RECURSIVE GAMES

FRANK THUIJSMAN
Maastricht University
Maastricht, The Netherlands

1. Introduction

Recursive games are stochastic games with the property that any non-zero-payoff is absorbing, i.e., play immediately moves to an absorbing state where each player has only one action available and these actions give this particular non-zero payoff at all further stages. By its structure, it is natural to examine such games using limiting average rewards, or total rewards on the assumption of stopping play as soon as a non-zero payoff occurs. Everett [1] introduced the recursive game model and immediately solved it for the zero-sum case. We shall briefly discuss his approach. Later Thuijsman and Vrieze [7] presented an asymptotic algebraic proof for the existence of stationary ε-optimal strategies for recursive[1] games, which can be derived from any arbitrary sequence of stationary λ-discounted optimal strategies, converging for λ going to 0. Because of their simple structure, one might hope that recursive games always allow for stationary ε-equilibria as well, but that such is not true can clearly be seen from an example in Flesch et al. [2]. However, if we wish to solve the general existence problem for ε-equilibria in stochastic games, then one should certainly be able to tackle the problem for recursive games. Such has indeed been done by Vieille [8], [9]. Vieille [8] shows that if one can exhibit the existence of ε-equilibria in recursive games, then it follows that ε-equilibria exist in any stochastic game.[2] Vieille [9] then shows that equilibria exist in recursive games. Hence the two papers together comprise a proof for the existence of ε-equilibria in any arbitrary stochastic game. These results we shall leave to him for discussion. Instead, based on the paper by Flesch et al. [2], we shall exhibit

[1]Unless otherwise specified we are discussing the situation for the case of a two-person stochastic game with finite state and action spaces.

[2]Actually his result is even stronger; one only needs to show existence for a specific type of recursive game.

the existence of stationary ε-equilibria for recursive repeated games with absorbing states, and we shall discuss some examples that illustrate the sharpness of this result. More precisely, stationary ε-equilibria fail to exist for three-person recursive repeated games with absorbing states, as was shown in Flesch et al. [3], as well as for two-person recursive games with more than one non-trivial state.

In Section 2 we shall briefly discuss the zero-sum model examined by Everett [1]. In Section 3 we discuss the approach of Thuijsman and Vrieze [7] for solving zero-sum recursive games. In Section 4 we deal with two non-zero-sum recursive repeated games with absorbing states and Section 5 discusses two examples on the impossibility of extending the latter result.

2. Everett's Recursive Games

Everett [1] introduces these games as follows:

> "A recursive game is a finite set of 'game elements,' which are games for which the outcome of a single game (payoff) is either a real number, or another game of the set, but not both."

In his model there are no assumptions on the number of actions in each state, i.e., the actions sets are not necessarily finite. So if we assume that there are k non-trivial states $1, 2, \ldots, k$, then for actions a and b for the respective players in any of these states z, we either move to a non-trivial state z' with probability $p(z'|z, a, b)$ with stage payoff 0, or we absorb in entry (a, b) with probability p^*_{ab} and player 1 receives at each further stage the (absorbing) payoff u^*_{ab} from player 2. Using these notations for the recursive game Γ we can introduce for each (non-trivial) state z the auxiliary game

$$\Gamma_z(\theta) := [p^*_{ab}u^*_{ab} + \sum_{z'=1}^{k} p(z'|z, a, b)\theta_{z'}]_{a\in A_z, b\in B_z}.$$

Everett [1] then proves the following.

Theorem 1
1) If for each z and θ the game $\Gamma_z(\theta)$ has a value, then the recursive game Γ has a value v and both players have ε-optimal strategies.
2) If the players have optimal strategies in $\Gamma_z(v)$ or if $v_z > 0$, then player 1 has a stationary ε-optimal strategy (and similarly for player 2).

