Cooperative Strategic Games

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Abstract

We examine a solution concept, called the "value," for n-person strategic games.

In applications, the value provides an a-priori assessment of the monetary worth of a player's position in a strategic game, comprising not only the player's contribution to the total payoff but also the player's ability to inflict losses on other players. A salient feature is that the value takes account of the costs that "spoilers" impose on themselves.

Our main result is an axiomatic characterization of the value.

For every subset, S, consider the zero-sum game played between S and its complement, where the players in each of these sets collaborate as a single player, and where the payoff is the difference between the sum of the payoffs to the players in S and the sum of payoffs to the players not in S. We say that S has an effective threat if the minmax value of this game is positive. The first axiom is that if no subset of players has an effective threat then all players are allocated the same amount.

The second axiom is that if the overall payoff to the players in a game is the sum of their payoffs in two unrelated games then the overall value is the sum of the values in these two games.

The remaining axioms are the strategic-game analogs of the classical coalitional-games axioms for the Shapley value: efficiency, symmetry, and null player.

Keyword: Strategic games, cooperative games, Nash bargaining, Shapley value, threats, bribes, corruption.

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1 Introduction

1.1 Purpose

We examine a solution concept that describes the a-priori value of each position in a strategic game.

The a-priori value takes into account possible side payments, e.g., bribes, which may arise from implicit or explicit threats. Thus in applications this concept sheds light on the economic value of a player's position that arises not only from the player's potential contribution to the total payoff, but also from the player's ability to inflict losses on other players.

Examples of "spoilers" with valuable positions abound: a small member of a lending syndicate who puts obstacles in the way to a restructuring agreement; a small shareholder who brings a derivative lawsuit against the directors of a public company; a municipal employee who imposes excessive costs on builders seeking construction permits; a purchasing agent for a local or national government who is in a position to influence the choice among competing suppliers. Indeed, the phenomenon of PEPs (politically exposed persons) abusing their positions is a major element of the endemic corruption in the public sectors of many countries.[1]

A salient feature of the solution is that it explicitly takes account of the costs imposed on the spoiler. These costs can be direct, such as monetary expenditures, or indirect, such as probabilistic expectations of fines or prison terms. Thus the solution may provide a tool for studying the effectiveness of various legal constraints and monitoring regimes in an attempt to design rules of the game that promote maximum social benefit.

Our main result is an axiomatic characterization of the solution. The axioms elucidate the underlying assumptions, thus making it possible to judge the reasonableness of an application in a specific context.

1.2 Overview

Game Theory is traditionally divided into two main branches – non cooperative and cooperative – each with its own solution concepts, e.g., minmax value and Nash equilibrium for non-cooperative games, core and Shapley value for cooperative games. However, most real-world economic and political interactions contain elements of both competition and cooperation.

[1] More than two thirds of the 176 countries and territories in the 2016 Corruption Perception Index fall below the midpoint of the index's scale of 0 (highly corrupt) to 100 (very clean).

The seminal work of Nash [6] pioneered the notion of a solution concept for strategic games that takes account of both their competitive and their cooperative aspects. Nash defined such a solution for two-person games and proved an existence and uniqueness theorem.

The solution is derived by means of "bargaining with variable threats." In an initial competitive stage, each player declares a "threat" strategy, to be used if negotiations break down; the outcome resulting from deployment of these strategies constitutes a "disagreement point." In a subsequent cooperative stage, the players coordinate their strategies to achieve a Pareto optimal outcome, and share the gains relative to the disagreement point; the sharing is done in accordance with principles of fairness.

Here, we build on the work of Nash [6], Harsanyi [5], Shapley [10], Aumann and Kurtz [1, 2], and Aumann, Kurtz, and Neyman [3, 4] to generalize this procedure to n-player games. We refer to the resulting solution concept as the *value* of the game.

The first step is to consider an alternative view of the two-player case. As was pointed out by Shapley [10], when players can transfer payoffs among themselves the outcome of "bargaining with variable threats" can be described very simply, as follows.[2]

Let s denote the maximal sum of the players' payoffs in any entry of the payoff matrix, and let d be the minmax value of the zero-sum game constructed by taking the difference between player 1's and 2's payoffs. Then the Nash solution splits the amount s in such a way that the difference in payoffs is d. Specifically, the payoffs to players 1 and 2 are, respectively, $\frac{1}{2}s + \frac{1}{2}d$ and $\frac{1}{2}s - \frac{1}{2}d$.

To get a sense of the Nash solution, consider the two-person game below.

Example 1.

$$\begin{bmatrix} 2,\ 1 & -1, -2 \\ -2, -1 & 1,\ 2 \end{bmatrix}.$$

At first blush the game looks entirely symmetrical. The set of feasible payoffs (the convex hull of the four entries in the matrix) is symmetrical; and the maximum payoff that each player can guarantee is the same, namely, 0. (Player 1's and 2's maxmin strategies are $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{2}{3}, \frac{1}{3})$, respectively.)

Thus one would expect the maximal sum of payoffs $s = 3$ to be shared equally, resulting in $(1.5, 1.5)$. However, the Nash analysis reveals a fundamental asymmetry.

[2] Kalai and Kalai [7] independently discovered Shapley's reformulation of "bargaining with variable threats" in two-person games and named the resulting solution concept the *coco value*. Their main result is an axiomatic characterization of the coco value. We discuss their work in Section 12.

In the zero-sum game of differences

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

the minmax value is not zero but rather $d = 1$. This indicates that the threat power of player 1 is greater than the threat power of player 2. The Nash solution reflects this advantage. It is $(1.5 + .5,\ 1.5 - .5) = (2, 1)$.

Following Harsanyi [5], we generalize the procedure as follows. Let s denote the maximal sum of payoffs in any entry of the payoff matrix, and let $d(S)$ be the minmax value of the zero-sum game between a subset of players, S, and its complement, $N\backslash S$, where the players in each of these subsets collaborate as a single player, and where the payoff to S is the difference between the sum of payoffs to the players in S and the sum of payoffs to the players in $N\backslash S$.

We define the value of the game by splitting the amount s among the players in such a way as to reflect the relative threat power of the subsets S. Specifically, the allocation to the players is the Shapley value of a coalitional game v with[3] $v(S) - v(N \setminus S) = d(S)$.

Since the above procedure is well defined and results in a single vector of payoffs, it follows that any n-person strategic game has a unique value.

Our main result is an axiomatic characterization of the value. This provides a conceptual foundation for what would otherwise be merely an ad hoc procedure.

Another result of interest is a formula for the value that makes its computation transparent.

We apply the formula in some simple examples. In one example, the economic output of a large number of individuals is predicated on the approval of a regulator. Computation of the value indicates that the regulator's position is worth 25% of the total output; and if approval is required from two regulators, then their combined positions are worth a full 42% of the total output. However, if approval is required from only one of the two regulators, then their combined positions are worth just 8.5% of the output.

In another example, it is assumed that making good on an implicit threat to deny approval (for no valid reason) exposes the regulator to potential punishment. If the expected cost of the punishment is a fraction c of the lost output, then the value of the regulator's position decreases by a factor of $(1 - c)^2$.

[3] The condition $v(S) - v(N\backslash S) = d(S)$ does not uniquely determine v; however, it does determine the Shapley value of v.

Such examples provide an indication of the strength of the temptation to use the power of approval in order to extract side payments.

