ZERO-SUM STOCHASTIC GAMES WITH BOREL STATE SPACES

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1. Introduction

The finite state space stochastic game model by Shapley [31] covered in [33] was generalized among others by Maitra and Parthasarathy [17], [18] who considered compact metric state spaces. They imposed rather strong regularity conditions on the reward and transition structure in the game and considered the discounted payoff criterion only. Their results have been generalized by many authors. For a good survey of the results which are not reported here the reader is referred to [14], [23], [25], [27], [13].

As explained in [32], a major difficulty arises when the game has a continuum of states and we provide here a general Borel space framework for zero-sum stochastic games. Our main goal is to present a property of the Shapley operator in a measurable framework and a corresponding measurable selection theorem (Theorem 1) which has natural applications to studying discounted and positive stochastic games with discontinuous payoffs [23] and also to studying general classes of Borel stochastic games with limsup payoffs [19], [20], [21] or more generally Borel payoff function. Some applications of this result are also given in Section 2. Universally measurable strategies for the players are natural if we deal with discontinuous games as in [23], [25] or when we apply stopping time techniques as in [19], [20].

Section 2 recalls the model and states the results. Measure-theoretical definitions and notions are recalled in Section 3, which also provides basic material used in the proof which is deferred to Section 4.

2. Discounted and Positive Zero-Sum Stochastic Games

Recall that a *zero-sum stochastic game* is described by the following objects: (i) S is a *set of states* for the game and is assumed to be a Borel space

(Section 3.1).
(ii) A and B are the *action spaces* for players 1 and 2, respectively, and are also assumed to be Borel spaces.
(iii) F and G are nonempty Borel subsets of $S \times A$ and $S \times B$, respectively. We assume that for each $s \in S$, the nonempty s-section

$$F(s) = \{a \in A : (s, a) \in F\}$$

of F represents the *set of actions available* to player 1 in state s. Analogously, we define $G(s)$ for each $s \in S$. Define

$$H = \{(s, a, b) : s \in S,\ a \in F(s) \text{ and } b \in G(s)\}$$

which is a Borel subset of $S \times A \times B$.

(*iv*) p is a Borel measurable transition probability from H to S, called the *law of motion among states*. If s is a state at some stage of the game and the players select actions $a \in F(s)$ and $b \in G(s)$, then $p(\cdot|s, a, b)$ is the probability distribution on S of the next state of the game.

(*v*) $r : H \mapsto \mathbb{R}$ is a Borel measurable *reward function* for player 1 (*cost function* for player 2).

Extending the definitions in [33], a *universally measurable strategy* for player 1 is a sequence $\pi = (\pi_1, \pi_2, \ldots)$, where each π_n is a universally measurable (see Section 3.1) conditional probability $\pi_n(\cdot|h_n)$ on A, given the entire history $h_n = (s_1, a_1, b_1, ..., s_{n-1}, a_{n-1},\ b_{n-1}, s_n)$ of the game up to its n-th stage such that $\pi_n(F(s_n)|h_n) = 1$. (Of course, if $n = 1$, then $h_1 = s_1$.) The class of strategies for player 1 will be denoted by Π. Let D_1 be the set of universally measurable transition probabilities f from S to A such that $f(s) \in \Delta(F(s))$ for each $s \in S$. It is well known that D_1 is nonempty and every $f \in D_1$ can be identified with a universally measurable mapping from S into $\Delta(A)$ (see Propositions 7.26 and 7.49 and Lemma 7.28 in [3]). A (universally measurable) *stationary strategy* for player 1 is as usual a sequence $\pi = (f, f, \ldots)$, where $f \in D_1$. Every stationary strategy $\pi = (f, f, \ldots)$ for player 1 can be identified with the mapping $f \in D_1$. Similarly, we define the set Σ (D_2) of universally measurable strategies (stationary strategies) for player 2.

Recall that $H^\infty = S \times A \times B \times S \times \ldots$ denotes the space of all infinite histories of the game endowed with the product σ-algebra. Also, for any $\pi \in \Pi$ and $\sigma \in \Sigma$ and every initial state $s_1 = s \in S$, a probability measure $P_s^{\pi\sigma}$ and a stochastic process $\{s_m, a_m, b_m\}$ are defined on H^∞ in a canonical way, where the random variables s_m, a_m and b_m describe the state and the action chosen by players 1 and 2, respectively, on the m-th stage of the game (see Proposition 7.45 in [3]). Thus, for each initial state $s \in S$ and any strategies $\pi \in \Pi$, $\sigma \in \Sigma$, the *expected discounted reward* to player 1 is

