorem. In a later section we will state a general structure theorem for semialgebraic sets from which the Tarski–Seidenberg theorem will follow.

The following is a statement of the Tarski–Seidenberg theorem in a geometric form.

Theorem 1 *Let $V \subset R^{n+m}$ be a semialgebraic set, and let $\pi : R^{n+m} \to R^n$ be the natural projection on the first n coordinates; i.e., $\pi(x_1, \ldots, x_{n+m}) = (x_1, \ldots, x_n)$. Then $\pi V \subset R^n$ is semialgebraic.*

Notice that the natural projection on the first n coordinates of a subset $V \subset R^{n+m}$ is the set $\{x \in R^n \mid \exists y \in R^m \text{ s.t. } (x,y) \in V\}$. Therefore, an equivalent statement of the Tarski–Seidenberg theorem asserts that for every semialgebraic set $V \subset R^{n+m}$, the set $\{x \in R^n \mid \exists y \in R^m \text{ s.t. } (x,y) \in V\}$ is semialgebraic. Similarly, the set $\{x \in R^n \mid \forall y \in R^m \ (x,y) \in V\}$ is the complement of $\{x \in R^n \mid \exists y \in R^m \text{ s.t. } (x,y) \in V^c\}$ (where V^c denotes the complement of V) and thus it is semialgebraic. We now state a corollary of the Tarski–Seidenberg theorem which extends the above observation to an arbitrary number of universal quantifiers.

Corollary 2 *Assume that $k, m_1, \ldots, m_k$ are positive integers, that $V_i \subset R^{m_i}$, $1 \le i \le k$, are semialgebraic sets, and that V is a semialgebraic subset of $R^{\sum_{i=1}^k m_i}$. Then, if for every $1 < i \le k$, Q_i stands for either $\exists x^i \in V_i$ s.t. or $\forall x^i \in V_i$, then the set*

$$V_Q = \{x^1 \in V_1 \mid Q_k \ldots Q_2 \ (x^1, \ldots, x^k) \in V\}$$

is semialgebraic.

Proof. The proof is by induction on k. For $k = 2$, the set $\{x^1 \in V_1 \mid \exists x^2 \in V_2 \text{ s.t. } (x^1, x^2) \in V\}$ is the projection on the first m_1 coordinates of the set $V \cap V_1 \times V_2$, and the complement in V_1 of the set $\{x^1 \in V_1 \mid \forall x^2 \in V_2 \ (x^1, x^2) \in V\}$ is the projection on the first m_1 coordinates of the semialgebraic set $V^c \cap V_1 \times V_2$. Therefore, if $k = 2$, the set V_Q is semialgebraic. Assume that $k > 2$. By the induction hypothesis the set

$$U = \{(x^1, x^k) \in V_1 \times V_k \mid Q_{k-1} \ldots Q_2 \ (x_1, \ldots, x_{k-1}, x_k) \in V\}$$

is semialgebraic, and

$$V_Q = \{x^1 \in V_1 \mid Q_k \ (x^1, x^k) \in U\}$$

and thus V_Q is semialgebraic. ∎

6. Applications of the Tarski–Seidenberg Theorem

A useful property of continuous semialgebraic functions $\varphi : (0, r) \to \mathbb{R}$ is stated in the next proposition. A more detailed property of semialgebraic functions $\varphi : (0, r) \to \mathbb{R}$ is stated later.

Proposition 1 *Let $\varphi : (0, r) \to \mathbb{R}$ be a continuous semialgebraic function. Then there exists $0 < \theta < r$ such that φ is monotonic on $(0, \theta)$.*

Proof. By Corollary 1, there is $f \in \mathbb{R}[X, Y]$ with $f \neq 0$ and $f(x, \varphi(x)) = 0$ for every $x \in (0, r)$. In case that the degree of the polynomial f with respect to the variable X, $\deg_X f$ is 0, the function φ is a constant. We prove the proposition by induction on $\deg_X f + \deg_Y f$ where $f \in \mathbb{R}[X, Y]$ with $f(x, \varphi(x)) = 0$ for every $x \in (0, r)$.