We end this introduction with a brief comment on the axioms that characterize the value.

In a two-player game, if the minmax value of the game between players 1 and 2 is zero, then the Nash solution allocates the same payoff to both players. This can be interpreted as saying that if neither player has threat power, then each receives the same payoff.

In a two-person game the only proper subsets are the two singletons, and so the only threats to consider are those by one player vs. the other. But in n-person games there are many proper subsets, each of which can threaten its complement.

Thus, if we wish to generalize the Nash solution to n-person games, then it seems reasonable to require that in games where no proper coalition has threat power, i.e., $d(S) = 0$ for all proper subsets of N, all players receive the same payoff. We formalize this requirement as the axiom of balanced threats.

We prove that the axiom of balanced threats, in conjunction with the strategic-game analogs of the classical axioms for the Shapley value in coaltional games, uniquely determines a solution! This solution is the value.

The paper is organized as follows. In Section 2 we define the games and the axioms. In Section 3 we state the main results, namely the axiomatic characterization and the formula for computing the value. In Section 4 we apply the formula in a number of examples. Section 5 provides the background on games of threats that is required in the sequel. Section 6 provides an alternative definition of the value that relies on the notion of games of threats. Section 7 presents preliminary results that are needed for proving the characterization theorem; some of these are of interest in their own right. Section 8 provides the proof of the main theorem. In Section 9 we present additional properties of the value and in Section 10 we prove a stronger version of the uniqueness theorem. In Section 11 we discuss Shapley's classical notion of value for strategic games, and in Section 12 we discuss the coco value of Kalai and Kalai. In the Appendix we demonstrate that the axioms for the value are tight.

There are quite a few remarks, but we wish to emphasize that none of them is essential for reading the paper.

2 The Axiomatic Characterization

A strategic game is a triple $G = (N, A, g)$, where

- $N = \{1, \ldots, n\}$ is a finite set of players,
- A^i is the finite set of player i's pure strategies, and $A = \prod_{1=1}^{n} A^i$,
- $g = (g^i)_{i \in N}$, where $g^i \colon A \to \mathbb{R}$ is player i's payoff function.[4]

We use the same notation, g, to denote the linear extension

- $g^i \colon \Delta(A) \to \mathbb{R}$,

where for any set K, $\Delta(K)$ denotes the probability distributions on K, and we denote

- $A^S = \prod_{i \in S} A^i$, and
- $X^S = \Delta(A^S)$ (correlated strategies of the players in S).

We define the *direct sum* of strategic games as follows.

Definition 1. *Let $G_1 = (N, A_1, g_1)$ and $G_2 = (N, A_2, g_2)$ be two strategic games. Then*

$G := G_1 \oplus G_2$ is the game $G = (N, A, g)$, where $A = A_1 \times A_2$ and $g(a) = g_1(a_1) + g_2(a_2)$.

Remark 1. *The game $G_1 \oplus G_2$ models a situation where the same set of players play two unrelated games.*

Remark 2. *It is easy to verify that the operation $\oplus$ is, informally, commutative and associative.*[5] *However, there is no natural notion of inverse. (In general $G \oplus (-G) \neq 0$.)*

Denote by $\mathbb{G}(N)$ the set of all n-player strategic games.

Let $\gamma : \mathbb{G}(N) \to \mathbb{R}^n$. It may be viewed as a map that assigns to any strategic game an allocation of payoffs to the players.

We consider a list of axioms on γ. To that purpose we first introduce a few definitions.

We say that S is a *proper* subset of N if $S \neq \emptyset$ and $S \neq N$.

[4]The assumption that the sets of players and strategies are finite is made for convenience. The results remain valid when the sets are infinite, provided the minmax value exists in the two-person zero-sum games defined in the sequel.

[5]Formally, $G_1 \oplus G_2$ is not the same game as $G_2 \oplus G_1$, because $A_1 \times A_2 \neq A_2 \times A_1$.

Definition 2. *Let S be a proper subset of N. We say that S has an effective threat if* [6]

$$\max_{x\in X^S}\min_{y\in X^{N\setminus S}}\left(\sum_{i\in S} g^i(x,y) - \sum_{i\notin S} g^i(x,y)\right) > 0.$$

We say that *i and j are substitutes in G* if $A^i = A^j$ and $g^i = g^j$; and for any $a, b \in A^N$, if $a^i = b^j, a^j = b^i$, and $a^k = b^k$ for all $k \neq i, j$, then $g(a) = g(b)$.

We say that i is a *null player* in G if $g^i(a) = 0$ for all a; and if $a^k = b^k$ for all $k \neq i$, then $g(a) = g(b)$.

Our list of axioms is as follows.

For all strategic games G,

- **Efficiency** $\sum_{i\in N}\gamma_i G = \max_{a\in A^N}(\sum_{i\in N} g^i(a))$.
- **Balanced threats** If no proper subset of players has an effective threat then $\gamma_i G = \gamma_j G$ for all $i, j \in N$.
- **Symmetry** If i and j are substitutes in G then $\gamma_i G = \gamma_j G$.
- **Null player** If i is a null player in G then $\gamma_i G = 0$.
- **Additivity** $\gamma(G_1 \oplus G_2) = \gamma G_1 + \gamma G_2$.

Efficiency says that the players are allocated the maximum available payoff.

Since the sum of the allocations is fixed, any demand for payoff by a player or a group of players must come, by necessity, at the expense of the remaining players. The axiom of balanced threats says that if no player – no matter the additional players that have joined him – can effectively threaten the remaining players, then all players receive the same amount.

Symmetry says that players whose payoffs are identical everywhere, and whose strategies can be switched without impacting any payoff, receive the same allocation.

The null-player axiom says that a player whose actions do not affect any player's payoff, and whose own payoff is identically zero, receives an allocation of zero.

[6]Expressions of the form max or min over the empty set should always be ignored

Additivity says that if the payoff to the players is the sum of their payoffs in two games that are unrelated to each other then the allocation to the players is the sum of their allocations in these two games.

Our main result is that *there exists a unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ satisfying the axioms of efficiency, balanced threats, symmetry, null player, and additivity.*

It is remarkable that no further axioms are required to determine the value uniquely.

There are many additional desirable properties of the value that we do not assume but rather deduce from the axioms. These include dependence on the reduced form of the game (removing strategies that are convex combinations of other strategies does not affect the value), independence of the utility scale ($\gamma(\alpha G) = \alpha\gamma G$ for $\alpha > 0$), time-consistency ($\gamma(\frac{1}{2}G_1 \oplus \frac{1}{2}G_2) = \frac{1}{2}\gamma G_1 + \frac{1}{2}\gamma G_2$, i.e., it does not matter if the allocation is determined before or after the resolution of uncertainty about the game), monotonicity in actions (removing a pure strategy of a player does not increase the player's value)[7], independence of the set of players (addition of null players does not affect the value of the existing players), shift-invariance (adding a constant payoff to a player increases the player's value by that constant), a stronger form of symmetry (the names of the players do not matter), and continuity ($\gamma(G_n) \to \gamma G$ whenever $G_n = (N, A, g_n), G = (N, A, g)$, and $g_n \to g$).

Remark 3. *We do not require, nor are we able to deduce, that $\gamma(\alpha G) = \alpha\gamma G$ for negative α. Such a requirement, which is natural in the context of coalitional games, would make no sense in the context of strategic games. The game $-G$ involves dramatically different strategic considerations than the game G, so there is no reason to expect a simple relationship between the allocations in the two games.*

In the next section we describe an explicit formula for the value.