$$\gamma_\lambda(\pi, \sigma)(s) = E_s^{\pi\sigma}[\sum_{m=1}^{\infty} \lambda(1-\lambda)^{m-1} r(s_m, a_m, b_m)],$$

where λ is a fixed real number in $(0, 1)$, called the *discount factor*, and $E_s^{\pi\sigma}$ means the expectation operator with respect to the probability measure $P_s^{\pi\sigma}$. (Later on we make assumptions on r which assure that all expectations considered here are well defined.) Because λ is fixed, we will drop the reference to it. Let

$$\underline{v}(s) = \sup_{\pi \in \Pi} \inf_{\sigma \in \Sigma} \gamma(\pi, \sigma)(s) \quad \text{and} \quad \overline{v}(s) = \inf_{\sigma \in \Sigma} \sup_{\pi \in \Pi} \gamma(\pi, \sigma)(s), \quad s \in S.$$

Recall that the discounted stochastic game has a *value* v iff $v = \overline{v} = \underline{v}$.

Let $\varepsilon \geq 0$ be given. As usual, a strategy $\pi^* \in \Pi$ is called ε*-optimal* for player 1 if

$$\overline{v} \leq \inf_{\sigma \in \Sigma} \gamma(\pi^*, \sigma) + \varepsilon$$

for each $s \in S$. Similarly, a strategy $\sigma^* \in \Sigma$ is called ε*-optimal* for player 2 if

$$\underline{v} \geq \sup_{\pi \in \Pi} \gamma(\pi, \sigma^*) - \varepsilon$$

for each $s \in S$. The 0-optimal strategies are called *optimal.*

Before formulating our assumptions and results, we introduce some helpful notation. For any Borel space X, $\Delta(X)$ stands for the space of all probability measures on all Borel subsets of X. Let $s \in S$, $\mu \in \Delta(F(s))$ and $\nu \in \Delta(G(s))$. We define the bilinear extensions

$$r(s, \mu, \nu) = \int_{F(s)} \int_{G(s)} r(s, a, b) \mu(da) \nu(db),$$

and, for any Borel set $D \subset S$, we put

$$p(D|s, \mu, \nu) = \int_{F(s)} \int_{G(s)} p(D|s, a, b) \mu(da) \nu(db).$$

Let $M_+(S)$ be the set of all nonnegative universally measurable functions on S, $\tilde{M}_+(S)$ be the set of all nonnegative upper semianalytic functions on S, and $\tilde{B}_+(S)$ be the set of all bounded functions in $\tilde{M}_+(S)$ (Section 3.1). Define also

$$K_1 = \{(s, \mu) : s \in S, \mu \in \Delta(F(s))\}, \quad K_2 = \{(s, \nu) : s \in S, \nu \in \Delta(G(s))\}$$

and

$$K = \{(s, \mu, \nu) : s \in S, \mu \in \Delta(F(s)), \nu \in \Delta(G(s))\}. \tag{1}$$

Some of the results will be stated in terms of the *Shapley operator* defined on $M_+(S)$, extending the definition of [33]. If $u \in M_+(S)$, we introduce the auxiliary game $\Gamma(u)[s]$ played on $F(s) \times G(s)$ with the payoff function $L(u)(s, ., .)$ such that, for $(s, a, b) \in H$,

$$L(u)(s, a, b) = \lambda r(s, a, b) + (1 - \lambda) \int_S u(z) p(dz|s, a, b).$$

Given $(s, \mu, \nu) \in K$, we define the bilinear extension

$$L(u)(s, \mu, \nu) = \int_{F(s)} \int_{G(s)} L(u)(s, a, b) \mu(da) \nu(db) \tag{2}$$

and

$$(Uu)(s) = \inf_{\mu \in \Delta(G(s))} \sup_{\nu \in \Delta(F(s))} L(u)(s, \mu, \nu).$$

As in [33], the aim is to prove that
(i) U is also a sup inf operator, hence that the Shapley operator $\mathbf{\Psi}$ is well defined as the value of the auxiliary game;
(ii) $\mathbf{\Psi}$ maps a complete subset of $M_+(S)$ to itself;
(iii) a measurable selection theorem allows us to construct ε-optimal strategies.

A basic setting is $F(s) = F_0$ and $G(s) = G_0$ are compact sets and r is bounded. For each $(s, a) \in F$, $r(s, a, \cdot)$ is continuous on G_0 and for each $(s, b) \in G$, $r(s, \cdot, b)$ is continuous on F_0. For every Borel set $D \subset S$, the function $p(D|s, a, \cdot)$ is continuous on G_0 for each $(s, a) \in F$ and the function $p(D|s, \cdot, b)$ is continuous on F_0 for each $(s, b) \in G$. Then Sion's minmax theorem implies (i), and since $\mathbf{\Psi}$ maps measurable functions on S to measurable functions on S, (ii), and since Borel measurable optimal strategies will exist (see [5]), (iii).

