COOPERATIVE STRATEGIC GAMES - EXPANDED VERSION

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Abstract. The *value* is a solution concept for n-person strategic games, developed by Nash, Shapley, and Harsanyi. The value provides an a priori evaluation of the economic worth of the position of each player, reflecting the players' strategic possibilities, including their ability to make threats against one another. Applications of the value in economics have been rare, at least in part because the existing definition (for games with more than two players) consists of an ad hoc scheme that does not easily lend itself to computation. This paper makes three contributions: We provide an axiomatic foundation for the value; we exhibit a simple formula for its computation; and we extend the value – its definition, axiomatic characterization, and computational formula – to Bayesian games. We then apply the value in simple models of corruption, oligopolistic competition, and information sharing.
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1. Introduction

1.1. **The value solution.** A *strategic game* is a model for a multi-person competitive interaction. Each player chooses a strategy, and the combined choices of all the players determine a payoff to each of them. A problem of obvious interest, and with a long history in game theory, is this: How to evaluate, in advance of playing a game, the economic worth of a player's position?[1] A "value" is a general solution, i.e., a method for evaluating the worth of any player in a given strategic game.

The value ought to reflect both the cooperative and the competitive aspects of the game. One may think of it as the expected payoff in a cooperative process that takes into account all the players' strategic possibilities, including their capacity to make threats against one another.

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[1] Alternatively: What would be the outcome if it were devised by an arbitrator?

We make the simplifying assumption that *utility is transferable*, i.e., that the players' payoffs are measured in units of a commodity that is freely exchangeable, like money.[2] Therefore it is reasonable to expect that the players will coordinate their strategic choices to maximize the sum of their payoffs, and that this maximal sum will be allocated in accordance with the threat powers of the players.

A value solution provides an a priori assessment of the cooperative allocation. Thus it is a powerful tool for studying a variety of economic phenomena where side payments – utility transfers between the players – are made in response to explicit or implicit threats.

Shapley [16] provided the original definition of a value for strategic games and Harsanyi [6] suggested a modification. We believe that Harsanyi's definition is preferable. Its essential advantage – highlighted in the examples below – is that it takes into account the potential damage of a threat not only to the threatened party but also to the party making the threat. Harsanyi calls his solution the modified Shapley value; others call it the Harsanyi–Shapley value; we call it simply *the value*.[3]

This paper makes three contributions: We provide an axiomatic foundation for the value; we exhibit a simple formula for its computation; and we extend the value – its definition, axiomatic characterization, and computational formula – to Bayesian games (Theorems 1 and 2.)

The axiomatic foundation delineates what assumptions must be made in order to justify use of the value solution. The formula makes it possible to compute the value much more easily than by following Harsanyi's original procedure, which is rather complex (see Appendix B). And the extension to Bayesian games opens the door to applications of the value in a wide class of games of interest in information economics, e.g., adverse selection, moral hazard, screening, and signaling.[4]

We postpone the description of the axioms and the formula in order to first present a few applications of the value.

[2]If one wishes to have a concrete model of a game with transferable utilities, then one may think of a single prize, desirable by all players, and a game where each player's payoff is the probability of receiving the prize. The value of a player is then the a-priori probability that a cooperative process (that may involve randomization) will allocate the prize to this player.

[3]The key idea underlying the Harsanyi modification is due to Nash [13]. An alternative name, then, would have been the Nash-Harsanyi-Shapley solution.

[4]It is somewhat surprising that Shapley [16], Harsanyi [6] and Myerson [12] focused only on the complete information case. However, Kalai and Kalai [7] defined their "coco value" for Bayesian games. In two-person Bayesian games, our definition of the value coincides with the coco value.

1.2. **The value in simple models of corruption.** In order to study the incentive of a public official to take bribes, it is instructive to determine the economic value inherent in the official's authority to make decisions in matters of financial importance to private individuals or companies.

In section 5.1.1, we consider a game between a public official with authority to grant building permits and a number of competing builders, each seeking a permit to build on the same site. In one version of the game, bribery is legal. The value solution then indicates the expected receipts from bribes, i.e., the economic worth of the authority to grant permits. In another version, bribery is illegal: if the official does not grant a permit that should have been granted or grants a permit that should not have been granted then he might be found out and penalized. The value solution will exhibit a reduction in the economic worth of the official. This reduction is a quantitative measure of the effectiveness of the disciplinary regime in reducing the temptation to engage in bribery. Example 5 is a simple numerical demonstration.