We would like to emphasize that Everett's result was stated and proved for recursive games with general action spaces and his proof does not apply to the uniform value. His approach is based on the selection of a fixed point of the map $\psi : \theta \mapsto val[\Gamma_z(\theta)]$. Generally, ψ does not have a unique fixed point. Therefore Everett introduces two sets C_1 and C_2 defined as follows.

$$C_1 = \{\theta | \text{ for each } z \text{ either } \psi(\theta)_z > \theta_z \text{ and } \theta_z > 0 \text{ or } \psi(\theta)_z = \theta_z \text{ and } \theta_z \leq 0\}$$

$$C_2 = \{\theta | \text{ for each } z \text{ either } \psi(\theta)_z < \theta_z \text{ and } \theta_z < 0 \text{ or } \psi(\theta)_z = \theta_z \text{ and } \theta_z \geq 0\}.$$

Then Everett [1] proves that, on condition that for each z and θ the game $\Gamma_z(\theta)$ has a value, the value v of the recursive game Γ is the unique point in the intersection of the closures of C_1 and C_2. That is:

$$\overline{C_1} \cap \overline{C_2} = \{v\}.$$

2.1. EXAMPLE

We examine the recursive game Γ defined by

$$\Gamma = \begin{array}{|l|l|}\hline 0 & 2 \quad * \\ \hline 1 \quad * & 0 \quad * \\ \hline\end{array}$$

For this game the auxiliary game is

$$\Gamma(\theta) = \begin{array}{|l|l|}\hline 0 & 2 \\ \hline 1 & 0 \\ \hline\end{array}$$

for which we find that

$$\psi(\theta) = \text{val } (\Gamma(\theta)) = \begin{cases} 2 & \text{for } \theta > 2 \\ \theta & \text{for } 1 \leq \theta \leq 2 \\ \frac{2}{3-\theta} & \text{for } \theta < 1. \end{cases}$$

Clearly the map ψ does not have a unique fixed point, since $\psi(\theta) = \theta$ for all $\theta \in [1, 2]$. If we examine the sets C_1 and C_2 introduced above, then we find that

$$C_1 = (-\infty, 1), \;\; C_2 = [1, \infty) \;\text{ and }\; \overline{C_1} \cap \overline{C_2} = \{1\}.$$

Hence, we find that the value v of Γ is 1. Moreover, the stationary strategy $(1-\varepsilon, \varepsilon)^\infty$ is ε-optimal for player 1 and $(1, 0)^\infty$ is optimal for player 2.

2.2. EXAMPLE

We now consider the following recursive game Γ.

0 $(0,1)$	1 $*$
1 $*$	0 $*$

1

0 (1,0)	-1 $*$
-1 $*$	0 $*$

2

For this game we find that:

$$\psi_1(\theta_1, \theta_2) = \text{val } (\Gamma(\theta_1, \theta_2)) = \text{val} \begin{bmatrix} \theta_2 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2-\theta_2} \quad \text{if } \theta_2 \leq 1;$$

$$\psi_2(\theta_1, \theta_2) = \text{val } (\Gamma(\theta_1, \theta_2)) = \text{val} \begin{bmatrix} \theta_1 & -1 \\ -1 & 0 \end{bmatrix} = \frac{-1}{2+\theta_1} \quad \text{if } \theta_1 \geq -1.$$

For this game the sets C_1 and C_2 are determined by the curves $\frac{1}{2-\theta_2} = \theta_1$ and $\frac{-1}{2+\theta_1} = \theta_2$ as presented in Figure 1. As can easily be computed we find that

$$\overline{C_1} \cap \overline{C_2} = \{(-1+\sqrt{2}, 1-\sqrt{2})\}$$

and therefore $(-1+\sqrt{2}, 1-\sqrt{2})$ is the value of this recursive game, where the first coordinate refers to the game starting in state 1 and the second for state 2. Please notice that neither $v \in C_1$ nor $v \in C_2$.