In the current framework our basic assumptions will be significantly weaker than above:
C1: For each $s \in S$, the set $G(s)$ is nonempty and compact.
C2: For each $(s, a) \in F$, $r(s, a, \cdot)$ is lower semicontinuous on $G(s)$.
C3: For each $(s, a) \in F$ and every Borel set $D \subset S$, the function $p(D|s, a, \cdot)$ is continuous on $G(s)$.

Our main result is:

Theorem 1 *Assume that r is bounded from below and* **C1** *through* **C3** *hold. Let u be a bounded from below upper semianalytic function on S such that $(Uu)(s) < \infty$ for each $s \in S$. Then $(Uu)(s)$ is the value of the auxiliary* *game $\Gamma(u)[s]$. Moreover, $(Uu)(\cdot)$ is upper semianalytic, player 2 has an optimal universally measurable strategy, and for any $\varepsilon > 0$ player 1 has an ε-optimal universally measurable strategy.*

The proof of Theorem 1 will be given in Section 4.

Theorem 2 *Assume* **C1** *through* **C3** *and that r is bounded and $\lambda \in (0,1)$. Then the discounted stochastic game has a value function v_λ, the function v_λ is bounded and upper semianalytic, and v_λ is the unique solution of the equation:*

$$v_\lambda = Uv_\lambda.$$

Moreover, player 2 has an optimal universally measurable stationary strategy, and for any $\varepsilon > 0$ player 1 has an ε-optimal universally measurable stationary strategy.

Proof. Using Theorem 1, we infer that U is equal to the Shapley operator $\mathbf{\Psi}$ and is a contraction mapping from $\tilde{B}_+(S)$ into itself. Thus, there exists a unique $v_\lambda \in \tilde{B}_+(S)$ such that $v_\lambda = Uv_\lambda$. The existence of ε-optimal strategies for the players can be proved by making use of Theorem 1 with $u = v_\lambda$ and the arguments given in [33], Theorem 1. ▪

The next result concerns total reward games in which no discounting is assumed.

Theorem 3 *Assume* **C1** *through* **C3** *and that r is bounded and nonnegative. Then the stochastic game with total reward is called positive and has a value v, the function v is upper semianalytic, and is the smallest nonnegative solution of the equation:*

$$v = Uv.$$

Moreover, player 2 has an optimal universally measurable stationary strategy, and for any $\varepsilon > 0$ player 1 has an ε-optimal universally measurable semi-stationary strategy which depends on both the current state and the initial state of the game.

For the proof of this result consult [23]. It is shown that $v = \lim_{\lambda\to 0} V_\lambda$ where V_λ is the unnormalized discounted value, that is, $V_\lambda = \frac{v_\lambda}{\lambda}$. We close this section with a result on stochastic games with *weakly continuous* transition probabilities. Assume that A and B are compact metric spaces. Let the family of all nonempty compact subsets of A (and also of B) be endowed with the Hausdorff metric (see [2] or [3]). We make the following further assumptions:

C4: The set valued mappings $s \mapsto F(s)$ and $s \mapsto G(s)$ are continuous.
C5: The function r is bounded and continuous on H.
C6: The transition probability $p(\cdot|s,a,b)$ depends continuously on (s,a,b)

if $\Delta(S)$ is endowed with the weak topology.

Using the theorem of Berge (see pp. 115–116 in [2]) and a result by Himmelberg and Van Vleck [10], it is easy to prove that the operator U (i) is equal to $\mathbf{\Psi}$ and (ii) is a contraction mapping from the Banach space of all bounded continuous real-valued functions on S into itself. Therefore, there exists a unique fixed point v_λ for U. Next, using Theorem 2 from [5] and the property of the Shapley operator, (iii) holds and one obtains the following result.

Theorem 4 *Assume* **C4** *through* **C6** *and* $\lambda \in (0, 1)$. *Then the discounted stochastic game has a bounded continuous value function* v_λ, *and* v_λ *is the unique solution of the equation:*

$$v_\lambda = U v_\lambda.$$

Moreover, both players have optimal Borel measurable stationary strategies.

Remark 1 Theorem 1 is a game-theoretic extension of Theorem 2 of Brown and Purves [5]. Similar selection theorems with the asymmetric conditions corresponding to the assumptions of the Fan minmax theorem [7] (Theorem 2) can be found in [22], [23], [25]. If we drop the semicontinuity condition in Theorem 1 then it is consistent with the usual axioms of set theory to assume that Uu is not universally measurable. This observation is based on (F6) [30], [22].