Section 5.1.2 presents additional examples of this nature. In these examples the economic output of a large number of individuals is predicated on the approval of a regulator. Computation of the value indicates that the regulator's position is worth 25% of the total output; and if approval is required from two regulators, then their combined positions are worth a full 42% of the total output. However, if approval is required from only one of the two regulators, then their combined positions are worth just 8.5% of the output. In another example, it is assumed that making good on an implicit threat to deny approval (for no valid reason) exposes the regulator to potential punishment. If the expected cost of the punishment is a fraction c of the lost output, then the value of the regulator's position decreases[5] by a factor of $(1-c)^2$.

1.3. **The value in a Cournot oligopoly.** [6]

Consider a Cournot oligopoly with inverse demand function $1 - \sum_1^n q_i$, where q_i is the quantity of firm i, and with constant unit costs $c_1 < c_2 < \ldots < c_n$, and assume that the firms intend to engage in a collusive arrangement. What is the profit that each firm should expect to receive? In other words, denoting the monopoly profit of

[5]The original value solution of Shapley gives different results. Most notably, the Shapley solution remains the same, regardless of the cost of a potential punishment.

[6]We are grateful to an anonymous referee for suggesting that we characterize the value solution in a Cournot oligopoly.

firm i by $M_i := \max_q(10 - q - c_i)q$, how is the maximal available profit, M_1, to be divided among the firms?

First consider a duopoly, i.e., $n = 2$. The Shapley value is $(\frac{M_1}{2}, \frac{M_1}{2})$. The rationale is that each firm can only guarantee zero on its own, since its rival can threaten to flood the market; since the amounts that the firms can guarantee are equal, so must be their shares of the total profit.

This solution does not seem to make sense – should the firm with the lower cost not receive a larger share of the profit? The reason for obtaining a "non-sensical" solution is precisely the difficulty mentioned above, namely that the Shapley value fails to take into account the damage that a threat inflicts on the party making the threat. And, obviously, the damage to a firm flooding the market is greater the greater is the firm's unit cost.

In contrast to the Shapley value, the value solution does take account of the unit costs. The solution is $(M_1 - \frac{M_2}{2},\ \frac{M_2}{2})$. Since $M_1 > M_2$, the value of firm 1 is, indeed, greater than the value of firm 2.

In the case of three firms, the value is $(M_1 - \frac{1}{2}M_2 - \frac{1}{6}M_3,\ \frac{1}{2}M_2 - \frac{1}{6}M_3,\ \frac{1}{3}M_3)$, which equals $\frac{1}{3}(M_3, M_3, M_3) + \frac{1}{2}(M_2 - M_3, M_2 - M_3, 0) + (M_1 - M_2, 0, 0)$. More generally, in the case of n firms the value may be described as follows: First M_n, the monopoly profit of the least efficient firm, is divided equally among all the firms. Next, $M_{n-1} - M_n$ is shared equally among firms $1, \ldots, n-1$. And so on, until finally $M_1 - M_2$ is received only by firm 1.

Section 5.2 provides the relevant computations.

1.4. **The value in a Bayesian game.** A *Bayesian game* (or a "strategic game with incomplete information") is a general model of a competitive interaction, where the payoff functions are not known with certainty, but rather each player has some private information (a "signal") about which of several possible functions is the true payoff function.

In a Bayesian game, a player can impact the side payments that she makes or receives not only through her strategic choices, but also by sharing or withholding information. Thus the value solution, by providing an a-priori assessment of the side payments, quantifies the economic worth of information in a competitive environment. Below is a numerical example.

First, consider a two-person strategic game with complete information: Firm 2 is developing a new product and firm 1 is developing an add-on product. Each firm makes a private choice about which one of two alternative technologies it will use; and the market for the add-on product may attain one of two unknown states that are equally likely.

Firm 2's profits will be the same, irrespective of the technology choices or the state of the market. (For simplicity, assume this profit is zero.) Firms 1's profit will be 4 if both firms choose the same technology and the market attains the state that is favorable to the chosen technology, and zero otherwise.