3 The main result

Let $G \in \mathbb{G}(N)$. Define

$$(\delta G)(S) := \max_{x\in X^S} \min_{y\in X^{N\setminus S}} \left(\sum_{i\in S} g^i(x, y) - \sum_{i\notin S} g^i(x, y) \right). \tag{1}$$

[7]These four properties follow from formula (3) and the corresponding properties of the minmax value of zero-sum games.

Theorem 1. *There is a unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of symmetry, null player, efficiency, balanced threats, and additivity. It may be described as follows.*

$$\gamma_i G = \frac{1}{n} \sum_{k=1}^{n} \delta_{i,k}, \tag{2}$$

where $\delta_{i,k}$ denotes the average of the $(\delta G)(S)$ over all k-player coalitions that include i.

We shall refer to the above map as the *value* for strategic games.

Remark 4. *Note that $\delta_{i,n} = \max_{x \in X^N} (\sum_{i \in N} g^i(x))$ is the maximum available payoff. The formula allocates to each player $\frac{1}{n}$th of this amount, adjusted according to the average threat power of the subsets that include the player.*

Remark 5. *Each player is allocated a weighted average of the $\delta G(S)$ over the coalitions S that include that player. The weight is the same for all coalitions of the same size but different for coalitions of different size. Specifically, for each $k = 1, \ldots, n$, the total weight of $\frac{1}{n}$ is divided among the $\binom{n-1}{k-1}$ coalitions of size k that include i. Thus the formula can be rewritten as follows.*

$$\gamma_i G = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\binom{n-1}{k-1}} \sum_{\substack{S: i \in S \\ |S| = k}} \delta G(S). \tag{3}$$

Remark 6. *The formula implies that the value of a game G depends only on δG. Note that this is not an assumption but rather a conclusion. (See Proposition 4 and Claim 4.)*

Remark 7. *In two-player games the value coincides with the Nash solution: in Example 1, $(\delta G)(1) = 1$, $(\delta G)(2) = -1$ and $(\delta G)(1,2) = 3$; therefore $\gamma_1 G = \frac{1}{2} \times 1 + \frac{1}{2} \times 3 = 2$, and $\gamma_2 G = \frac{1}{2} \times (-1) + \frac{1}{2} \times 3 = 1$.*

The proof of Theorem 1 requires the notion of games of threats [8]. We provide the relevant definitions and results in Section 5.

4 Examples

In each of the examples below we apply formula (2) to determine the value, γ.

Example 2. *This is a three-player game. Player* 1 *chooses the row, players* 2 *chooses the column, and player* 3 *has only a single strategy. The payoff matrix is* [8]

$$G = \begin{bmatrix} 2,2,2 & 0,0,0 \\ 0,0,0 & 1,1,1 \end{bmatrix}.$$

Now,

$$\delta G(1) = \text{minmax} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = -\frac{2}{3},$$

$$\delta G(1,2) = \max(2,0,1) = 2,$$

$$\delta G(1,3) = \text{minmax} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{2}{3}, \quad \text{and}$$

$$\delta G(1,2,3) = \max(6,0,3) = 6.$$

Thus $\gamma_1 G = \frac{1}{3} \times (-\frac{2}{3}) + \frac{1}{3} \times \frac{2+\frac{2}{3}}{2} + \frac{1}{3} \times 6 = 2\frac{2}{9}$, and therefore $\gamma G = (2\frac{2}{9},\ 2\frac{2}{9},\ 1\frac{5}{9})$.

When player 3 is dropped, the game becomes

$$G' = \begin{bmatrix} 2,2 & 0,0 \\ 0,0 & 1,1 \end{bmatrix},$$

and $\gamma G' = (2,2)$.

Example 3. *This is a two-player game. Player* 1 *can choose one of two rows. Player* 2 *has a single strategy. The payoff matrix is*

$$G_2 = \begin{bmatrix} 1,1 \\ 0,0 \end{bmatrix},$$

and $\gamma_1 = \frac{1}{2} \times 0 + \frac{1}{2} \times 2 = 1$. *Thus* $\gamma G_2 = (1,1)$.

[8] Player 1 and player 2's payoffs are identical and their strategies can be switched without impacting any payoff. This is an example of substitute players.

Example 4. *This is an n-player version of the game G_2 above. Player 1 can choose one of two rows. Players $2, \ldots, n$ each have but a single strategy. The payoff matrix is*

$$G_n = \begin{bmatrix} 1, 1, \ldots, 1, 1 \\ 0, 0, \ldots, 0, 0 \end{bmatrix}.$$

Now $\delta_{1,k} = \max(0,\ k - (n-k))$.

By formula (2), $\gamma_1 G_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{i,k} = \frac{1}{n} \sum_{k=1}^{n} \max(0,\ 2k - n)$ and, by symmetry and efficiency, $\gamma_i G_n = \frac{n-\gamma_1}{n-1}$ for $i = 2, \ldots, n$.

Plugging in $n = 3$ *and* $= 4$ we see that $\gamma G_3 = \frac{1}{6}(8, 5, 5)$ and $\gamma G_4 = \frac{1}{6}(9, 5, 5, 5)$, and it is straightforward to verify that as the number of players, n, becomes large $\gamma G_n \sim \frac{1}{4}(n, 3, \ldots, 3)$.

Thus the value of player 1 is approximately one fourth of the total feasible output. In effect, each of the remaining players concedes one fourth of their equal share to player 1.

These examples highlight the power of player 1's threat to reduce everyone's payoff to zero. The greater the number of other players, the greater is the power of this threat. The value, γ, reflects this.

Example 5. *This is a variant of game G_4 above. Now there is a cost, $c > 0$, to the spoiler.*

$$G_{4,c} = \begin{bmatrix} 1, 1, 1, 1 \\ -c, 0, 0, 0 \end{bmatrix}.$$

If $c \geq 2$ then $\gamma = (1, 1, 1, 1)$; but if $c \leq 2$ then $\gamma_1 = \frac{1}{4} \times (-c) + \frac{1}{4} \times 0 + \frac{1}{4} \times 2 + \frac{1}{4} \times 4 = \frac{3}{2} - \frac{c}{4}$; thus $\gamma = (\frac{3}{2} - \frac{c}{4}, \frac{5}{6} + \frac{c}{12}, \frac{5}{6} + \frac{c}{12}, \frac{5}{6} + \frac{c}{12})$.

Note that the value of player 1 in this game is $1.5 - \frac{c}{4}$, as compared with 1.5 in the game G_4. This demonstrates that the economic value of the ability to spoil is diminished when there is a cost to the spoiler.

Example 6.

$$G_{n,c} = \begin{bmatrix} 1, 1, \ldots, 1, 1 \\ -c, 0, \ldots, 0, 0 \end{bmatrix}.$$

It is straightforward to verify that as the number of players, n, becomes large, the value of player 1 becomes approximately one fourth of the total payoff, the same as in Example 4. However, if the cost to the spoiler is proportional to the damage imposed on the others, say $c = c_0 n$, where $c_0 < 1$, then as the number of players becomes large, the value of player 1 in the game $G_{n,c_0 n}$ becomes approximately $\frac{(1-c_0)^2}{4}$ of the total feasible output.