Remark 2 (a) Theorem 1 has some relevance to studying general classes of Borel stochastic games with limsup payoffs [19], [20], [21].

(b) Universally measurable strategies were also used to study stochastic games with complete information in [15]. However, no counterpart of Theorem 1 is stated there.

(c) Player 1 need not have ε-optimal stationary strategies in a positive stochastic game even if the state space is countable and the value function v is bounded [28].

(d) If the transition probability structure satisfies some stochastic stability conditions, then the existence of optimal stationary strategies can be extended to a class of zero–sum stochastic games with the expected limiting average payoff criterion [14], [27]. Also some results on sensitive optimal strategies (related to turnpike theorems in economics) can be obtained for games with additive transition and reward structure (ARAT games). For example, the existence of 1-optimal strategies for the players [26] in a class of Borel state space ergodic ARAT games can be proved by combining a recent result by Jaśkiewicz [12] and Theorem 5 in [27]. (We point out that in [26], [27], [12] the immediate payoff function r is not multiplied by λ as is usually done in papers by many authors.)

(e) Zero-sum ergodic semi-Markov games with Borel state spaces were recently studied in [13].

Remark 3 Some versions of Theorem 4 are stated in [6], [30]. Rieder [30] also considered positive stochastic games. An extension of his result to zero–sum stochastic games with general lower semicontinuous payoff function on H^∞ is given in [24], [25], where persistently (subgame perfect) optimal strategies are also studied.

3. Measure-Theoretical Tools

3.1. BOREL AND ANALYTIC SETS. SEMIANALYTIC FUNCTIONS

A separable metric space X is called a *Borel space* or a *Borel set* if X is a Borel subset of some *Polish space,* i.e., a complete separable metric space, and is endowed with the σ-algebra $\mathcal{B}(X)$ of all its Borel subsets.

We shall need the following facts.

(F1) Let X and Y be Borel spaces and E be a Borel subset of $X \times Y$ such that the set $E(x) = \{y \in Y : (x, y) \in E\}$ is nonempty and compact for each $x \in X$. Then by [11] (Theorem 3) and [9] (Theorem 5.6), there is a sequence $\{f_n\}$ of Borel measurable functions on X into Y such that

$$E(x) = \mathrm{cl}\{f_n(x)\} \quad \text{for each } x \in X,$$

where cl denotes the closure operation in Y.

(F2) If X and Y are Borel spaces, then the product space $X \times Y$ endowed with the product topology is also a Borel space and $\mathcal{B}(X \times Y)$ equals the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ on $X \times Y$ [3], Proposition 7.13.

Let N^N be the set of sequences of positive integers, endowed with the product topology. So N^N is a Polish space. Let X be a separable metric space. Then X is called an *analytic space* or an *analytic set* provided there is a continuous function f on N^N whose range $f(N^N)$ is X.

In this section, we list some properties of analytic sets that we shall be using.

(F3) Every Borel set is analytic [3], Proposition 7.36.

(F4) The countable union, intersection and product of analytic sets is analytic [3], Corollary 7.35.2.

(F5) Let E be an analytic subset of an analytic space X. Then E is *universally measurable,* that is, if μ is any probability measure on the Borel subsets of X, then E is in the completion of the Borel σ-algebra with respect to μ [3], Corollary 7.42.1.

The complement of an analytic set relative to a Borel space is called *complementary analytic.* We have the following fact.

(F6) According to Gödel [8], it is consistent with the usual axioms of set theory to assume that there is a complementary analytic subset of the unit square whose projection on the horizontal axis is not universally measurable.

For any Borel space X, we denote by $\mathcal{U}(X)$ the σ-algebra of all universally measurable subsets of X. Let X and Y be Borel spaces. A function $f : X \mapsto Y$ is *universally measurable* if $f^{-1}(B) \in \mathcal{U}(X)$ for every $B \in \mathcal{B}(Y)$. By Theorem 5.5 of Leese [16] we have:
(F7) Let X and Y be Borel spaces and $C \in \mathcal{U}(X) \otimes \mathcal{B}(Y)$. Then the projection $\mathrm{proj}_X C$ of C on X belongs to $\mathcal{U}(X)$ and, moreover, there is a universally measurable function $f : X \mapsto Y$ such that $(x, f(x)) \in C$ for every $x \in \mathrm{proj}_X C$.