The maximal sum of (expected) payoffs in this game is 2. (The firms optimize for the same state; thus firm 1 gets the payoff 4 with probability 50%.) What is the side payment that player 2 ought to receive for its cooperation? The value solution is $(1.5, 0.5)$, i.e., the side payment is 0.5. This makes intuitive sense, as player 2 can threaten to deprive firm 1 of 50% of its payoff (of 2) by randomizing with probabilities $(.5, .5)$; thus the magnitude of the threat is 1, and player 1 concedes one half this amount.

Next, consider an additional firm, 3, that specializes in market research. Firm 3 has no strategic choices but it knows which state will occur. (Note that the introduction of a player with differential information has turned the example into a Bayesian game.[7]) Clearly, the maximal sum of payoffs is 4. But what are the side payments? The value solution is (2, 1, 1). It is interesting to note that the value of each one of the firms reflects a different consideration. Firm 1's value derives from its potential payoff of 4; firm 2's value derives from its threat to reduce the payoff to firm 1; and firm 3's value derives from its knowledge of the true state.

Section 5.2 provides the relevant computations.

1.5. **The axioms for the value.** We consider the following axioms: efficiency, symmetry, additivity, null player, balanced threats, and individual rationality.

[7]Formally, the two-player game is also a game of incomplete information. However, as noted earlier, if the players' information is symmetric, then we might as well view the game as a game of complete information.

Efficiency says that the sum of the values of all the players is the maximum available payoff.[8]

Symmetry says that two players whose payoffs are identical everywhere, and whose strategies can be switched without impacting any payoff, have the same value.

Additivity says that if the payoff to each player is the sum of her payoffs in two games that are unrelated to each other then the player's value is the sum of her values in those two games.[9]

The null-player axiom says that a player whose actions do not affect any player's payoff, and whose own payoff is identically zero, has value zero.

Individual rationality says that a player's value is at least her security level – the maximal payoff that the player can guarantee unilaterally, irrespective of the strategies of the other players.

Efficiency, symmetry, additivity, and null-player are strategic-game analogs of the classic Shapley axioms for the value of coalitional games. The individual rationality axiom is standard for strategic games. But the axiom of balanced threats is new. One way to motivate the axiom is to consider a public official, who may attempt to exploit his authority to shift rewards from one group of players to another. The axiom of balanced threats essentially stipulates that if the public official cannot shift rewards then his value is zero. In fact, the axiom requires less – it stipulates that if no player can shift rewards then every player's value is zero. We now turn to the formal definition.

1.6. **The axiom of balanced threats.** The minmax theorem of von Neumann says that for any two-person zero-sum game there exists a number, v, called the *minmax value* of the game, such that player 1 can guarantee to receive a payoff of at least v and player 2 can guarantee[10] to receive a payoff of at least $-v$. Thus, the evaluation of player 1's position must be greater than or equal to v and the evaluation of player

[8] Efficiency seems to be a reasonable axiom for the evaluation of a cooperative outcome. But one can imagine models where this axiom is rejected. Important examples are Ray and Vohra [14] and Maskin [10].

[9] Note that the rationale for this axiom does not depend on a cooperative point of view. For example, the mapping from strategic games to their Nash equilibrium payoffs, viewed as a set function, satisfies additivity.

[10] That is, player 1 (respectively, 2) has a strategy that yields a payoff of at least v (resp., $-v$), regardless of the strategy chosen by her opponent.

2's position must be greater than or equal to $-v$. Since the sum of the evaluations cannot exceed zero, they must be v and $-v$, respectively.

Similarly, in a two-person constant sum game, where the sum of the payoffs of the two players is always c, there is a number v such that player 1 can guarantee to receive a payoff of at least v and player 2 can guarantee to receive a payoff of at least $c - v$; thus the evaluation of the players' positions must be $(v, c - v)$.[11]

In a general-sum two-person game it is less clear how to evaluate the players' positions. But in a seminal paper, Nash [13] proposed a scheme for doing just that. While Nash's scheme applies more generally, for our purposes it is sufficient to consider the special case of games with transferable utility:

Nash envisions a process of "bargaining with variable threats:" In an initial competitive stage, each player declares a "threat strategy," to be used if negotiations break down. The players' payoffs resulting from the deployment of these strategies constitute a "disagreement point." In a subsequent cooperative stage, the players coordinate their strategies to maximize the sum of their payoffs, and share the gains relative to the disagreement point equally.