For every choice of signs $\epsilon = (\epsilon_1, \epsilon_2) \in \{-1, 0, 1\}^2$, the set

$$A_\epsilon = \{x \in (0, r) \mid \text{sign} \frac{\partial f}{\partial X}(x, \varphi(x)) = \epsilon_1 \text{ and } \text{sign} \frac{\partial f}{\partial Y}(x, \varphi(x)) = \epsilon_2\}$$

is semialgebraic and $\cup_\epsilon A_\epsilon = (0, r)$. As every semialgebraic subset of $\mathbb{R}$ is the union of finitely many open intervals and finitely many points, there is ϵ and $0 < \theta < r$ such that $A_\epsilon \supset (0, \theta)$. In case that either ϵ_1 or ϵ_2 equals 0, the monotonicity of φ follows from the induction hypothesis. As the functions $\frac{\partial f}{\partial X}$ and $\frac{\partial f}{\partial Y}$ are continuous on $\mathbb{R}^2$ and the function φ is continuous on $(0, \theta)$, the function φ is monotonic decreasing on $(0, \theta)$ whenever $\epsilon_1 \epsilon_2 > 0$ and it is monotonic increasing whenever $\epsilon_1 \epsilon_2 < 0$. ∎

The above monotonicity (in a sufficiently small right neighborhood of 0) of a semialgebraic function $\varphi : (0, r) \to \mathbb{R}$ holds also for an arbitrary real closed field R, and moreover, the continuity assumption is not needed. However, when the field is either $\mathbb{R}$ or $\mathbb{R}_{alg}$, the result is derived also from the following more detailed property of a real semialgebraic (or real algebraic) function:

Theorem 2 *Let $\varphi : (0, \varepsilon) \to R$ be a semialgebraic function, where R stands for either the field of real numbers $\mathbb{R}$ or the field of real algebraic numbers $\mathbb{R}_{alg}$. Then, there exist a positive integer M, an integer k, a positive constant $\delta > 0$ and a sequence of real numbers $a_k, a_{k+1}, \ldots \in R$ such that $\sum_{i=k}^{\infty} a_i x^{i/M}$ converges and equals $\varphi(x)$ for $0 < x < \delta$.*

Proof.[1] It follows from Corollary 1 that there is a polynomial $P(X, Y)$, $P \in R[X, Y]$, such that $P(x, \varphi(x)) = 0$. Consider the polynomial P as a

[1] An alternative proof (see [6], Lemma 6.2) to the one presented here relies on the theory of algebraic functions (see, for instance, [3], Theorem 13.1).

polynomial Q in the variable Y and with coefficients in $R[x]$. Assume that the degree of the polynomial P with respect to Y is n. It follows that P can be represented as $P(X,Y) = \sum_{i=1}^{n} P_i(X)Y^i$ with $P_n \neq 0$ and thus we can identify $P \in R[X,Y]$ with $Q \in (R[X])[Y]$. Note that $R[X]$ is identified canonically with a subset of the algebraically closed field F of convergent fractional power series with coefficients in R. Therefore, there are distinct elements $f_1, \ldots, f_k \in F$, $k \leq n$, and positive integers $n_1, \ldots, n_k$ such that $Q(Y) = P_n(X)\Pi_{i=1}^{k}(Y - f_i)^{n_i}$ and thus for every $x > 0$ sufficiently small (so that $P_n(x) \neq 0$ and all the series defined by f_i, $i = 1, \ldots, k$, converge), there is $1 \leq i \leq k$ such that $\varphi(x) = f_i(x)$. Let $r > 0$ be sufficiently small so that for all $0 < x < r$ the series defined by $f_i(x)$, $i = 1, \ldots, k$, converge and $f_i(x) \neq f_j(x)$ whenever $1 \leq i < j \leq k$, $P_n(x) \neq 0$, and the function φ is continuous on $(0, r)$. It follows that on the interval $(0, r)$ the function φ coincides with one of the functions $x \mapsto f_i(x)$, $1 \leq i \leq k$. As φ is real-valued all the coefficients of the Puiseux series f_i are necessarily real. Moreover, if $f_i = \sum_{j=k}^{\infty} a_j X^{j/M}$ is a root of Q and all coefficients of Q are polynomials in $\mathbb{R}_{alg}[X]$, it follows by induction on $j - k$ that $a_j \in \mathbb{R}_{alg}$. ∎