If X is an analytic space and f is an extended real-valued function on X, then we say that f is *upper semianalytic* (u.s.a.) if the set $\{x \in X : f(x) > c\}$ (equivalently, $\{x \in X : f(x) \geq c\}$) is analytic for each real number c. By (F3) every Borel measurable function is u.s.a., and by (F5) every u.s.a. function is universally measurable.

3.2. AUXILIARY MEASURE-THEORETICAL FACTS

Let X be a separable metric space, endowed with the σ-algebra $\mathcal{B}(X)$ of all its Borel subsets. We write $C(X)$ for the set of all bounded uniformly continuous real-valued functions on X. Recall that $\Delta(X)$ is the set of all probability measures on $\mathcal{B}(X)$. The *weak topology* on $\Delta(X)$ is the coarsest topology in which all mappings $\mu \mapsto \int u(x)\mu(dx)$, $u \in C(X)$, are continuous.
(F8) By embedding X in a countable product of unit intervals and using the fact that the unit ball in the space of uniformly continuous functions on a totally bounded metric space (with the supremum norm $\|\cdot\|$) is separable we get: there is a sequence $\{u_n\}$ of real-valued continuous functions on X with $\|u_n\| \leq 1$, $n \in N$, such that the metric ρ defined on $\Delta(X)$ by

$$\rho(\mu, \lambda) = \sum_{n=1}^{\infty} 2^{-n} \left| \int u_n(x)\mu(dx) - \int u_n(x)\lambda(dx) \right|, \quad \mu, \lambda \in \Delta(X), \quad (3)$$

is equivalent to the weak topology on $\Delta(X)$ [29] (page 47).
(F9) If X is a Borel space, then $\Delta(X)$ is a Borel space too [3], Corollary 7.25.1.
(F10) If X is compact, so is $\Delta(X)$ [3], Proposition 7.22.
(F11) The σ-algebra $\mathcal{B}(\Delta(X))$ of all Borel subsets of $\Delta(X)$ coincides with the smallest σ-algebra on $\Delta(X)$ for which the mapping $\mu \mapsto \mu(E)$ is measurable for each $E \in \mathcal{B}(X)$ [3], Proposition 7.25.
(F12) Let u be a bounded below real-valued lower semicontinuous function

on X. Then $\mu \mapsto \int u(x)\mu(dx)$ is an extended real-valued lower semicontinuous function on $\Delta(X)$. This fact follows from the theorem of Baire [1] (page 390) and the monotone convergence theorem.

(F13) Let X and Y be analytic spaces and u be a bounded below extended real-valued u.s.a. function on $X \times Y$. Then from Corollary 31 of [4] or Proposition 7.48 in [3], it follows that $(x,p) \mapsto \int u(x,y)p(dy)$ is an extended real-valued u.s.a. function on $X \times P(Y)$.

If X and Y are Borel spaces, $t(\cdot|x)$ is a probability measure on $\mathcal{B}(Y)$ for each $x \in X$, and the function $t(B|\cdot)$ from X into $[0,1]$ is Borel (universally) measurable for each $B \in \mathcal{B}(Y)$; we say that t is a Borel (universally) measurable *transition probability* from X into Y. It can be shown that t is a Borel (universally) measurable transition probability from X into Y if and only if the mapping $x \mapsto t(\cdot|x)$ from X into $\Delta(Y)$ is Borel (universally) measurable (see Proposition 7.26 and Lemma 7.28 in [3]). By a modification of Lemma 29 of [4] (see Proposition 7.46 in [3]) we can obtain the following fact.

(F14) If f is a real-valued universally measurable (respectively, u.s.a., Borel measurable) function on $X \times Y$ which is bounded below, and $t : X \mapsto \Delta(Y)$ is universally measurable (respectively, Borel measurable, Borel measurable), then $x \mapsto \int f(x,y)t(dy|x)$ is an extended real-valued universally measurable (respectively, u.s.a., Borel measurable) function on X.

Finally, we give the following fact.

(F15) Let f be a bounded real-valued universally measurable function on a Borel space Y, and t be a Borel measurable transition probability from a Borel space X into Y such that $t(B|\cdot)$ is continuous on X for each $B \in \mathcal{B}(Y)$. Then the function $x \mapsto \int f(y)t(dy|x)$ is continuous on X.