Nash observes that what matters in the disagreement point is only the difference between the players' payoffs: If the disagreement point is (π_1, π_2) then after the cooperative stage player 1's payoff is $\pi_1 + \frac{1}{2}(s - (\pi_1 + \pi_2)) = \frac{1}{2}s + \frac{1}{2}(\pi_1 - \pi_2)$, and similarly player 2's payoff is $\frac{1}{2}s - \frac{1}{2}(\pi_1 - \pi_2)$, where s denotes the maximal sum of the players' payoffs in any entry of the payoff matrix. Thus, player 1 strives to maximize $\pi_1 - \pi_2$, while player 2 strives to minimize the same expression.

Nash then constructs an auxiliary (zero-sum) game by taking the difference between player 1's and player 2's payoffs. If δ denotes the minmax value of the auxiliary game, then players 1 and 2 can guarantee, at the end of the cooperative stage,

$$\frac{1}{2}s + \frac{1}{2}\delta \text{ and } \frac{1}{2}s - \frac{1}{2}\delta, \tag{1}$$

respectively. The above pair of numbers is the Nash solution.[12]

[11] Note that in a two-person constant-sum game the two axioms of efficiency and individual rationality define a uniqu value solution.

[12] The simple definition, by means of formula (1), for the Nash bargaining solution in games with transferable utilities is due to Shapley [17]. Kalai and Kalai [7] independently discovered the formula and used it to define their competitive-cooperative solution concept, the *coco value*; furthermore, Kalai and Kalai provide an axiomatic characterization of the Nash solution in games with transferable utilities.

For a numerical example, consider the following game.

Example 1.

$$\begin{bmatrix} 1,5 & 2,4 \\ 0,0 & 0,0 \end{bmatrix}.$$

The game of differences is

$$\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix},$$

and its minmax value is zero. (Player 1 can guarantee zero by playing the bottom row, and she can obviously not guarantee any higher payoff.) Thus $\delta = 0$ and $s = 6$; hence, by formula (1), the Nash solution is $(3, 3)$.

The Nash solution provides a compelling method for evaluating the cooperative outcome in a two-person game. The challenge, then, is to extend the solution to n-person games. Harsanyi [6] and Myerson [12] both approached this challenge by generalizing the scheme of "bargaining with variable threats." These n-person schemes are quite complex. (See Appendix B.) By contrast, we focus on a basic property of the Nash solution and adopt it as an axiom that can be generalized to n-person games.

The basic property is this: *If* $\delta \geq 0$ *then the value of player* 1 *is greater than or equal to the value of player* 2. (This follows from equation (1).)

In order to generalize this property to n-player games, we proceed as follows. Consider, for any subset S of the set of players, N, an auxiliary two-player zero-sum game between S and its complement, $N\backslash S$, where the players in each of these subsets coordinate their strategies (and pool their information) to act as a single player, and where the payoff to player S is the difference between the sum of the (original game) payoffs to the players in S and the sum of the payoffs to the players in $N \setminus S$; and define $\delta(S)$ as the minmax value of this game.

Now, consider two players, i and j, and assume that $\delta(S) \geq 0$ for any subset S that includes i but not j. Then – by the same logic as in a two-player game, – in the cooperative outcome player i's payoff should be greater than or equal to player 2's payoff. Therefore, if $\delta(S) = 0$ for any subset S that includes i but not j, and consequently – by the minmax theorem – $\delta(S) = 0$ for any subset S that includes j but not i, the payoffs to i and j should be equal. This, essentially, is the assumption of "balanced threats."

In fact, to prove our uniqueness result, we require less. We assume that only if the above holds for any pair i and j, i.e., if $\delta(S) = 0$ for any proper[13] subset of N, then the payoffs to all players should be equal. Furthermore, we weaken the axiom even more[14] by requiring the condition only in games of pure transfers, where $\delta(N)$ – the maximum sum of the players' payoffs[15] – is equal to zero. Thus, the axiom of balanced threats is defined as follows: *if $\delta(S) = 0$ for all $S \subseteq N$ then the value of each player is zero.*

Further discussion of the axiom of balanced threats is provided in Appendix D.1.