From the auxiliary games

$$\Gamma_1(v_1, v_2) = \begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \Gamma_2(v_1, v_2) = \begin{bmatrix} -1+\sqrt{2} & -1 \\ -1 & 0 \end{bmatrix}$$

we can derive, by symmetry, that for each player the stationary strategy

$$((\frac{1}{1+\sqrt{2}}, \frac{\sqrt{2}}{1+\sqrt{2}}), (\frac{1}{1+\sqrt{2}}, \frac{\sqrt{2}}{1+\sqrt{2}}))^{\infty},$$

which consists of optimal mixed actions in the auxiliary games, is optimal for the game Γ.

Generally, in Everett's paper the stationary ε-optimal strategies for player 1 in the recursive game consist of optimal mixed actions α_z in the auxiliary games $\Gamma_z(\theta'_\varepsilon)$, where θ'_ε is an arbitrary element of C_1 sufficiently close to v, meaning that $||\theta'_\varepsilon - v|| < \varepsilon$.

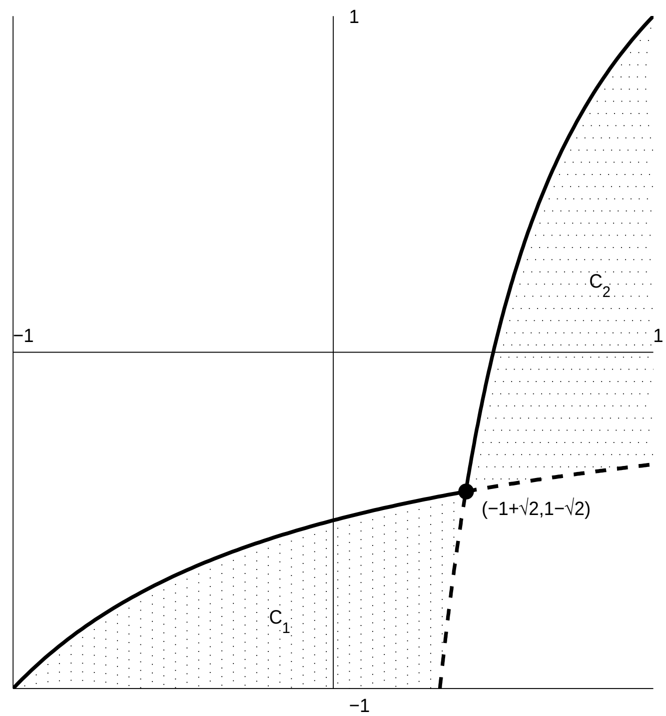

Figure 1. The sets C_1 and C_2, respectively left-below and right-up from the point $(-1+\sqrt{2}, 1-\sqrt{2})$

3. Asymptotic Approach to Zero-Sum Recursive Games

For finite state and action spaces Thuijsman and Vrieze [7] approach the zero-sum limiting average reward recursive games by examining the solutions for the λ-discounted games as λ goes to 0.

If we let v_λ be the λ-discounted value for $\lambda \in (0,1)$ and α^λ a stationary λ-discounted optimal strategy for player 1, then, by examining some sequence of λ's going to 0, we can assume that $\lim_{\lambda \downarrow 0} v_\lambda$ and $\lim_{\lambda \downarrow 0} \alpha^\lambda$ converge to v and α respectively. We define

$$\overline{\alpha}_z^\lambda = \begin{cases} \alpha_z & \text{if } v_z \leq 0 \\ \alpha_z^\lambda & \text{if } v_z > 0. \end{cases}$$

Thuijsman and Vrieze [7] prove the following result.

Theorem 2 *The limiting average value of* Γ *is* v *and for player 1 the stationary strategy* $\overline{\alpha}^\lambda$ *is limiting average* ε*-optimal for* λ *sufficiently close to 0.*

This theorem is proved as follows. First it is observed that there does exist a strategy β, which is a pure stationary limiting average best reply for player 2 against $\overline{\alpha}^\lambda$ for all λ sufficiently close to 0. This strategy exists because, for λ sufficiently close to 0, the ergodicity properties of the Markov decision problem player 2 faces when trying to play a best reply to $\overline{\alpha}^\lambda$, no longer depend on λ. Next the result is derived by careful examination of the recursion equations for the strategy pairs $(\overline{\alpha}^\lambda, \beta)$.