Example 7. *This is a variant of the game G_n where there is more than one player whose approval is required for all players to receive* 1*. If one of the distinguished players disapproves then all players receive zero.*

It is easy to compute the asymptotic behavior as $n \to \infty$. In the case of two distinguished players, the payoff to each of them divided by n – the total feasible output – converges, as $n \to \infty$, to $\int_{1/2}^{1}(2x-1)xdx = \frac{5}{24}$, which is about 21%. Thus the two spoilers receive 42% of the total feasible output, compared with 25% in the case of a single spoiler.

In the case of k distinguished players the payoff to each of them divided by n converges, as $n \to \infty$, to $\int_{1/2}^{1}(2x-1)x^{k-1}dx$. Since $k\int_{1/2}^{1}(2x-1)x^{k-1}dx$ converges, as $k \to \infty$, to 1, we see that when there are many spoilers essentially all of the economic output goes to them.

Example 8. *This is a variant of the game G_n in which there are k distinguished players; if any one of these players approves, then all players receive* 1*.*

The asymptotic behavior as $n \to \infty$ is as follows. The payoff to each distinguished player divided by n – the total feasible output – converges, as $n \to \infty$, to $\frac{2^{-k}}{k(k+1)}$. Thus the combined payoff to all the distinguished players is $\frac{2^{-k}}{(k+1)}$.

When $k = 1$ this amounts to $\frac{1}{4}$, as we have seen in Example 4. When $k = 2$ this amounts to $\frac{1}{12}$. Thus, when there are two distinguished players, only one of whose approvals is required, the fraction of the total value that they command is about 8.5%; this is in contrast to 42% in the case where both approvals are required, as in Example 7.

Remark 8. *The examples demonstrate that games with more than two players exhibit phenomena that are not present in two-player games. These arise from a player's ability to play off some of the other players against one another.*

5 Games of Threats

A *coalitional game of threats* is a pair (N, d), where

- $N = \{1, \ldots, n\}$ is a finite set of players.
- $d\colon 2^N \to \mathbb{R}$ is a function such that $d(S) = -d(N \backslash S)$ for all $S \subseteq N$.

Remark 9. *A game of threats need not be a coalitional game as $d(\emptyset) = -d(N)$ may be non-zero.*

Remark 10. *If d is a game of threats then so is $-d$.*

Denote by $\mathbb{D}(N)$ the set of all coalitional games of threats.

Let $\psi\colon \mathbb{D}(N) \to \mathbb{R}^n$. It may be viewed as a map that associates with any game of threats an allocation of payoffs to the players. Following Shapley ([9]), we consider the following axioms.

For all games of threat $(N, d_1), (N, d_2)$, and for all players i, j,

- *Symmetry* $\psi_i(d) = \psi_j(d)$ if i and j are substitutes in d (i.e., if $d(S \cup i) = d(S \cup j)$ $\forall S \subseteq N \backslash \{i, j\}$).
- *Null player* $\psi_i d = 0$ if i is a null player in d (i.e., if $d(S \cup i) = d(S)$ $\forall S \subseteq N$).
- *Efficiency* $\sum_{i \in N} \psi_i d = d(N)$.
- *Additivity* $\psi(d_1 + d_2) = \psi d_1 + \psi d_2$.

Below are two results from ([8]) that will be needed in the sequel.

Proposition 1. *There exists a unique map $\psi\colon \mathbb{D}(N) \to \mathbb{R}^n$ satisfying the axioms of symmetry, null player, efficiency, and additivity. It may be described as follows.*

$$\psi_i d = \frac{1}{n} \sum_{k=1}^{n} d_{i,k}, \tag{4}$$

where $d_{i,k}$ denotes the average of the $d(S)$ over all k-player coalitions that include i.

We refer to this map as the *Shapley value for games of threats.*

Definition 3. *Let $T \subseteq N$, $T \neq \emptyset$. The unanimity game, $u_T \in \mathbb{D}(N)$, is defined by*

$$u_T(S) = \begin{cases} |T| & \text{if } S \supseteq T, \\ -|T| & \text{if } S \subseteq N \backslash T, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2. *Every game of threats is a linear combination of the unanimity games u_T.*

6 An alternative definition of the value

Using the notion of games of threats we can provide an alternative definition of the value:

Proposition 3. *The value of a strategic game G is the Shapley value of the game of threats associated with G, i.e., $\gamma = \psi \circ \delta$, where $\gamma \colon \mathbb{G}(N) \to \mathbb{R}^n$, $\psi \colon \mathbb{D}(N) \to \mathbb{R}^n$, and $\delta \colon \mathbb{G}(N) \to \mathbb{D}(N)$ are as in (3), (4), and (1), respectively.*

Proof. Formula (2) is the same as formula (4), applied to the game of threats $d = \delta G$. □

Thus, Theorem 1 can be rephrased as follows. *$\gamma = \psi \circ \delta$ is the unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of symmetry, null player, efficiency, balanced threats, and additivity.*

7 Preliminary results

In this section we present properties of the mapping $\delta \colon \mathbb{G}(N) \to \mathbb{D}(N)$ that are needed for the proof of the main result.

Let $G \in \mathbb{G}(N)$. For any $S \subseteq N$, let $\delta G(S)$ be as in (1).

Lemma 1. *δG is a game of threats.*

Proof. By the minmax theorem $\delta G(S) = -\delta G(N \backslash S)$ for any $S \subseteq N$. □

We refer to δG as the game of threats associated with G.

Lemma 2. $\delta : \mathbb{G}(N) \to \mathbb{D}(N)$ *satisfies:*

- $\delta(G_1 \oplus G_2) = \delta G_1 + \delta G_2$ *for any* $G_1, G_2 \in \mathbb{G}(N)$.
- $\delta(\alpha G) = \alpha \delta G$ *for any* $G \in \mathbb{G}(N)$ *and* $\alpha \geq 0$.

Proof. Let $\text{val}(G)$ denote the minmax value of the two-person zero-sum strategic game G. Then $\text{val}(G_1 \oplus G_2) = \text{val}(G_1) + \text{val}(G_2)$.

To see this, note that by playing an optimal strategy in G_1 as well as an optimal strategy in G_2, each player guarantees the payoff $\text{val}(G_1) + \text{val}(G_2)$.

Now apply the above to all the two-person zero-sum games played between a coalition S and its complement $N \backslash S$, as indicated in (1). □

The next lemma is an immediate consequence of the definition of δ:

Lemma 3. $\delta : \mathbb{G}(\mathbb{N}) \to \mathbb{D}(\mathbb{N})$ *satisfies:*

- $\delta G(N) = \max_{a \in A^N} (\sum_{i \in N} g^i(a))$.
- *If* i *and* j *are substitutes in* G *then* i *and* j *are substitutes in* δG.
- *If* i *is a null player in* G *then* i *is a null player in* δG.

Denote by $1_T \in \mathbb{R}^n$ the indicator vector of a subset $T \subseteq N$, i.e., $(1_T)_i = 1$ or 0 according to whether $i \in T$ or $i \notin T$.

Definition 4. *Let* $T \subseteq N$, $T \neq \emptyset$. *The unanimity game on* T *is* $U_T = (N, A, g_T)$, *where*

$A^i = \{0, 1\}$ *for all* $i \in N$,

$g_T(a) = 1_T$ *if* $a^i = 1$ *for all* $i \in T$, *and* $g_T(a) = 0$ *otherwise.*

That is, if all the members of T consent then they each receive 1; however, if even one member dissents, then all receive zero; the players outside T always receive zero.