We now state an important implication of the above results to two-person zero-sum stochastic games:

Theorem 3 (Bewley and Kohlberg [2]) *For any two-player zero-sum stochastic game with finitely many states and actions, the functions $\lambda \mapsto v_\lambda$, $v_\lambda = (v_\lambda(z))_{z \in S}$, where v_λ is the λ-discounted value, are monotonic (and thus in particular of bounded variation) in a right neighborhood of 0. Moreover, these functions are given, for sufficiently small values of λ, by convergent series in fractional powers of λ. I.e., there are 1) a positive integer M, 2) series of real numbers $(a_i(z))_{i=1}^{\infty}$, $z \in S$, and 3) $\lambda_0 > 0$, such that the series $\sum_{i=1}^{\infty} a_i(z)\lambda^{i/M}$ converges for every $0 < \lambda < \lambda_0$ to $v_\lambda(z)$.*

Proof. For each fixed initial state z, the map $\lambda \mapsto v_\lambda(z)$ is semialgebraic; its graph is the projection of a semialgebraic set (see E.6). The result follows from the previous theorem. ∎

We continue with classic applications of the Tarski–Seidenberg theorem. The first corollary is often called the Tarski–Seidenberg theorem.

Corollary 3 *Let $X \subset R^n$ and $Y \subset R^m$. Assume that $f : X \to Y$ is a semialgebraic map. Then X and the image $f(X) \subset Y$ are semialgebraic sets.*

Proof. Let G be the graph of f. As f is semialgebraic, G is a semialgebraic subset of R^{n+m}. Note that $f(X)$ coincides with $\pi(G)$ where $\pi : R^n \times R^m \to R^m$ is the natural projection of $R^n \times R^m$ to R^m and X coincides with the natural projection of G on R^n. ∎

Corollary 4 *The composition of semialgebraic functions is a semialgebraic function.*

Proof. Assume $X \subset R^n$, and $\varphi : X \to Y \subset R^m$ and $\psi : Y \to Z \subset R^k$ are semialgebraic. Therefore the set $\{(x, y, z) \in R^{n+m+k} \mid x \in X,\ y = \varphi(x) \text{ and } z = \psi(y)\}$ is the intersection of two semialgebraic sets and thus it is semialgebraic. Its projection on $R^n \times R^k$ is the graph of the composition $\psi \circ \varphi$ and by the Tarski–Seidenberg theorem it is semialgebraic. ∎

Corollary 5 *Let $V \subset R^k$ and let $f : V \to R^n$, $g : V \to R^n$ and $h : V \to R$ be semialgebraic functions. Then $f + g : V \to R^n$ and $hf : V \to R^n$ are semialgebraic functions.*

Proof. The graph of the function $f + g$ is the projection on the first $k + n$ coordinates of the semialgebraic set $\{(v, x, y, z) \in R^{k+n+n+n} \mid v \in V,\ x = y + z,\ f(v) = y \text{ and } g(v) = z\}$. The graph of the function hf is the projection on the first $k + n$ coordinates of the semialgebraic set $\{(v, x, y, r) \in R^{k+n+n+1} \mid v \in V,\ f(v) = y,\ h(v) = r, \text{ and } x = ry\ \}$. ∎

The next corollary deals with the closure and interior of a semialgebraic set $V \subset R^n$ where the closure and interior are with respect to the Euclidean topology on R^n which extends the classical Euclidean topology on $\mathbb{R}^n$.

The Euclidean norm of an element $x = (x_1, \ldots, x_n) \in R^n$ is defined as $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$. The Euclidean topology on R^n is defined as the topology for which the open balls, $\{y \in R^n \mid \|y - x\| < r\}$ where $x = (x_1, \ldots, x_n) \in R^n$ and $0 < r \in R$, form a basis of open subsets. Continuity of R^n-valued semialgebraic functions is defined as continuous functions with respect to the Euclidean topology. In other words, using the ε, δ language, we observe that a semialgebraic function $\varphi : V \to R^n$, where $V \subset R^k$ is continuous at $x \in V$ if for every $0 < \varepsilon$ in R there is $0 < \delta$ in R such that for every $y \in V$ with $\|y - x\| < \delta$, $\|\varphi(y) - \varphi(x)\| < \varepsilon$.