Henceforth assume that we have fixed such a strategy β, and also assume without loss of generality that the ergodic structure for $(\overline{\alpha}^\lambda, \beta)$ is independent of λ. The following lemma shows the optimality of $\overline{\alpha}^\lambda$ for initial states that are recurrent with respect to $(\overline{\alpha}^\lambda, \beta)$. Obviously our concern is with the non-absorbing recurrent initial states, since for absorbing initial states there is nothing to prove.

We now sketch the part of the proof of Theorem 2 for transient initial states.

Lemma 3 *If z is recurrent with respect to $(\overline{\alpha}^\lambda, \beta)$, then*

$$\gamma^\lambda(z, \overline{\alpha}^\lambda, \beta) = \gamma(z, \overline{\alpha}^\lambda, \beta) = 0 = v_z.$$

Proof. If z is recurrent, then play never reaches an absorbing state and therefore the corresponding limiting average reward and λ-discounted reward are both 0. It remains to show that the limiting average value v_z, for initial state z, is equal to 0 as well. First of all, notice that $0 \leq v_z$, because β is a limiting average best reply to $\overline{\alpha}^\lambda$ and apparently player 2 cannot force absorption with an expected negative yield for player 1. Secondly, suppose now that for all states in the ergodic set that z belongs to, the limiting average value is strictly positive; then we would have that in all these states $\overline{\alpha}^\lambda_z = \alpha^\lambda_z$ and hence we find

$$0 = \gamma^\lambda(z, \overline{\alpha}^\lambda, \beta) = \gamma^\lambda(z, \alpha^\lambda, \beta) \geq v^\lambda > 0,$$

which is a contradiction. Thirdly, if $v_z > 0$ and there are states in the ergodic set that z belongs to, for which the limiting average value is 0, then any play for (α^λ, β) will lead, with probability 1, to a state with limiting average value 0. But that contradicts the λ-discounted optimality of α^λ, since v^λ_z is bounded away from 0 for λ sufficiently small. Therefore we conclude that $v_z > 0$ is impossible, and hence $v_z = 0$. ∎

In order to complete the proof of Theorem 2, the only remaining initial states for which we have to show the ε-optimality of $\overline{\alpha}^\lambda$ (for λ close to 0) are the ones that are transient with respect to $(\overline{\alpha}^\lambda, \beta)$. Let us call this set of transient states T, while R shall denote the set of recurrent states (including the absorbing states). By the λ-discounted optimality of α^λ we have the following inequality:

$$v_T^\lambda \leq (1-\lambda)P(\alpha^\lambda,\beta)_{TT}v_T^\lambda + (1-\lambda)P(\alpha^\lambda,\beta)_{TR}v_R^\lambda. \qquad (*)$$

In case the limiting average value of all states in T is positive, we have $\overline{\alpha}_T^\lambda = \alpha_T^\lambda$ and hence

$$v_T^\lambda \leq P(\overline{\alpha}^\lambda,\beta)_{TT}v_T^\lambda + P(\overline{\alpha}^\lambda,\beta)_{TR}(v_R + \varepsilon 1_R),$$

from which we get

$$\begin{aligned} v_T - \varepsilon 1_T \leq v_T^\lambda &\leq (I_{TT} - P(\overline{\alpha}^\lambda,\beta)_{TT})^{-1}P(\overline{\alpha}^\lambda,\beta)_{TR}(v_R + \varepsilon 1_R) \\ &\leq \gamma(\overline{\alpha}^\lambda,\beta) + \varepsilon 1_T. \end{aligned}$$