Lemma 4. *Let* $T \neq \emptyset$, *and let* $U_T \in \mathbb{G}(N)$ *and* $u_T \in \mathbb{D}(N)$ *be the unanimity games on* T. *Then* $\delta U_T = u_T$.

Proof. Consider the two-person zero-sum game between S and $N \setminus S$.

If $S \cap T \neq \emptyset, T$ then both S and $N \backslash S$ include a player in T. If these players dissent then all players receive 0. Thus the minmax value, $\delta U_T(S)$, is 0.

If $S \cap T = T$ then, by consenting, the players in S can guarantee a payoff of 1 to each player in T and 0 to all the others. Thus $\delta U_T(S) = |T|$.

If $S \cap T = \emptyset$ then, by consenting, the players in $N \setminus S$ can guarantee a payoff of 1 to each player in $N \setminus T$ and 0 to all the others. Thus $\delta U_T(S) = |T|$.

By definition 3 , $\delta U_T = u_T$. □

Definition 5. *The anti-unanimity game on T is $V_T = (N, A, g)$, where*

$A^i = \{S \subseteq T : S \neq \emptyset\}$,

$g(S_1, \ldots, S_n) = \sum_{i \in T} -1_{S_i}$.

That is, each player chooses a non-empty subset of T where each members loses 1. Thus the payoff to player i is minus the number of players in T whose chosen set includes i.

Lemma 5. $\delta V_T = -u_T$.

Proof. Let S be a subset of N such that $S \cap T \neq \emptyset$ and $T \setminus S \neq \emptyset$. In the zero-sum game between a proper subset S and its complement, the minmax strategies are for the players in S to choose $T \setminus S$ and for the players in $N \setminus S$ to choose $S \cap T$. The resulting payoff is $s(|T| - s) - (|T| - s)s = 0$, where s is the number of elements of $T \cap S$. Thus $\delta V_N(S) = 0$.

When all $T \subseteq S$, the players in S collaborate: they each choose a subset of size 1. Thus $\delta V_T(S) = -|T|$.

Therefore, $\delta V_T = -u_T$. □

Lemma 6. *For every game of threats $d \in \mathbb{D}(N)$ there exists a strategic game $U \in \mathbb{G}(N)$ such that $\delta U = d$. Moreover, there exists such a game that can be expressed as a direct sum of non-negative multiples of the unanimity games $\{U_T\}_{T \subseteq N}$ and the anti-unanimity games $\{V_T\}_{T \subseteq N}$.*

Proof. By Proposition 2, d is a linear combination of the unanimity games u_T.

$$d = \sum_T \alpha_T u_T - \sum_T \beta_T u_T \text{ where } \alpha_T, \beta_T \geq 0 \text{ for all } T.$$

By Lemmas 4 and 5,

$$d = \sum_T \delta(\alpha_T U_T) + \sum_T \delta(\beta_T V_T),$$

and, by Lemma 2,

$$d = \delta((\bigoplus_{T \subseteq N} \alpha_T U_T) \oplus (\bigoplus_{T \subseteq N} \beta_T V_T)),$$

where $\bigoplus_T$ stands for the direct sum of the games parameterized by T. □

Remark 11. *In particular, Lemma 6 establishes that the mapping* $\delta \colon \mathbb{G}(N) \to \mathbb{D}(N)$ *is onto.*

As was pointed out earlier, the operation $\oplus$ does not have a natural inverse. However, we have the following:

Lemma 7. *For every* $G \in \mathbb{G}(N)$ *there exists a* $\delta-$*inverse, i.e.,* $U \in \mathbb{G}(N)$ *such that* $\delta(G \oplus U) = 0$*. Moreover, if* $G' \in \mathbb{G}(N)$ *is such that* $\delta G' = \delta G$ *then there exists* $U \in \mathbb{G}(N)$ *that is a* $\delta-$*inverse of both* G *and* G'*.*

Proof. Consider $-\delta G \in \mathbb{D}(N)$. By Lemma 6, there exists $U \in \mathbb{G}(N)$ such that $-\delta G = \delta U$. By Lemma 2, $\delta(G \oplus U) = 0$.

And if G' is such that $\delta G' = \delta G$ then, by the same argument, $\delta(G' \oplus U) = 0$. □

Lemma 8. *Assume that* $\gamma \colon \mathbb{G}(N) \to \mathbb{R}^n$ *satisfies the axioms of balanced threats and efficiency. If* $\delta G = 0$ *then* $\gamma G = 0$*.*

Proof. Since $(\delta G)(S) = 0$ for all proper subsets of N, the axiom of balanced threats implies that all the $\gamma_i G$ are the same. By efficiency, their sum is equal to $\max_{a \in A^N}(\sum_{i \in N} g^i(a)) = \delta G(N) = 0$. Thus each of the $\gamma_i G$ is zero. □

Proposition 4. *If* $\gamma \colon \mathbb{G}(N) \to \mathbb{R}^n$ *satisfies the axioms of balanced threats, efficiency, and additivity then* γG *is a function of* δG*.*

Proof. Let $G, G' \in \mathbb{G}(N)$ be such that $\delta G = \delta G'$. We must show that $\gamma G = \gamma G'$.
By Lemma 7, there exists $U \in \mathbb{G}(N)$ such that $\delta(G \oplus U) = 0 = \delta(G' \oplus U)$.
By Lemma 8, $\gamma(G \oplus U) = 0 = \gamma(G' \oplus U)$.
Thus, by the additivity axiom, $\gamma G = -\gamma U = \gamma G'$. □

Lemma 9. *For any $T \neq \emptyset$ and $\alpha \geq 0$, the axioms of symmetry, null player, and efficiency determine γ on the game αU_T. Specifically, $\gamma(\alpha U_T) = \alpha 1_T$.*

Proof. Any $i \notin T$ is a null player in U_T, and so $\gamma_i = 0$. Any $i, j \in T$ are substitutes in U_T, and so $\gamma_i = \gamma_j$. By efficiency, the sum of the γ_i is the maximum total payoff, which, since $\alpha > 0$, is $\alpha|T|$. Thus each of the $|T|$ non-zero γ_i is equal to α. □

Lemma 10. *For any $\alpha \geq 0$, the axioms (of symmetry, null player, additivity, balanced threats, and efficiency) determine γ on the game αV_T. Specifically, $\gamma(\alpha V_T) = -\alpha 1_T$.*

Proof. By Lemma 9 the axioms determine $\gamma(\alpha U_T) = \alpha 1_T$. By Lemmas 4 and 5, $\delta(\alpha V_T \oplus \alpha U_T) = 0$. Therefore, by Lemma 8, $\gamma(\alpha V_T \oplus \alpha U_N) = 0$. Thus, by additivity, $\gamma(\alpha V_T) = -\gamma(\alpha U_T) = -\alpha 1_T$. □

Remark 12. *We cannot prove the lemma by appealing to symmetry and efficiency. In the game V_T, it is not true that any two players, $i, j \in T$, are substitutes.*

8 Proof of the main result

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Proof of Theorem 1.

We first prove uniqueness.