Proof. Let $x_n \to x_0$ as $n \to \infty$. For each $m \geq 0$, there is a Borel measurable function f_m on Y and there is a Borel subset B_m of Y such that $f(y) = f_m(y)$ for all $y \in B_m$ and $t(B_m|x_m) = 1$ (see Lemma 7.27 in [3]). Let $B = \cup_{m=0}^{\infty} B_m$. Then $t(B|x_m) = 1$ for each $m \geq 0$, and since f is bounded we have

$$\int_Y f(y)t(dy|x_n) = \int_B f(y)t(dy|x_n) \to \int_B f(y)t(dy|x_0) = \int_Y f(y)t(dy|x_0)$$

as $n \to \infty$, which terminates the proof. ∎

3.3. MEASURABLE SELECTIONS OF EXTREMA

Let X and Y be Borel spaces, and $E \subset X \times Y$ be such that $E(x) = \{y \in Y : (x,y) \in E\} \neq \emptyset$ for each $x \in X$. Let $u : E \mapsto R$ be such that

$$u^*(x) = \sup_{y \in E(x)} u(x,y) < \infty \quad \text{for each} \quad x \in X.$$

Define $\mathcal{Q} = \{x \in X : u^*(x) = u(x, y_x)$ for some $y_x \in E(x)\}$. A function $f : X \mapsto Y$ is called an *ε-maximizer* of u if $(x, f(x)) \in E$ for each $x \in X$ and

$$u^*(x) = u(x, f(x)) \quad \text{for} \quad x \in \mathcal{Q},$$

and

$$u^*(x) < u(x, f(x)) + \varepsilon \quad \text{for} \quad x \in X \setminus \mathcal{Q}.$$

If $\mathcal{Q} = X$ then an ε-maximizer is called a *maximizer* of u.

We shall need the following results:

Lemma 1 (see [3], Proposition 7.50) *Assume that E is an analytic set and u is an upper semianalytic function on E. Then u^* is upper semianalytic, $\mathcal{Q} \in \mathcal{U}(X)$, and for any $\varepsilon > 0$ there is a universally measurable ε-maximizer of u.*

Lemma 2 *Assume that $E \in \mathcal{U}(X) \otimes \mathcal{B}(Y)$ and u is a $\mathcal{U}(X) \otimes \mathcal{B}(Y)$-measurable function. Then u^* is universally measurable, $\mathcal{Q} \in \mathcal{U}(X)$, and for any $\varepsilon > 0$ there is a universally measurable ε-maximizer of u.*

Proof. Note that, for each real number c,

$$C = \{x \in X : u^*(x) > c\} = \text{proj}_X\{(x, y) \in E : u(x, y) > c\}.$$

By (F7) the set C belongs to $\mathcal{U}(X)$. This obviously proves the measurability of u^*.
Define

$$D_0 = \{(x, y) \in E : u^*(x) = u(x, y)\},$$

and, for any given $\varepsilon > 0$,

$$D = \{(x, y) \in E : u^*(x) < u(x, y) + \varepsilon\} \setminus D_0.$$

It is clear that D_0 and D belong to $\mathcal{U}(X) \otimes \mathcal{B}(Y)$, and $\mathcal{Q} = \text{proj}_X D_0$. Now the lemma follows from (F7). ∎

4. The Proof

To prove Theorem 1, we state some auxiliary lemmas. One can easily prove the following:

Lemma 3 *Let Y be a compact metric space and $u_n : Y \mapsto \mathbb{R}$, $n \in N$. Assume that $u_n \leq u_{n+1}$, and u_n is lower semicontinuous on Y for each n. Then*

$$\lim_n \inf_{y \in Y} u_n(y) = \inf_{y \in Y} \lim_n u_n(y).$$

Lemma 4 *Assume* **C1** *through* **C3**. *Then*

(a) The set K defined in (1) is Borel, and $\Delta(G(s))$ is compact for each $s \in S$.

(b) For any $u \in \tilde{M}_+(S)$ the (extended real-valued) function $L(u)(\cdot,\cdot,\cdot)$ defined on K by (2) *is upper semianalytic.*

(c) If r is bounded and $r(s,a,\cdot)$ is continuous on $G(s)$, $(s,a) \in F$, and $u \in \tilde{B}_+(S)$, then the function $L(u)(s,\mu,\cdot)$ is continuous on $\Delta(G(s))$, $(s,\mu) \in K_1$.

(d) For any $u \in \tilde{M}_+(S)$, the function $L(s,\mu,\cdot)(u)$ is lower semicontinuous on $\Delta(G(s))$, $(s,\mu) \in K_1$.