Corollary 6 *Let V be a semialgebraic set in R^n. Then the closure $\bar{V}$ of V, its interior and its frontier are semialgebraic sets.*

Proof. Note that as the family of semialgebraic sets is closed under complementation and finite intersection, it is sufficient to prove that the closure of a semialgebraic set is semialgebraic. Set $U = \{(x, r, y) \in R^{n+1+n} : \|x - y\|^2 < r^2\}$. The set U is semialgebraic and $\bar{V} = \{x \in R^n : \forall r > 0\ \exists y \in V \text{ s.t. } (x, r, y) \in U\}$. Thus by Corollary 2, $\bar{V}$ is semialgebraic. ∎

We continue with a less classic application of the Tarski–Seidenberg theorem. The functions defined in the lemma below use the infimum (largest lower bound) and supremum (smallest upper bound) of semialgebraic sets. A bounded set of a real closed field need not have a least upper bound. However, as every semialgebraic subset V of a real closed field R is the finite union of open intervals and single points, every semialgebraic subset

V of a real closed field F that is bounded from above (bounded from below) has a least upper bound (a largest lower bound).

Lemma 1 *Let V be a nonempty semialgebraic subset of R^{n+m+k}, and assume that $f : V \to R$ is a bounded semialgebraic function. Let $V_1(z)$, z in the image of the natural projection of V on the last k coordinates, denote the semialgebraic set of all elements $x \in R^n$ for which there is $y \in R^m$ with $(x, y, z) \in V$. Then,*

$$z \mapsto \inf_{x \in V_1(z)} \sup_{\{y|(x,y,z) \in V\}} f(x, y, z) = \overline{v}_f(z)$$

and

$$z \mapsto \sup_{x \in V_1(z)} \inf_{\{y|(x,y,z) \in V\}} f(x, y, z) = \underline{v}_f(z)$$

are R-valued semialgebraic functions defined on the natural projection of V on R^k.

Proof. As $\underline{v}_f = -\overline{v}_{-f}$, it suffices to prove that $\overline{v}_f$ is semialgebraic. Let V_2 be the projection of $V \subset R^{n+m+k}$ on R^{n+k} ($(x, y, z) \mapsto (x, z)$), and let $g : V_2 \to R$ be the function defined on V_2 by $g(x, z) = \sup_{\{y|(x,y,z) \in V\}} f(x, y, z)$. The graph of g is the semialgebraic set $\{(x, z, r) \mid (x, z) \in V_2,\ \forall \varepsilon > 0\ \exists y \in R^m$ s.t. $(x, y, z) \in V$ and $f(x, y, z) \geq r - \varepsilon$ and $f(x, y, z) \leq r \forall y \in R^m$ s.t. $(x, y, z) \in V\}$. The graph of the function $\overline{v}$ is the following semialgebraic set: $\{(z, v) \in R^{k+1} : \forall \varepsilon > 0\ \exists x \in R^n$ s.t. $(x, z) \in V_2$ and $g(x, z) \leq v + \varepsilon,$ and $\forall x \in R^n$ with $(x, z) \in V_2,\ g(x, z) \geq v\}$. ∎

The next theorem asserts that given a semialgebraic nonempty-valued correspondence Γ, it is always possible to select a semialgebraic function f whose graph is a subset of the graph of Γ.

Theorem 4 *Let $X \subset R^n \times R^m$ be a semialgebraic set, and let π be the natural projection of $R^n \times R^m$ onto R^n. Then there is a semialgebraic function $f : \pi(X) \to R^m$ whose graph is a subset of X.*