Let $G \in \mathbb{G}(N)$. Consider $\delta G \in \mathbb{D}(N)$; by Lemma 6 there exists a game $U \in \mathbb{G}(N)$ that is a direct sum of non-negative multiples of the unanimity games $\{U_T\}_{T\subseteq N}$ and the anti-unanimity games $\{V_T\}_{T\subseteq N}$, such that $\delta G = \delta U$.

By Proposition 4, $\gamma G = \gamma U$ and so it suffices to show that γU is determined by the axioms.

Now, by Lemmas 9 and 10 , γ is determined on non-negative multiples of the unanimity games $\{U_T\}_{T\subseteq N}$ and the anti-unanimity games $\{V_T\}_{T\subseteq N}$. It then follows from the axiom of additivity that γ is determined on U.

To prove existence we show that the value, $\gamma = \psi \circ \delta$, satisfies the axioms.

Efficiency, symmetry, and the null player axiom follow from Lemma 3 and the corresponding properties of the Shapley value ψ.

Additivity follows from Lemma 2 and the linearity of the Shapley value.

To see that γ satisfies the axiom of balanced threats, assume that $(\delta G)S = 0$ for any proper subset of N. Then $\delta_{i,k}$, the average of the $(\delta G)(S)$ over all k-player coalitions that include i, is zero for any $k < n$. It then follows from (2) that $\gamma_i G = \frac{1}{n}\delta G(N)$; thus $\gamma_i G = \gamma_j G$ for all i, j. □

9 Additional Properties of the value

Another axiom of interest is the following.

Small worlds If the set of players is the union of two disjoint subsets such that the payoffs to the players in each subset are unaffected by the actions of the players in the other subset, then the value of each player is the same as it would be in the game restricted to the subset that includes the player.

Proposition 5. *The value satisfies the small-worlds axiom.*

Proof. Let $G = (N, A, g)$, where $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, and where the actions of players in N_1 do not affect the payoffs to players in N_2, and vice versa.

Assume, w.l.o.g., that $1 \in A_2^i$ for all $i \in N$. Define $G_1 \in \mathbb{G}(N)$ by modifying G as follows. Restrict the set of pure strategies of each player in N_2 to $\{1\}$ and define $g_1^i = g^i$ for $i \in N_1$, $g_1^i = 0$ for $i \in N_2$; and define G_2 in a similar way.

By the definition (1) of δ,

$$\delta G = \delta G_1 + \delta G_2.$$

Recall that $\gamma = \psi \circ \delta$, where ψ is the Shapley value for games of threats (Proposition 3). Since ψ is additive,

$$\gamma G = \psi \circ \delta(G) = \psi \circ \delta(G_1 + G_2) = \psi \circ \delta G_1 + \psi \circ \delta G_2 = \gamma G_1 + \gamma G_2.$$

Since any $i \in N_1$ is a null player in G_2, it follows from the null-player axiom that $\gamma_i G_2 = 0$. Thus

$$\gamma_i G = \gamma_i G_1 + \gamma_i G_2 = \gamma_i G_1 \text{ for all } i \in N_1.$$

Similarly, $\gamma_i G = \gamma_i G_2$ for all $i \in N_2$.

Thus, for $i \in N_1$, the value of G is the same as the value of G_1, which may be viewed as the restriction of G to N_1. And similarly for $i \in N_2$. □

Remark 13. *It is insufficient to assume that the payoffs to the players in* N_1 *are unaffected by the actions of the players in* N_2*. In example 2, player* 3 *has only one strategy and so he obviously cannot affect the payoffs of the other players. Yet when player* 3 *is dropped, the values for players* 1 *and* 2 *change.*

Remark 14. *The small-world axiom may be viewed as an instance of the more general statement, that the additivity of the value extends to games over two different sets of players. Let* $G_1 \in \mathbb{G}(N_1)$ *and* $G_2 \in \mathbb{G}(N_2)$*. By adding the members of* $N_2 \setminus N_1$ *as dummy players in* G_1*, and the members of* $N_1 \setminus N_2$ *as dummy players in* G_2*, we may view both* G_1 *and* G_2 *as games in* $\mathbb{G}(N_1 \cup N_2)$*. Thus* $\gamma(G_1 \oplus G_2) = \gamma G_1 + \gamma G_2$*. Since the value of the existing players is unaffected by the addition of dummy players,* $\gamma_i(G_1 \oplus G_2) = \gamma_i G_1$ *for all* $i \in N_1 \setminus N_2$ *and* $\gamma_i(G_1 \oplus G_2) = \gamma_i G_2$ *for all* $i \in N_2 \setminus N_1$*. The small-worlds axiom corresponds to the case where* N_1 *and* N_2 *are disjoint.*

Recall that the axiom of balanced threats says that if $\delta G(S) = 0$ for any proper subset S, then $\gamma_i G = \gamma_j G$ for all i, j.

We now consider a stronger version of this axiom:

Strong axiom of balanced threats If $\delta G(S) = 0$ for all subsets S such that $S \ni i$ and $S \not\ni j$ then $\gamma_i G = \gamma_j G$.

Proposition 6. *The value satisfies the strong axiom of balanced-threats.*

Proof. Since $(\delta G)S = 0$ for subsets S that include i but do not include j, formula (3) becomes

$$\gamma_i G = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\binom{n-1}{k-1}} \sum_{\substack{S:\{i,j\} \subseteq S \\ |S|=k}} \delta G(S).$$

But the r.h.s. is the same for $\gamma_j G$. □

Recall that our axiom of symmetry says that if two players, i and j, are substitutes in the game G then $\gamma_i G = \gamma_j G$. By contrast, the classical axiom requires that the value be invariant to permutations of the players' names. Since every permutation of N consists of a sequence of pairwise exchanges, the axiom can be stated as follows.

Axiom of full symmetry Let $G = (N, A, g)$ and let $\hat{G} = (N, \hat{A}, \hat{g})$ be such that $\hat{A}^i = A^j$, $\hat{A}^j = A^i$ and $\hat{g}^i = g^j$, $\hat{g}^j = g^i$, then $\gamma_i \hat{G} = \gamma_j G$, $\gamma_j \hat{G} = \gamma_i G$, and $\gamma_k \hat{G} = \gamma_k G$ for $k \neq i, j$.

Clearly, this axiom is stronger than our symmetry axiom. Still, formula (2) establishes that

Proposition 7. *The value satisfies the axiom of full symmetry.*

Given a game $G = (N, A, g)$ and $\alpha \in \mathbb{R}^n$, let $G + \alpha$ be the game obtained from G by adding α to each payoff entry, namely, $G + \alpha = (N, A, g + \alpha)$.

Axiom of shift invariance $\gamma(G + \alpha) = \gamma G + \alpha$.