In case the limiting average value of all states in T is non-positive, we have $\overline{\alpha}_T^\lambda = \alpha_T^\lambda$ and hence, by taking limits for λ to 0 in equation (*), we get

$$\begin{aligned} v_T &\leq P(\alpha,\beta)_{TT}v_T + P(\alpha,\beta)_{TR}v_R \\ &= P(\overline{\alpha}^\lambda,\beta)_{TT}v_T + P(\overline{\alpha}^\lambda,\beta)_{TR}v_R \end{aligned}$$

from which we get

$$v_T \leq (I_{TT} - P(\overline{\alpha}^\lambda,\beta)_{TT})^{-1}P(\overline{\alpha}^\lambda,\beta)_{TR}v_R = \gamma(\overline{\alpha}^\lambda,\beta).$$

The situation where some transient states have a positive value and some have a non-positive value can be examined in a similar, though slightly more complicated, way. We refer to the original paper by Thuijsman and Vrieze [7] for the details. ∎

3.1. EXAMPLE

If we want to compute the λ-discounted solution for the recursive game of Example 2.1, then we should observe that there is neither a pure stationary λ-discounted optimal strategy for player 1, nor for player 2. We focus on player 1. He should play some stationary strategy $(x, 1-x)^\infty$ for which player 2's stationary strategies $(1,0)^\infty$ and $(0,1)^\infty$ yield the same λ-discounted reward.[3]

By the Shapley equation we have that

$$\gamma^\lambda((x,1-x)^\infty,(0,1)^\infty) = 2x; \text{ and}$$

$$\begin{aligned} \gamma^\lambda((x,1-x)^\infty,(1,0)^\infty) \quad &= \quad (1-x)\lambda + (1-x)(1-\lambda) \\ &\quad\; + x(1-\lambda)\gamma^\lambda((x,1-x)^\infty,(1,0)^\infty) \end{aligned}$$

[3] x depends on λ. To keep notations simple we write x instead of x^λ.

from which the latter gives

$$\gamma^\lambda((x,1-x)^\infty,(1,0)^\infty) = \frac{1-x}{1-x(1-\lambda)}.$$

Therefore, we derive v^λ and x by solving

$$v^\lambda = 2x = \frac{1-x}{1-x(1-\lambda)}$$

which leads to

$$v^\lambda = \frac{3-\sqrt{1+8\lambda}}{2(1-\lambda)} \quad \text{and} \quad x = \frac{3-\sqrt{1+8\lambda}}{4(1-\lambda)}.$$

For this x the strategy $(x,1-x)^\infty$ is λ-discounted optimal, and limiting average ε-optimal for λ close to 0.

4. Non-Zero-Sum Recursive Repeated Games with Absorbing States

In this section we examine non-zero-sum recursive repeated games with absorbing states. These are recursive games with just one non-absorbing state. Again we shall consider only the case of finite state and action spaces. Flesch et al. [2] prove the following theorem.

Theorem 4 *In any two-person non-zero-sum recursive repeated game with absorbing states there exists a stationary limiting average ε-equilibrium.*

Here a limiting average ε-equilibrium ($\varepsilon > 0$) is a pair of strategies $(\sigma_\varepsilon,\tau_\varepsilon)$, such that for all σ and τ we have $\gamma_1(\sigma,\tau_\varepsilon) \le \gamma_1(\sigma_\varepsilon,\tau_\varepsilon)+\varepsilon$ and $\gamma_2(\sigma_\varepsilon,\tau) \le \gamma_2(\sigma_\varepsilon,\tau_\varepsilon)+\varepsilon$, i.e., σ_ε and τ_ε are ε-best replies to each other.

Before sketching a proof for this theorem, we wish to remark that this theorem does not follow in some straightforward way from the approach that Vrieze and Thuijsman [10] developed for repeated games with absorbing states, in which they showed the existence of (generally history-dependent) limiting average ε-equilibria for the latter class. Furthermore, examples in the next section will show that the result of the above theorem can neither be extended to the situation of more than one non-absorbing state, nor to the situation of more than two players.