Proof. Part (a) follows from (F2) and (F9)–(F11). To prove (b) it is sufficient to use (F2), (F3), (F13), and (F14). Part (c) follows immediately from (F15). To prove (d), let $u_n = \min\{u,n\}, n \in N$. Then by (F12) and (F15) each function $L(u_n)(s,\mu,\cdot)$ is lower semicontinuous on $\Delta(G(s))$, $n \in N$, and by monotone convergence theorem

$$L(u_n)(s,\mu,\cdot) \uparrow L(u)(s,\mu,\cdot), \quad (s,\mu) \in K_1.$$

This obviously implies (d). ∎

Proof of Theorem 1. Without loss of generality, we shall assume in this proof that both the functions r and u are nonnegative. The fact that Uu is the value function of the one-stage game with terminal reward u follows from compactness of the sets $\Delta(G(s))$, $s \in S$, Lemma 4(d), and the Fan minmax theorem [7] (Theorem 2).

Define

$$\Phi(s,\mu) = \inf_{\nu \in \Delta(G(s))} L(u)(s,\mu,\nu), \quad (s,\mu) \in K_1.$$

Note that

$$(Uu)(s) = \sup_{\mu \in \Delta(F(s))} \Phi(s,\mu), \quad s \in S.$$

To prove that Uu is u.s.a., and player 1 has an ε-optimal universally measurable strategy for each $\varepsilon > 0$, it is sufficient to show that Φ is u.s.a. on the Borel space K_1 and apply Lemma 1. In order to show that Φ is u.s.a. on K_1, we construct an auxiliary sequence $\{\Phi_n\}$ of u.s.a. functions on K_1 that converges to Φ. Thus, Φ becomes a u.s.a. function on K_1.

Let

$$\varphi_n = \min\{\psi_n, n\}, \quad n \in N,$$

where

$$\psi_n(s,a,b) = \inf_{y \in G(s)} [\lambda r(s,a,y) + nd(b,y)], \quad (s,a,b) \in F \times B, \quad n \in N,$$

and d is the metric in B.

By [5] (Theorem 2), ψ_n is a Borel measurable function on $F \times B$, and so is φ_n, $n \in N$. It is easy to check that $\varphi_n(s, a, \cdot)$ is continuous on B for each $(s, a) \in F$, $n \in N$. By the proof of the theorem of Baire [1] (page 390), $\psi_n \uparrow \lambda r$ on H. Hence $\varphi_n \uparrow \lambda r$ on H.

Let $L_n(u_n)(\cdot, \cdot, \cdot)$ be defined by (2) where the function λr is replaced by φ_n, and u is replaced by $u_n = \min\{u, n\}$. Clearly, the facts listed in Lemma 4 for $L(u)(\cdot, \cdot, \cdot)$ carry over to $L_n(u_n)(\cdot, \cdot, \cdot)$.

Define

$$\Phi_n(s, \mu) = \inf_{\nu \in \Delta(G(s))} L_n(u_n)(s, \mu, \nu), \quad (s, \mu) \in K_1, n \in N.$$

Because $\varphi_n \uparrow \lambda r$ on H, and $u_n \uparrow u$ on S, from the monotone convergence theorem we get

$$L_n(u_n)(\cdot, \cdot, \cdot) \uparrow L(u)(\cdot, \cdot, \cdot) \quad \text{on } K. \tag{4}$$

This fact, the compactness of $\Delta(G(s))$, $s \in S$, Lemma 4(c), and Lemma 3 imply that $\Phi_n \uparrow \Phi$ on K_1 for each $n \in N$.

By Lemma 4(a) and (F1) there is a sequence $\{g_k\}$ of Borel measurable mappings $g_k : S \mapsto \Delta(B)$ such that

$$\Delta(G(s)) = \mathrm{cl}\{g_k(s)\} \quad \text{for each} \quad s \in S, \tag{5}$$

where cl denotes the closure in the weak topology on $\Delta(B)$. This together with Lemma 4(c) implies

$$\Phi_n(s, \mu) = \inf_k L_n(u_n)(s, \mu, g_k(s)), \quad (s, \mu) \in K_1 \quad n \in N. \tag{6}$$

From the Borel measurability of g_k and (F14), we infer that the function

$$(s, a) \mapsto L_n(u_n)(s, \mu_a, g_k(s)), \quad (s, a) \in F,$$

where $\mu_a(\{a\}) = 1$, is u.s.a. on F for each $k, n \in N$. Using these facts and (F13) we can easily show that $L_n(u_n)(\cdot, \cdot, g_k(\cdot))$ is u.s.a. on K_1, for each $k, n \in N$, which together with (6) and (F4) (used for the intersection) implies that so is Φ_n, $n \in N$.