1.7. **The uniquess result and the formula for the value.** Theorems 1 and 2 state our main results – that the axioms of efficiency, balanced threats, symmetry, null player, and additivity imply a unique value solution for strategic games with complete information as well as for Bayesian games, and that the value satisfies individual rationality; furthermore, the theorems provide a formula for computing the value.

The formula says that the value of a player in an n-person strategic game or Bayesian game is an average of the threat powers, $\delta(S)$, of the subsets of which the player is a member. Specifically, *if $\delta_{i,k}$ denotes the average of $\delta(S)$ over all k-player subsets that include i, then the value of player i is the average of $\delta_{i,k}$ over $k = 1, 2, \ldots, n$.*

Remark 1. In a two-player game with complete information the value coincides with the Nash variable-threats solution. Indeed, the formula says that the value of player 1 is $\frac{1}{2}\delta(1) + \frac{1}{2}\delta(1, 2)$, which is the same as equation (1). In a two-player Bayesian game the value coincides with the Kalai and Kalai's coco value.

1.8. **The impact of inferior strategies.** We end this introduction by emphasizing the fundamental distinction between the cooperative-competitive approach underlying the idea of a value and the purely competitive approach underlying the concept of equilibrium. Consider the two-player game of Example 1.

[13]A proper subset of N is a subset that is neither $\emptyset$ nor N.

[14]This means that the uniqueness theorem is stronger.

[15]Note that $\delta(N)$ is the minmax value of a redundant two-person game between the all-player set N and the empty set; it is therefore natural to think of $\delta(N)$ as the maximum sum of the players' payoffs.

$$\begin{bmatrix} 1,5 & 2,4 \\ 0,0 & 0,0 \end{bmatrix}.$$

In purely-competitive analysis, the strategy "down" is viewed as an *incredible threat*; thus the availability of this strategy does not affect the equilibrium outcome, $(1,5)$. But in an explicit or implicit cooperative process, player 1's threat to play "down" is a source of power. Indeed, the Nash solution – $(3,3)$ – exhibits a side payment from player 2 to player 1, as does Shapley's original notion of value (see Section 2 for details.) Other authors, e.g., Green [5], have also emphasized the impact of inferior strategies on the cooperative-competitive outcome.

1.9. **Organization of the paper.** Section 2 discusses the historical development of the ideas. In Sections 3 and 4 we define the axioms and state the main results – the axiomatic characterization and the formula for computing the value. In Section 5 we apply the formula in a number of examples. In Section 6 we present a characterization of the von Neumann–Morgenstern–Shapley value that parallels the characterization of the value, and in Section 7 we provide the proofs of our main results. In Appendix A we discuss the "coco value" of Kalai and Kalai, and in Appendix B we describe the n-player generalizations of Nash's "bargaining with variable threats" by Harsanyi and by Myerson. Appendixes C and D provide additional properties of the value and alternative versions of the axioms. And Appendix E shows that all the axioms are tight; i.e., if any of them is dropped then the uniqueness theorem is no longer valid.

2. History of the Concepts

This section reviews the historical development of the ideas at the foundation of the notion of value. Skipping this section will not affect the reader's understanding of the rest of the paper.

In the classic work [18] of von Neumann and Morgenstern (vNM), the starting point for the cooperative analysis of strategic games is to reduce every such game to a characteristic function, nowadays called a *coalitional game*, which assigns to every subset of players ("coalition") S a single number, $v(S)$, defined as the total payoff that the members of S can guarantee, i.e, the maxmin of the sum of the payoffs to the members of S, where the max is over all the correlated strategies of S and the min is over the correlated strategies of the complement of S. Having reduced strategic games to coalitional games, vNM focused on developing their solution concept for coalitional games, the "stable set."

In contrast to vNM's set-valued solution, Shapley highlighted the need to define a single-valued function that assigns to each strategic game a vector of payoffs, representing the value of each role in the game. Shapley accepted the vNM approach of reducing strategic games to coalitional games; thus he addressed the problem of defining a value function for coalitional games. In a seminal paper [16] he formulated properties ("axioms") that would be desirable in such a function and proved that – remarkably – a mere four of them uniquely imply one particular function, the "Shapley value."