4.1. EXAMPLE

We now consider the following recursive game

$\Gamma =$

0,0	-2,1 *
-1,2 *	-1,1 *

For this game we find that, for all $\lambda \in (0,1)$, the unique stationary λ-discounted equilibrium is given by

$$(\alpha^\lambda, \beta^\lambda) = ((\frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda})^\infty, (\frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda})^\infty)$$

for which we have

$$\gamma^\lambda(\alpha^\lambda, \beta^\lambda) = (-1, 1).$$

If we recall the method developed by Vrieze and Thuijsman [10] for repeated games with absorbing states (see also [5]), then we should examine

$$(\alpha, \beta) = \lim_{\lambda \downarrow 0}(\alpha^\lambda, \beta^\lambda) = ((1,0)^\infty, (1,0)^\infty)$$
$$V = \lim_{\lambda \downarrow 0} \gamma^\lambda(\alpha^\lambda, \beta^\lambda) = (-1, 1)$$

and notice that

$$\gamma_2(\alpha, \beta) = 0 < V_2.$$

The method of the chapter on repeated games with absorbing states tells us that then player 1 should play some strategy α^* (in this case $(0,1)^\infty$) and play $(1-\mu)\alpha + \mu\alpha^*$ for some μ sufficiently small, thereby checking whether or not player 2 is really (credibly) playing according to β. In this example we would have that $((1-\mu)\alpha + \mu\alpha^*, \beta) = ((1-\mu, \mu)^\infty, (1,0)^\infty)$ which is not a stationary ε-equilibrium since, against $(1,0)^\infty$, player 1 would rather play $(1,0)^\infty$ than $(1-\mu,\mu)^\infty$ and gain 1.

We shall now sketch the proof of Theorem 4, which uses the notion of *proper pairs* of strategies.

Definition 5 *A pair of stationary strategies* (x_δ, y_δ) *is called* δ-proper *for* $\delta > 0$ *if*
1) $x_\delta(a) > 0$ *for all actions* a *of player 1 and* $x_\delta(b) > 0$ *for all actions* b *of player 2.*
2) If $\gamma_1(a, y_\delta) > \gamma_1(\widetilde{a}, y_\delta)$ *then* $x_\delta(\widetilde{a}) < \delta x_\delta(a)$ *and if* $\gamma_2(x_\delta, b) > \gamma_2(x_\delta, \widetilde{b})$, *then* $x_\delta(\widetilde{b}) < \delta x_\delta(b)$.
A pair of stationary strategies (x, y) *is called* proper *if there is a (discrete) sequence of* δ*-proper pairs* (x_δ, y_δ) *such that* $(x,y) = \lim_{\delta \downarrow 0}(x_\delta, y_\delta)$.

We wish to remark that for a pair of strategies neither properness nor ε-properness implies that the pair is an ε-equilibrium. An example below illustrates this statement for the case of proper pairs. We refer to Flesch et al. [2] for the case of ε-proper pairs.

In Flesch et al. [2] Theorem 4 is proved by examining some specific cases, as is done in Theorem 6 below. We refer to the original paper for the proofs.

Theorem 6
1) There exist a proper pair $(\widetilde{x},\widetilde{y})$ *and a sequence of* δ*-proper pairs* (x_δ, y_δ) *such that* $(\widetilde{x},\widetilde{y}) = \lim_{\delta\downarrow 0}(x_\delta, y_\delta)$.
2) If $(\widetilde{x},\widetilde{y})$ *is absorbing, then* (x_δ, y_δ) *is a limiting average* ε*-equilibrium for small* δ.
3) If $(\widetilde{x},\widetilde{y})$ *is non-absorbing, then at least one of the pairs* $(\widetilde{x},\widetilde{y})$, $(x_\delta,\widetilde{y})$, $(\widetilde{x}, y_\delta)$ *is a limiting average* ε*-equilibrium for small* δ.

4.2. EXAMPLE

We now consider the following recursive game Γ.