Proof. We first provide a simple proof for the special case where $R = \mathbb{R}$ and $\pi^{-1}(x) := \{y \in \mathbb{R}^m \mid (x, y) \in X\}$ is compact for every $x \in \pi(X)$. This special case is often used in applications to stochastic games. Define inductively the decreasing sequence of semialgebraic sets $X_0, X_1, \ldots, X_m$ by $X_0 = X$ and $X_i = \{(x, y) \in X_{i-1} \mid \forall y' \in \mathbb{R}^m$ with $(x, y') \in X_{i-1}\ y'_i \leq y_i\}$. It follows by induction that X_i is a semialgebraic subset of X, $\pi(X_i) = \pi(X)$, $\pi^{-1}(x) := \{y \in \mathbb{R}^m \mid (x, y) \in X_i\}$ is compact for every $x \in \pi(X_i)$, and that X_m is the graph of a function $f : \pi(X) \to \mathbb{R}^m$. The proof of the general case uses the structure theorem that is stated in the last section and is by induction on m. For $m = 1$ it follows from the structure theorem. Indeed, the structure theorem asserts that there is a partition of R^n into finitely many (connected) semialgebraic sets, $\mathcal{T}$, such that for any $A \in \mathcal{T}$ there is a nonnegative integer s_A and (continuous) semialgebraic functions $f_i^A : A \to R$, $i = 1, \ldots, s_A$ with $f_i^A < f_{i+1}^A$, such that setting $T_i = \{(x,t) \mid f_i^A(x) = t\}$, $i = 1, \ldots, s_A$, $S_0 = \{(x,t) \mid t < f_1^T(x)\}$, $S_{s_A} = \{(x,t) \mid t > f_{s_A}^A(x)\}$ and for $1 \leq j < s_A$, $S_j = \{(x,t) | f_j^A(x) < t < f_{j+1}^A\}$, such that

$$X \cap (A \times R) = (\cup_{i:T_i \subset X} T_i) \cup (\cup_{j:S_j \subset X} S_j).$$

Therefore, for every $A \in \mathcal{T}$, either $A \subset \pi(X)$ or $A \cap \pi(X) = \emptyset$. For every $A \in \mathcal{T}$ with $A \subset \pi(X)$ we define $f_T : T \to R$ as follows. If $A \in \mathcal{T}$ then there is either $1 \leq i \leq s_A$ with $T_i \subset X$ and then we define $f_A(x) = f_i^T(x)$. Otherwise, there is $0 \leq j \leq s_A$ with $S_j \subset X$. If $S_0 \subset X$ we define $f_A(x) = f_1^A(x) - 1$, or otherwise if $S_{s_A} \subset X$ we define $f_A(x) = f_{s_A}^A(x) + 1$, or otherwise there is $1 \leq i < s_A$ such that $S_i \subset X$ and in that case we define $f_A(x) = (f_i^T(x) + f_{i+1}^T(x))/2$. Next we define $f : \pi(X) \to R$ as the collation of all functions f_A with $A \subset \pi(X)$, i.e., $f(y) = f_A(y)$ if $y \in A$.

Assume the result is true for m and let $X \subset R^n \times R^m \times R$ be semialgebraic. Let π be the projection of $R^n \times R^m \times R$ onto R^n. Let π_2 be the projection of $R^n \times R^m \times R$ onto $R^n \times R^m$ and let π_1 be the projection of $R^n \times R^m$ onto R^n. Note that $\pi = \pi_1 \pi_2$. Then $\pi_2(X)$ is a semialgebraic subset of $R^n \times R^m$ and by the induction hypothesis there are semialgebraic functions $f_1 : \pi_1(\pi_2(X)) \to R^m$ and $f_2 : \pi_2(X) \to R$ whose graphs are subsets of $\pi_2(X)$ and X respectively. The function $f : \pi(X) \to R^m \times R$ defined by $f(y) = (f_1(y), f_2(y, f_1(y)))$ is semialgebraic and its graph is a subset of X. ∎

We now state an application of the algebraic tools — the semialgebraic selection theorem (Theorem 4) and Theorem 1 — to the λ-discounted equilibrium correspondence. If $\lambda = (\lambda_i)_{i \in N}$ is a profile of discount rates, a λ-discounted equilibrium is a profile of strategies σ such that for every player i and every strategy τ^i of player i we have

$$E_{\sigma^{-i},\tau^i}(\sum_{n=1}^{\infty} \lambda_i (1-\lambda_i)^{n-1} r^i(z_n, a_n)) \leq v_\lambda^i(z_1)$$

where

$$v_\lambda^i(z_1) = E_\sigma(\sum_{n=1}^{\infty} \lambda_i (1-\lambda_i)^{n-1} r^i(z_n, a_n)).$$

Every λ-discounted game with a profile of discount rates λ has an equilibrium in stationary strategies.