It would seem, then, that Shapley's goal of defining a value function for strategic games had been accomplished: given a strategic game, transform it to its vNM coalitional form, then apply the Shapley value. But there were doubts. The doubts, centering on the adequacy of the vNM coalitional game, were raised by vNM[16] and Shapley themselves, as well as by Luce and Raiffa [9], Harsanyi [6], and Myerson [12]. As Shapley [16] wrote: "The difficulty, intuitively, is that the characteristic function does not distinguish between threats that damage just the threatened party and threats that damage both parties."

The difficulty with the vNM coalitional game – the reason that it does not properly reflect the threat powers of the players – arises because a coalition is allowed to deploy two different strategies, one for maximizing its own payoff and the other for minimizing the complementary coalition's payoff. Consider Example 1. The vNM coalitional game is $v(1) = 1$, $v(2) = 0$ and $v(1, 2) = 6$. (The players' security levels – the maximum payoff that each one of them can guarantee regardless of the strategies of the opponent – are 1 and 0, respectively.) Note that player 1 plays Top in order to maximize her own payoff, but plays Bottom in order to minimize player 2's payoff.

Harsanyi [6] proposed a modification in the definition of the value that is motivated by Nash's "bargaining with variable threats." It is this modification that we call "the value." We describe Harsanyi's method in Appendix B. Here we provide a simpler description, as follows.

[16]von Neumann and Morgenstern [18] wrote: "In a general [-sum] game the advantage of one group of players need not be synonymous with the disadvantage of the others. In such a game moves – or rather changes in strategy – may exist which are advantageous to both groups. . . . Does our approach not disregard this aspect?"

Instead of considering two separate zero-sum games between a coalition S and its complement, one that focuses on the payoff to S and the other on the payoff to $N\backslash S$, we consider a single game that focuses on the *difference* between these payoffs; and we assign to each coalition S a single number, $\delta(S)$, defined as the maximal difference between the total payoffs to S and to $N\backslash S$ that the members of S can guarantee. By the minmax theorem, $\delta(S) = -\delta(N \setminus S)$.

Now δ is not a coalitional game. It may fail to satisfy the single condition required of a set function to qualify as a coalitional game, namely $\delta(\emptyset) = 0$. This condition is essential for the formula of the Shapley value, which assigns to each player i an average of her marginal contributions, including the marginal contribution $v(i) - v(\emptyset) = v(i)$. However, we show in [8] that an appropriate modification of the definition of the Shapley value applies to set functions such as δ, which satisfy the condition that $\delta(S) = -\delta(N\backslash S)$ for all $S \subseteq N$, and which we call "games of threats." The value of the strategic game is then obtained by taking the Shapley value of δ. We refer to this modification by Harsanyi of Shapley's original notion as *the value* of a strategic game.

It is easy to verify that the value coincides with the Nash variable-threats solution in two-player games and that the value coincides with the vNM–Shapley value in constant-sum games and in pure-exchange economies. (Proposition 2.)

We wish to emphasize that neither Shapley's original definition of a value for strategic games nor Harsanyi's modification rest on an axiomatic foundation, as the first step – that of reducing the strategic game to a coalitional form – is arbitrary.

We end this section with an example that demonstrates the different responses of the two notions of value to an increase in the cost of carrying out a threat. Consider once again the two-player game of Example 1.

$$\begin{bmatrix} 1,5 & 2,4 \\ 0,0 & 0,0 \end{bmatrix}.$$

The vNM coalitional game is $v(1) = 1$, $v(2) = 0$ and $v(1,2) = 6$.

Thus the vNM–Shapley value is $(3.5, 2.5)$. (In a two-person game, the Shapley value of player i is $\frac{1}{2}v(i) + \frac{1}{2}v(1,2)$.) As we have seen, the value (i.e., the Nash solution) is $(3,3)$.

Now consider the following variant.