To prove that player 2 has an optimal universally measurable strategy we define the function

$$\Xi(s, \nu) = \sup_{\mu \in \Delta(F(s))} L(u)(s, \mu, \nu), \quad (s, \nu) \in K_2.$$

Note that

$$(Uu)(s) = \inf_{\nu \in \Delta(G(s))} \Xi(s, \nu) = \Xi(s, \nu_s) \tag{7}$$

for each $s \in S$ and some $\nu_s \in \Delta(G(s))$. The last equality follows from the compactness of the sets $\Delta(G(s))$, $s \in S$, and Lemma 4(d). We shall show that Ξ is a $\mathcal{U}(S) \otimes \mathcal{B}(\Delta(B))$-measurable function on K_2 (being a Borel space). Then the existence of the required strategy for player 2 follows immediately from (7) and Lemma 2.

In order to prove the measurability of Ξ we use the following sequences of functions:

$$\Xi_n(s,\nu) = \sup_{\mu \in \Delta(F(s))} L_n(u_n)(s,\mu,\nu), \quad (s,\nu) \in K_2, n \in N,$$

and

$$\Theta_{nm}(s,\nu) = \inf_{\eta \in \Delta(G(s))} [\Xi_n(s,\eta) + m\rho(\eta,\nu)], \quad n,m \in N,$$

where $(s,\nu) \in S \times \Delta(B)$, and ρ is the metric on $\Delta(B)$ defined according to (3). Let $n, m \in N$ be arbitrary. Note that $\Theta_{nm}(s,\cdot)$ is continuous on $\Delta(B)$ for each $s \in S$. We shall prove that $\Theta_{nm}(\cdot,\nu)$ is u.s.a. on S for each $\nu \in \Delta(B)$.

Denote

$$w_{nm}(s,\mu,\eta,\nu) = L_n(s,\mu,\eta)(u_n) + m\rho(\eta,\nu),$$

where $(s,\mu,\eta) \in K$ and $\nu \in \Delta(B)$. From the properties of $L_n(u_n)(s,\cdot,\cdot)$ and (3) we infer that $w_{nm}(s,\cdot,\eta,\nu)$ is linear on a convex set $\Delta(F(s))$ and $w_{nm}(s,\mu,\cdot,\nu)$ is convex and continuous on $\Delta(G(s))$, which is a compact convex space. Applying the Fan minmax theorem [7] (Theorem 2) to the function $w_{nm}(s,\cdot,\cdot,\nu)$ we get:

$$\Theta_{nm}(s,\nu) = \sup_{\mu \in \Delta(F(s))} \inf_{\eta \in \Delta(G(s))} w_{nm}(s,\mu,\eta,\nu), \quad (s,\nu) \in S \times \Delta(B). \tag{8}$$

It is clear that $w_{nm}(\cdot,\cdot,\cdot,\nu)$ is u.s.a. on K, and since $w_{nm}(s,\mu,\cdot,\nu)$ is continuous on $\Delta(G(s))$, $(s,\mu) \in K_1$, we can show, using the sequence $\{g_k\}$ satisfying (5), that the function

$$(s,\mu) \mapsto \inf_{\eta \in \Delta(G(s))} w_{nm}(s,\mu,\eta,\nu) \quad \text{is u.s.a. on } K_1.$$

This fact together with (8) and Lemma 1 implies that $\Theta_{nm}(\cdot,\nu)$ is u.s.a. on S. Thus, we have shown that $\Theta_{nm}(\cdot,\nu)$ is $\mathcal{U}(S)$-measurable on S for each $\nu \in \Delta(B)$ and $\Theta_{nm}(s,\cdot)$ is continuous on $\Delta(B)$ for each $s \in S$. By [9] (Theorem 6.1), the function Θ_{nm} is $\mathcal{U}(S) \otimes \mathcal{B}(\Delta(B))$-measurable on $S \times \Delta(B)$ because $\Delta(B)$ endowed with the weak topology is a separable metric space.

Now observe that $\Xi_n(s,\cdot)$ is lower semicontinuous on $\Delta(G(s))$ for each $s \in S$, $n \in N$. By the proof of the theorem of Baire [1] (page 390) we obtain $\Theta_{nm} \uparrow \Xi_n$ as $m \to \infty$. Hence, it follows that Ξ_n is $\mathcal{U}(S) \otimes \mathcal{B}(\Delta(B))$-measurable on K_2 for each $n \in N$, and from (4) we can easily derive that $\Xi_n \uparrow \Xi$ as $n \to \infty$. Thus, Ξ is also a $\mathcal{U}(S) \otimes \mathcal{B}(\Delta(B))$-measurable function on K_2, which terminates the proof. ∎

Remark 4 The Borel measurability of the functions $\{g_k\}$ is very important in the proof of Theorem 1 because of the fact that the composition of two analytically measurable functions need not be analytically measurable (see [3] (page 187) or [4] (Example 24)).

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