Theorem 5 *For every stochastic game with finitely many players, states and actions, if* $t \mapsto \lambda(t) = (\lambda_i(t))_{i \in N}$ *is a semialgebraic function from* $(0,1)$

to $(0,1)$*, then there are semialgebraic maps* $t \mapsto x^i_{\lambda(t)}(z) \in \Delta(A^i(z))$*,* $i \in N$ *and* $z \in S$*, and* $t \mapsto v^i_{\lambda(t)}(z)$ *such that* $x^i_{\lambda(t)}$ *is a stationary equilibrium of the* $\lambda(t)$*-discounted stochastic game with equilibrium payoffs* $v^i_{\lambda(t)}(z)$ *to player* $i \in N$ *when the starting state is* z*.*

7. Minmax and Maxmin in Stochastic Games

We start with the introduction of constrained stochastic games. Assume that in addition to the ordinary description of a stochastic game we have for every state z and player i a subset $X^i(z)$ of the mixed actions in state z of player i. We study the stochastic game in which player i is restricted to behavioral strategies σ^i such that for every history $(z_1, a_1, \dots, z_n)$, the mixed action $\sigma^i(z_1, a_1, \dots, z_n)$ is in $X^i(z)$. Such a strategy is called a $(X^i(z))_{z\in S}$-*constrained strategy*, or a X^i-*constrained strategy* for short. Consider a two-person constrained stochastic game. We say that $\underline{v}_\lambda$ is the *maxmin* of the λ-discounted constrained stochastic game if for every $\varepsilon > 0$,
1) there is a X^1-constrained strategy σ of player 1 such that for every X^2-constrained strategy τ of player 2 and every initial state z_1

$$E^{z_1}_{\sigma,\tau}\left(\sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} r_n\right) \geq \underline{v}_\lambda(z_1) - \varepsilon$$

where $r_n = r(z_n, a_n)$, and
2) for every X^1-constrained strategy σ of player 1 there is a X^2-constrained strategy τ of player 2 such that for every initial state z_1

$$E_{\sigma,\tau}\left(\sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} r_n\right) \leq \underline{v}_\lambda(z_1) + \varepsilon.$$

Let Φ be the map $\Phi : (0,1) \times \mathbb{R}^S \to \mathbb{R}^S$ defined by

$$[\Phi(\lambda, v)](z) = \sup_{x\in X^1(z)} \inf_{y\in X^2(z)} \left[\lambda r(z,x,y) + (1-\lambda)\sum_{z'\in S} p(z' \mid z,x,y) v(z')\right]$$

where for $x \in \Delta(A^1(z))$ and $y \in \Delta(A^2(z))$, $r(z,x,y)$ and $p(z' \mid z,x,y)$ are the multilinear extensions of r and p respectively, i.e.,

$$r(z,x,y) = \sum_{a\in A^1(z)} \sum_{b\in A^2(z)} x(a)y(b)r(z,a,b)$$

and

$$p(z' \mid z,x,y) = \sum_{a\in A^1(z)} \sum_{b\in A^2(z)} x(a)y(b)p(z' \mid z,a,b).$$

For each fixed $0 < \lambda < 1$ the map $v \mapsto Tv := \Phi(\lambda, v)$ is monotonic and $T(v+\alpha 1_S) = Tv + (1-\lambda)\alpha 1_S$ and therefore $\|Tv - Tu\|_\infty \leq (1-\lambda)\|v-u\|_\infty$ and thus T has a unique fixed point $\underline{w}_\lambda$. For every $\varepsilon > 0$ and every state z let $x(z) \in X^i(z)$ be such that for every $y \in X^2(z)$,

$$\lambda r(z, x(z), y) + (1-\lambda)\sum_{z' \in S} p(z' \mid z, x(z), y)\underline{w}_\lambda(z') \geq [T\underline{w}_\lambda](z) - \varepsilon\lambda.$$