Example 2.

$$\begin{bmatrix} 1,5 & 2,4 \\ -1,0 & 0,0 \end{bmatrix}.$$

The security levels of the players are unchanged. Thus the vNM coalitional game is unchanged and the vNM–Shapley value is still $(3.5, 2.5)$. But the game of differences is now

$$\begin{bmatrix} -4 & -2 \\ -1 & 0 \end{bmatrix}$$

and its minmax value is -1. Thus $\delta = -1$ and $s = 6$; hence, by formula (1), the value is $(2.5, 3.5)$: The increased cost of the threat has had an impact on the value solution.

3. The Axioms

A strategic game is a triple $G = (N, A, g)$, where

- $N = \{1, \dots, n\}$ is a finite set of players,
- A^i is the finite[17] set of player i's pure strategies, and $A = \prod_{1=1}^{n} A^i$,
- $g^i \colon A \to \mathbb{R}$ is player i's payoff function, and $g = (g^i)_{i \in N}$.

We use the same notation, g, to denote the linear extension

- $g^i \colon \Delta(A) \to \mathbb{R}$,

where for any set K, $\Delta(K)$ denotes the probability distributions on K, and we denote

- $A^S = \prod_{i \in S} A^i$, and
- $X^S = \Delta(A^S)$ (correlated strategies of the players in S).

We define the *direct sum* of strategic games as follows.[18]

Definition 1. *Let $G_1 = (N, A_1, g_1)$ and $G_2 = (N, A_2, g_2)$ be two strategic games. Then*

$G := G_1 \oplus G_2$ is the game $G = (N, A, g)$, where $A = A_1 \times A_2$ and $g(a) = g_1(a_1) + g_2(a_2)$.

[17]The assumption that the sets of players and strategies are finite is made for convenience. The results remain valid when the sets are infinite, provided the minmax value exists in the two-person zero-sum games defined in the sequel.

[18]von Neumann and Morgenstern [18], Section 27.6.2, refer to this operation as the *superposition* of games.

Remark 2. The game $G_1 \oplus G_2$ is a model for a competitive interaction where the same set of players simultaneously play two games that are independent, i.e., where the moves in one game do not influence the other game.

Remark 3. It is easy to verify that the operation $\oplus$ is, informally, commutative and associative.[19] However, there is no natural notion of inverse. (In general, $G \oplus (-G) \neq 0$.)

Denote by $\mathbb{G}(N)$ the set of all n-player strategic games. Let $\gamma : \mathbb{G}(N) \to \mathbb{R}^n$. This may be viewed as a map that associates with any strategic game an allocation of payoffs to the players. We consider a list of axioms for γ. To that end we first introduce a few definitions.

Let $G \in \mathbb{G}(N)$. We define the *threat power* of coalition S as follows:[20]

$$(\delta G)(S) := \max_{x \in X^S} \min_{y \in X^{N \setminus S}} \left(\sum_{i \in S} g^i(x, y) - \sum_{i \notin S} g^i(x, y) \right). \tag{2}$$

We say that i and j are *interchangeable* in G if $A^i = A^j$ and $g^i = g^j$; and for any $a, b \in A^N$, if $a^i = b^j, a^j = b^i$, and $a^k = b^k$ for all $k \neq i, j$, then $g(a) = g(b)$.

We say that i is a *null player* in G if $g^i(a) = 0$ for all a; and if $a^k = b^k$ for all $k \neq i$, then $g(a) = g(b)$.

We consider the following axioms. For all strategic games G,

- **Efficiency** $\sum_{i \in N} \gamma_i G = \max_{a \in A^N} (\sum_{i \in N} g^i(a))$.
- **Balanced threats** If $(\delta G)(S) = 0$ for all $S \subseteq N$ then $\gamma_i = 0$ for all $i \in N$.
- **Symmetry** If i and j are interchangeable in G then $\gamma_i G = \gamma_j G$.
- **Null player** If i is a null player in G then $\gamma_i G = 0$.
- **Additivity** $\gamma(G_1 \oplus G_2) = \gamma G_1 + \gamma G_2$.
- **Individual rationality** $\gamma_i(G) \geq \max_{x \in X^i} \min_{y \in X^{N \setminus i}} g^i(x, y)$.