Proposition 8. *The value satisfies the axiom of shift invariance.*

Proof. The definition (1) of δ implies that

$$\delta(G + \alpha)(S) = (\delta G)(S) + \sum_{j \in S} \alpha_j - \sum_{j \in N \setminus S} \alpha_j.$$

Therefore, if $i \in S$ then $\delta(G+\alpha)(S) + \delta(G+\alpha)(i \cup (N \setminus S)) = (\delta G)(S) + \sum_{j \in S} \alpha_j - \sum_{j \in N \setminus S} \alpha_j + (\delta G)(i \cup (N \setminus S)) + \sum_{j \in i \cup N \setminus S} \alpha_j - \sum_{i \neq j \in S} \alpha_j = (\delta G)(S) + (\delta G)(i \cup (N \setminus S)) + 2\alpha_i$. As the map from subsets S of size k that contain player i, defined by $S \mapsto i \cup (N \setminus S)$, is 1-1 and onto the subsets of size $n-k+1$ that contain player i, we deduce that $\delta_{i,k}(G + \alpha) + \delta_{i,n-k+1}(G + \alpha) = \delta_{i,k}(G) + \delta_{i,n-k+1}(G) + 2\alpha_i$. Therefore, by the formula (2) for the value, $\gamma(G + \alpha) = \gamma G + \alpha$. □

Remark 15. *Let I_α be a game where the payoff is the constant α. The game $G+\alpha$ is strategically equivalent to $G \oplus I_\alpha$. As the value of two strategically equivalent games coincide, it would have been sufficient to prove that $\gamma(G \oplus I_\alpha) = \gamma G + \alpha$. For this equality one need not rely on the axiom of balanced threats.*

Proposition 9. *A map $\gamma : \mathbb{G}(N) \to \mathbb{R}^n$ that satisfies the additivity, efficiency, and null player axioms, satisfies the axiom of shift invariance.*

Proof. We prove that $\gamma I_\alpha = \alpha$.

Note that a player i is a null player in I_α if and only if $\alpha_i = 0$. If all the players in I_α are null players then $\alpha = 0$ and $\gamma(I_\alpha) = 0$ by the null-player axiom. Assume that there is one non-null player in I_α, say player i. Then, $\alpha_i \neq 0$ and $\forall j \neq i$, $\alpha_j = 0$, and by the null-player axiom $\gamma_j(I_\alpha) = 0$, and by the efficiency axiom $\gamma_i(I_\alpha) = \alpha_i$. Therefore, $\gamma(I_\alpha) = \alpha$.

We continue by induction on the number of non-null players in I_α. If there are $k > 1$ non-null players in I_α, then let $\alpha(1)$ and $\alpha(2)$ be such that $\alpha = \alpha(1) + \alpha(2)$ and in each game $I_{\alpha(1)}$ and $I_{-\alpha(2)}$ there are fewer than k non-null players. By the additivity axiom, $\gamma(I_\alpha \oplus I_{-\alpha(2)}) = \gamma(I_\alpha) + \gamma(I_{-\alpha(2)}$, and by the induction hypothesis $\gamma(I_\alpha \oplus I_{-\alpha(2)}) = \alpha(1)$ and $\gamma(I_{-\alpha(2)}) = -\alpha(2)$. We conclude that $\gamma(I_\alpha) = \alpha(1) + \alpha(2) = \alpha$.

Therefore, $\gamma(G \oplus I_\alpha) = \gamma G + \gamma I_\alpha = \gamma G + \alpha$, where the first equality follows from the axiom of additivity and the second equality from the previously proved $\gamma I_\alpha = \alpha$. □

10 A stronger version of the uniqueness result

In this section we present a stronger version of the main result. Specifically, we show that the axiom of balanced threats can be replaced by a less restrictive variant without impacting Theorem 1.

Consider the following axiom:

Weak axiom of balanced threats If no subset of players has an effective threat then $\gamma_i G = \gamma_j G$ for all $i, j \in N$.

Recall that the standard balanced-threat axiom assumes that proper subsets of N have no effective threat. Here there is an additional assumption, namely, that $\emptyset$ and N have no effective threat as well, i.e., that $\max_{a \in A^N}(\sum_{i \in N} g^i(a) = 0$.

Thus the weak axiom of balanced threats, in conjunction with the axiom of efficiency, can be interpreted as saying that in a game of pure division, where the total available is zero, if no player has an effective threat then all players receive zero.

The appeal of the axiom is that it does not make a distinction between $\emptyset$ and N and the other subsets of N. More importantly, the requirement that the absence of effective threats implies equal allocations seems more convincing in a zero-sum context.

We are now ready to state the stronger version of the uniqueness result.

Proposition 10. *The value is the unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of symmetry, null player, efficiency, weak balanced threats, and additivity.*

Proposition 10 follows directly from the proof of Theorem 1.

11 The Shapley value of strategic games

An alternative solution concept, which we shall refer to as the *Shapley value for strategic games*, goes back to Shapley's [9] original paper. It is defined as the Shapley value of the coalitional game v, where

$$(vG)(S) := \max_{x \in X^S} \min_{y \in X^{N \setminus S}} \sum_{i \in S} g^i(x, y). \qquad (5)$$

This concept is similar to the notion of value advanced here, as both are obtained by taking the Shapley value of a coalitional game derived from the strategic game. The crucial difference is the manner in which the coalitional game is defined, i.e., equation (5) vs equation (1). The former is the maximal payoff to its own members that a coalition can guarantee, while the latter is the maximal difference in payoff between its members and the others that a coalition can guarantee.

We do not believe that the Shapley value is appropriate for analyzing the phenomenon of threats. The difficulty is that the strength of the claim by a coalition, S, and by its complement, $N \setminus S$, for a piece of the total payoff, is based on two different zero-sum games: one where the focus is exclusively on the payoff to S, the other where the focus is exclusively on the payoff to $N \setminus S$. But, as the total payoff is fixed, any payoff received by S comes at the expense of $N \setminus S$. It would therefore seem natural to determine the strengths of S and of $N \setminus S$ by a *single* game in which each of these coalitions strives to increase its own payoff and to decrease the payoff of its complement, i.e., to maximize the difference in payoffs. This leads to equation (1).[9,10]

Example 1 is a case in point. While the value is $(2, 1)$, the Shapley value is $(1.5, 1.5)$; i.e., it does not reflect the threat power of player 1.[11]

The Shapley value quantifies the economic benefit of a player's ability to spoil; however, it does not take into account the associated costs to the spoiler.

Consider the following variant of Example 3.

Example 9.

$$\begin{bmatrix} 3,3 \\ 0,1 \end{bmatrix}.$$

[9]In the case that the strategic game is constant-sum, there is no tension between maximizing the payoff to S and minimizing the payoff to $N \setminus S$; thus the two notions of value coincide.

[10]Shapley [9] has written: "Serious doubt has been raised as to the adequacy with which the characteristic function describes the strategic possibilities of a general-sum game. The difficulty, intuitively, is that the characteristic function does not distinguish between threats that damage just the threatened party and threats that damage both parties. This criticism, however, does not apply with any force to the constant-sum case."

[11]Note that the minmax strategies for player 2 are different in the game that focuses on 1's payoff and in the game that focuses on 2's payoff.

The Shapley value is $(4, 2)$. The extra payoff to player 1 is due to the threat of choosing the bottom row. But this threat is not credible, as the loss inflicted on the player himself exceeds the loss inflicted on player 2. The value, by contrast, is $(3, 3)$.

A similar phenomenon occurs in Examples 5 and 6, where the Shapley value remains the same, irrespective of the cost of spoiling.

12 The coco value

Kalai and Kalai [7] introduced the "coco value," which coincides with the value of two-person strategic games. Their main result is an axiomatic characterization of the coco value.