Let $\mathcal{H}_n$ denote the σ-algebra (of sets of plays of the stochastic game) generated by the sequence of states and actions $(z_1, a_1, \ldots, z_n)$. (Note that $\mathcal{H}_n$ is an algebra whenever the game has finitely many states and actions.) Let σ be the behavioral strategy of player 1 such that $\sigma(z_1, a_1, \ldots, z_n) = x(z_n)$. Then for every $(X^2(z))_{z\in S}$-constrained strategy τ of player 2,

$$E_{\sigma,\tau}(\lambda r_n + (1-\lambda)\underline{w}_\lambda(z_{n+1}) \mid \mathcal{H}_n) \geq \underline{w}_\lambda(z_n) - \varepsilon\lambda$$

and therefore by taking expectations in the above inequality and multiplying it by $(1-\lambda)^{n-1}$ we have

$$\begin{aligned} E_{\sigma,\tau}(\lambda(1-\lambda)^{n-1} r_n) + (1-\lambda)^n E_{\sigma,\tau}(\underline{w}_\lambda(z_{n+1})) \;\; \geq \;\; & (1-\lambda)^{n-1} E_{\sigma,\tau}(\underline{w}_\lambda(z_n)) \\ & - \varepsilon\lambda.(1-\lambda)^{n-1} \end{aligned}$$

Summing the above inequalities over $n = 1, 2 \ldots$, we conclude that

$$E^{z_1}_{\sigma,\tau}\left(\sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} r_n\right) \geq \underline{w}_\lambda(z_1) - \varepsilon.$$

Similarly, for every $(X^1(z))_{z\in S}$-constrained strategy σ of player 1 and $\varepsilon > 0$, let τ be the $(X^2(z))_{z\in S}$-constrained strategy of player 2 such that for every history $(z_1, a_1, \ldots, z_n)$, $\tau(z_1, a_1, \ldots, z_n)$ is an element $y \in X^2(z_n)$ such that

$$\lambda r(z, x, y) + (1-\lambda)\sum_{z' \in S} p(z' \mid z, x, y)\underline{w}_\lambda(z) \leq \underline{w}_\lambda(z) + \varepsilon\lambda,$$

where $x = \sigma(z_1, a_1, \ldots, z_n)$. It follows that for every positive integer n

$$E_{\sigma,\tau}(\lambda r_n + (1-\lambda)\underline{w}_\lambda(z_{n+1}) \mid \mathcal{H}_n) \leq \underline{w}_\lambda(z_n) + \varepsilon\lambda.$$

Multiplying the above inequality by $(1-\lambda)^{n-1}$ and summing the resulting inequalities over $n = 1, 2, \ldots$, we deduce that

$$E^{z_1}_{\sigma,\tau}\left(\sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} r_n\right) \leq \underline{w}_\lambda(z_1) + \varepsilon,$$

and therefore $\underline{w}_\lambda$ is the maxmin value of player 1 in the two-player constrained λ-discounted stochastic games.

Similarly, define the map $\Psi : (0,1) \times \mathbb{R}^S \to \mathbb{R}^S$ by

$$[\Psi(\lambda, v)](z) = \inf_{y \in X^2(z)} \sup_{x \in X^1(z)} \left[\lambda r(z,x,y) + (1-\lambda) \sum_{z' \in S} p(z' \mid z, x, y) v(z) \right].$$

The map $v \mapsto \Psi(\lambda, v)$ has a unique fixed point $\bar{w}_\lambda$ which is the minmax value of the constrained stochastic game, i.e., for every $\varepsilon > 0$,
1) there is a X^2-constrained strategy τ of player 2 such that for every X^1-constrained strategy σ of player 1 and every initial state z_1

$$E_{\sigma,\tau}\left(\sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} r_n\right) \leq \underline{w}_\lambda(z_1) + \varepsilon,$$

and
2) for every X^2-constrained strategy τ of player 2 there is a X^1-constrained strategy σ of player 1 such that for every initial state z_1

$$E^{z_1}_{\sigma,\tau}\left(\sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} r_n\right) \geq \underline{w}_\lambda(z_1) - \varepsilon.$$

Therefore the minmax of the λ-discounted constrained stochastic game, $\bar{v}_\lambda$, exists and equals $\bar{w}_\lambda$.