0,0	4,-3 *
3,-2 *	1,-4 *
1,-4 *	3,-2 *

For this game it can be verified that the pairs

$$(x_\delta, y_\delta) = ((1-\delta^2-\delta^4, \delta^4, \delta^2)^\infty, (\delta^2, 1-\delta^2)^\infty)$$

are δ-proper. Clearly

$$(\widetilde{x},\widetilde{y}) = \lim_{\delta\downarrow 0}(x_\delta, y_\delta) = ((1,0,0)^\infty, (0,1)^\infty)$$

is a proper pair, but obviously $(\widetilde{x},\widetilde{y})$ is no limiting average ε-equilibrium, since player 2 would rather play his first column against $\widetilde{x}$.

5. Impossibility of Generalization

In this section we show that neither two-person recursive games with more than one non-trivial state, nor three-person recursive repeated games with an absorbing state, need to have stationary limiting average ε-equilibria.

5.1. EXAMPLE

We now consider the following two-person recursive game Γ

0,0 (0,0,0,1)
0,0 $(0,\frac{1}{2},\frac{1}{2},0)$

1

0,0 (1,0,0,0)	0,0 (0,1,0,0)

2

3	4
3,-1 (0,0,1,0)	2,1 (0,0,0,1)

This game is a recursive perfect information game for which there is no stationary limiting average ε-equilibrium. One can prove this as follows. Suppose player 2 puts positive weight on Left in state 2, then player 1's only stationary limiting average ε-best replies are those that put weight at most $\frac{\varepsilon}{2-\varepsilon}$ on Top in state 1; against any of these strategies, player 2's only stationary limiting average ε-best replies are those that put weight 0 on Left in state 2. So there is no stationary limiting average ε-equilibrium where player 2 puts positive weight on Left in state 2. But neither is there a stationary limiting average ε-equilibrium where player 2 puts weight 0 on Left in state 2, since then player 1 should put at most 2ε weight on Bottom in state 1, which would in turn contradict player 2's putting weight 0 on Left. In Thuijsman and Raghavan [6] existence of limiting average 0-equilibria is shown for arbitrary n-person games with perfect information. The notion of perfect information stands for the fact that in any state there is at most one player with a non-trivial action space.

5.2. EXAMPLE

We now consider the following three-person recursive game Γ

	Near: Left	Near: Right	Far: Left	Far: Right
Top	0, 0, 0	0, 1, 3 *	3, 0, 1 *	1, 1, 0 *
Bottom	1, 3, 0 *	1,0,1 *	0, 1, 1 *	0, 0, 0 *

This is a three-person recursive repeated game with absorbing states, where an asterisk in any particular entry denotes a transition to an absorbing state with the same payoff as in this particular entry. There is only one entry for which play will remain in the non-trivial initial state. One should picture the game as a $2 \times 2 \times 2$ cube, where the layers belonging to the actions of player 3 (Near and Far) are represented separately. As before, player 1 chooses Top or Bottom and player 2 chooses Left or Right. The entry (T, L, N) is the only non-absorbing entry for the initial state. Hence, as long as play is in the initial state the only possible history is the one where entry (T, L, N) was played at all previous stages. This rules out the use of any non-trivial history-dependent strategy for this game. Therefore, the players have only Markov strategies at their disposal. In Flesch et al. [3] it is shown that, although (cyclic) Markov limiting average 0-equilibria exist for this game, there are no stationary limiting average ε-equilibria. Moreover, the set of all limiting average equilibria is characterized completely. An example of a Markov equilibrium for this game is (f, g, h), where f is defined by: at stages $1, 4, 7, 10, \ldots$ play T with probability $\frac{1}{2}$ and at all other stages play T with probability 1. Similarly, g is defined by: at stages $2, 5, 8, 11, \ldots$ play L with probability $\frac{1}{2}$ and at all other stages play L with probability 1. Likewise, h is defined by: at stages $3, 6, 9, 12, \ldots$ play N with probability $\frac{1}{2}$ and at all other stages play N with probability 1. The limiting average reward corresponding to this equilibrium is $(1, 2, 1)$. For a further discussion on three-person repeated games with absorbing states we refer to [4].

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