The axioms that they consider are the following: efficiency, shift invariance (if G_α is a modification of G obtained by adding to the payoff of one player, say player 1, an amount α everywhere, then $\gamma G_\alpha = \gamma G + (\alpha, 0)$), invariance to redundant strategies (removing a duplicate row or column in the payoff matrix does not affect the value), monotonicity in actions (removing a pure strategy of a player cannot increase the player's value), and payoff dominance (if player 1's payoff is everywhere strictly greater than player 2's payoff then $\gamma_1 \geq \gamma_2$). They prove that there is a unique map from $\mathbb{G}(2)$ to $\mathbb{R}^2$ that satisfies these axioms.

Since Kalai and Kalai characterize the same concept for two-person games as we do, their axioms are equivalent to ours. To see a direct connection between the two sets of axioms, it may be helpful to note that in two-person games the general additivity axiom can be replaced by the requirement that the solution be additive over the direct sum of a game and a trivial game, which amounts to shift invariance.

In games with more than two players the value still satisfies all the Kalai and Kalai axioms other than payoff dominance. This follows from Remark 6 and Proposition 8. However, the value does not satisfy payoff dominance. This is a reflection of the more complex considerations in games with more than two players. In Example 5, player 1's payoff is everywhere smaller than player 2's, but 1's value is greater. This is so because of player 1's ability to play off some of his opponents against each other.

Remark 16. *Kalai and Kalai [7] extend their axiomatization to two-player Bayesian games. In such games, monotonicity in actions is no longer equivalent to monotonicity in strategies. (An action is a strategy for a single "type" of a player while a strategy is an indication of an action for every possible type of the player.) Therefore* *they require an additional axiom, monotonicity in information (strictly reducing a player's information cannot increase the player's value.)*

13 Appendix: The axioms for the value are tight

In this section we show that the axioms for the value are tight, i.e., if any one of them is dropped then the uniqueness theorem is no longer valid. Furthermore, the axioms are tight even if balanced threats and symmetry are replaced by their more restrictive versions (strong balanced threats and full symmetry, respectively).

Let, for all $i \in N$,

$$\gamma_i G = \frac{1}{n}(\delta G)(N). \tag{6}$$

It is easy to verify that

Claim 1. *The mapping $\gamma\colon \mathbb{G}(N) \to \mathbb{R}^n$ defined by (6) satisfies all the axioms except for the null-player axiom.*

Let, for all $i \in N$,

$$\gamma_i G = 0. \tag{7}$$

It is easy to verify that

Claim 2. *The mapping $\gamma\colon \mathbb{G}(N) \to \mathbb{R}^n$ defined by (7) satisfies all the axioms except for efficiency.*

For each integer $1 \leq k \leq n$, let π_k be the order $k, k+1, \ldots, n, 1, \ldots, k-1$, and let, for all $i \in N$,

$$\gamma_i G = \frac{1}{2n} \sum_{k=1}^{n} (\delta G(\mathcal{P}_i^{\pi_k} \cup i) - \delta G(\mathcal{P}_i^{\pi_k})), \tag{8}$$

where $P_i^{\pi_k}$ consists of all players j that precede i in the order π_k.

Claim 3. *The mapping* $\gamma\colon \mathbb{G}(N) \to \mathbb{R}^n$ *defined in (8) satisfies all the axioms except for symmetry.*

Proof. It is easy to verify that the axioms of null player, balanced threats, and additivity are satisfied. As for efficiency, it is sufficient to verify it for G such that δG is a unanimity game in $\mathbb{D}(N)$.

Let then δG be the unanimity game on T, i.e., $\delta G(S) = 1$ if $S \supseteq T$, -1 if $S \subseteq N \backslash T$, and zero otherwise.

For $i \in T$, $\delta G(\mathcal{P}_i^{\pi_k} \cup i) = 1$ if $\mathcal{P}_i^{\pi_k} \cup i \supseteq T$, i.e., if in the order π_k, i is the last among the members of T, and zero otherwise. Thus

$$\sum_{i \in T} \frac{1}{n} \sum_{k=1}^{n} \delta G(\mathcal{P}_i^{\pi_k} \cup i) = \frac{1}{n} \sum_{k=1}^{n} \sum_{i \in T} \delta G(\mathcal{P}_i^{\pi_k} \cup i) = \frac{1}{n} \sum_{k=1}^{n} 1 = 1,$$

where the third equality follows from the fact that in each order π_k exactly one $i \in T$ is last among the members of T.

Similarly, for $i \in T$, $\delta G(\mathcal{P}_i^{\pi_k}) = -1$ if $\mathcal{P}_i^{\pi_k} \subseteq N \backslash T$, i.e., if in the order π_k, i is the first among the members of T, and zero otherwise. Since in each order π_k exactly one $i \in T$ is first among the members of T, we have

$$\sum_{i \in T} \frac{1}{n} \sum_{k=1}^{n} \delta G(\mathcal{P}_i^{\pi_k}) = \frac{1}{n} \sum_{k=1}^{n} \sum_{i \in T} \delta G(\mathcal{P}_i^{\pi_k}) = \frac{1}{n} \sum_{k=1}^{n} (-1) = -1.$$

By (8), $\sum_{i \in T} \gamma_i = \frac{1}{2}(1+1) = 1$.

For $i \notin T$, $\mathcal{P}_i^{\pi_k} \cup i \subseteq T$ if and only if $\mathcal{P}_i^{\pi_k} \subseteq T$, and $\mathcal{P}_i^{\pi_k} \cup i \subseteq N \backslash T$ if and only if $\mathcal{P}_i^{\pi_k} \subseteq N \backslash T$. By (8) then, $\gamma_i G = 0$.

Thus $\sum_{i=1}^{n} \gamma_i = 1 = \delta G(N)$, completing the proof of efficiency.

To see that γ of equation (8) does not satisfy the symmetry axiom, consider the unanimity game on $\{1,2,5\}$ in the game with player set $\{1,\ldots,5\}$.

Player 1 is first in T for the order π_1 and last in T for the order π_2. Thus $\gamma_1 = \frac{1}{10}(1-(-1)) = \frac{1}{5}$.

Player 2 is first in T for the order π_2 and last in T for the orders π_3, π_4 and π_5. Thus $\gamma_2 = \frac{1}{10}(1-(-3)) = \frac{2}{5}$.

But 1 and 2 are substitutes. □

Next, observe that the Shapley value for strategic games, as defined in Section 11, satisfies all the axioms except for the strong balanced-threats axiom. (It does not even satisfy the standard balanced-threats axiom.)

Finally, consider the following map. All dummy players in G receive the same as in the value (equation 2), and the others share equally the remainder vs. $G(N)$. It is easy to verify that this solution satisfies all the axioms except for additivity. (It does not even satisfy consensus-shift invariance.)

We conclude this section by commenting on the axioms required to imply that the value, γG, is a function of δG. Now the axiom of balanced threats says that if $(\delta G)(S) = 0$ for any subset S then $\gamma G = 0$. It would seem then that this axiom alone would suffice. However, that is not the case.

Let δG and vG be as defined in (1) and (5), respectively, and fix $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t. $f(0, y) = f(x, 0) = 0$, and $f(x, x) = x \ \ \forall x, y$.

Define $\gamma(G)$ as the Shapley value of the coalitional game u with $u(S) := f(d(G)(S), v(G)(S))$.

Claim 4. *The mapping $\gamma \colon \mathbb{G}(N) \to \mathbb{R}^n$ defined above satisfies the axioms of balanced threats, symmetry, efficiency, and null player, but it is not a function of δG.*

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