If the constrained sets $X^i(z)$ are semialgebraic, so are the maps Φ and Ψ, and therefore the maps $\lambda \mapsto \underline{w}_\lambda$ and $\lambda \mapsto \bar{w}_\lambda$ are semialgebraic. Moreover, for every $\varepsilon > 0$ there is a semialgebraic function mapping a discount factor λ to a $(X^i(z))_{z\in S}$-constrained stationary strategy σ_λ such that for every $(X^2(z))_{z\in S}$-constrained strategy τ of player 2, $E_{\sigma_\lambda,\tau}(\sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} r_n) \geq \underline{v}_\lambda(z_1) - \varepsilon$. In addition, if the supremum in the definition of $[\Phi(\lambda, \underline{v}_\lambda)](z)$ is attained, there is such a function $\lambda \mapsto \sigma_\lambda$ which is independent of ε. The following theorem is a partial summary of the above.

Theorem 6 *The maxmin $\underline{v}_\lambda$ and the minmax $\bar{v}_\lambda$ of a λ-discounted two-player constrained stochastic game with finitely many states and actions exist. Moreover, if the constraining sets $X^i(z)$ are semialgebraic subsets of $\Delta(A^i(z))$, then the maps $\lambda \mapsto \underline{v}_\lambda$ and $\lambda \mapsto \bar{v}_\lambda$ are semialgebraic.*

In an n-player λ-discounted stochastic game with finitely many states and actions, the maxmin $\underline{v}^i_\lambda(z)$ and the minmax $\bar{v}^i_\lambda(z)$ of player i, as a function of the initial state z, are equal to

$$\max_{\sigma^i} \min_{\sigma^{-i}} E^z_{\sigma^i,\sigma^{-i}}\left(\sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} r^i_n\right)$$

and

$$\min_{\sigma^{-i}} \max_{\sigma^i} E^z_{\sigma^i,\sigma^{-i}} \left(\sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} r^i_n \right)$$

respectively, where the max is over all strategies σ^i of player i and the min is over all $N \setminus \{i\}$ tuples of strategies σ^{-i} of the other players, and r^i_n is the payoff $r^i(z_n, a_n)$ to player i at stage n as a function of the state z_n and action profile a_n at stage n. These maxmin and minmax of player i are the maxmin and minmax of a two-person zero-sum constrained stochastic game: player 1, the maximizer, is player i with a constrained set $X^1(z) = \Delta(A^i(z))$, and player 2, the minimizer, is the set of players $N \setminus \{i\}$ with a constrained set $X^2(z) = \times_{j \neq i} \Delta(A^j(z))$. Thus, a special case of Theorem 6 is:

Corollary 7 *The maxmin $\underline{v}^i_\lambda$ and the minmax $\bar{v}^i_\lambda$ of player i in an n-player λ-discounted stochastic game with finitely many states and actions exist, and the functions $\lambda \mapsto \underline{v}^i_\lambda$ and $\lambda \mapsto \bar{v}^i_\lambda$ are semialgebraic.*

8. A Structure Theorem

We here state a structure theorem for semialgebraic sets (see [1] for the case $R = \mathbb{R}$).

Theorem 7 *Let V be a semialgebraic set in R^n. Then*

a) V has a finite number of connected components and each such component is semialgebraic.

b) There exists a partition of R^{n-1} into finitely many connected semialgebraic sets, such that for any element A of the partition there is a nonnegative integer s_A and functions

$$f^A_k : A \to \bar{R} \text{ (where } \bar{R} = R \cup \{\infty\} \cup \{-\infty\})$$

$k = 0, 1, \ldots, s_A, s_A + 1$ such that

i) $f^A_0 = -\infty$, $f^A_{s_A+1} = \infty$;

ii) $f^A_k : A \to R$, $k = 1, \ldots, s_A$, is a continuous function and, for every $x \in A$, $f^A_k(x) < f^A_{k+1}(x)$;

iii) all the sets of the form

$$\{(x,t) \in R^n : x \in A, f^A_k(x) < t < f^A_{k+1}\},\ k = 0, 1, \ldots, s_A,$$

or

$$\{(x,t) \in R^n : x \in A, f^A_k(x) = t\},\ k = 1, \ldots, s_A,$$

are semialgebraic; and

iv) the subcollection of all sets defined in part iii) which are contained in V makes a partition of V.

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