Remark 4. There are many additional desirable properties of the value that we do not assume but rather deduce from the axioms. These include dependence on the reduced form of the game (removing strategies that are convex combinations of other

[19] Formally, $G_1 \oplus G_2$ is not the same game as $G_2 \oplus G_1$, because $A_1 \times A_2 \neq A_2 \times A_1$.
[20] Expressions of the form max or min over the empty set should always be ignored.

strategies does not affect the value), homogeneity of degree one ($\gamma(\alpha G) = \alpha\gamma G$ for $\alpha > 0$), time-consistency ($\gamma(\frac{1}{2}G_1 \oplus \frac{1}{2}G_2) = \frac{1}{2}\gamma G_1 + \frac{1}{2}\gamma G_2$; i.e., it does not matter if the allocation is determined before or after the resolution of uncertainty about the game), monotonicity in actions (removing a pure strategy of a player does not increase the player's value),[21] independence of the set of players (addition of null players does not affect the value of the existing players), shift-invariance (adding a constant payoff to a player increases the player's value by that constant), a stronger form of symmetry (the names of the players do not matter), and continuity ($\gamma(G_n) \to \gamma G$ whenever $G_n = (N, A, g_n), G = (N, A, g)$, and $g_n \to g$).

Remark 5. We do not require, nor are we able to deduce, that $\gamma(\alpha G) = \alpha\gamma G$ for negative α. Such a requirement, which is natural in the context of coalitional games, would make no sense in the context of strategic games. The game $-G$ involves dramatically different strategic considerations than the game G, and so there is no reason to expect a simple relationship between the values of the two games.

4. The main results

4.1. **The value of strategic games.** Our main result for strategic games is as follows.

Theorem 1. *There is a unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of efficiency, balanced threats, symmetry, additivity, and null player. It may be described as follows.*

$$\gamma_i G = \frac{1}{n}\sum_{k=1}^{n} \delta_{i,k}, \tag{3}$$

where $\delta_{i,k}$ denotes the average of $(\delta G)(S)$ over all k-player coalitions S that include i. Furthermore, this map satisfies the axiom of individual rationality.

We shall refer to the above map as the *value* for strategic games.

[21]These four properties follow from formula (3) below and the corresponding properties of the minmax value of zero-sum games.

Example 3. This is a three-player game. Player 1 chooses the row, player 2 chooses the column, and player 3 has only a single strategy. The payoff matrix is[22]

$$\begin{bmatrix} 2,2,2 & 0,0,0 \\ 0,0,0 & 1,1,1 \end{bmatrix}.$$

Now, denoting $\delta G = \delta$,

$$\delta(1) = \text{maxmin} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = -\frac{2}{3},$$

$$\delta(1,3) = \text{maxmin} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{2}{3},$$

$\delta(1,2) = \max(2,0,1) = 2$, and $\delta(1,2,3) = \max(6,0,3) = 6$.

Thus $\gamma_1 = \frac{1}{3}(\delta(1) + \frac{\delta(1,2)+\delta(1,3)}{2} + \delta(1,2,3)) = \frac{1}{3} \times (-\frac{2}{3}) + \frac{1}{3} \times \frac{2+\frac{2}{3}}{2} + \frac{1}{3} \times 6 = 2\frac{2}{9}$, and similarly $\gamma_2 = 2\frac{2}{9}$; therefore $\gamma = (2\frac{2}{9},\ 2\frac{2}{9},\ 1\frac{5}{9})$. Players 1 and 2 each receive a side payment of $\frac{2}{9}$ from player 3.

Remark 6. There is only one n-player coalition, namely, N. Thus $\delta_{i,n} = (\delta G)(N)$, the maximum total payoff. The formula allocates to each player her equitable payoff, which is $\frac{1}{n}$th of this amount, adjusted according to the average threat powers of the proper subsets that include the player.

Remark 7. Formula (3) implies that the value of G depends only on the threats, $((\delta G)(S))_{S \subseteq N}$. We wish to emphasize that this is not an assumption but rather a conclusion. Indeed, a key step in proving the main result is the derivation of this conclusion from the axioms. (Proposition 7.)

Remark 8. At first blush it might appear that a value solution ought to satisfy the following consistency condition: If a player who is a strategic dummy (i.e., a player who has no strategic options) is dropped from the game, then the value of the remaining players remains the same. However, further reflection shows that this requirement is unwarranted: a player can exert influence on the outcome not only through her strategic choices, but also through her willingness to make side payments. (Recall

[22]Player 1's and player 2's payoffs are identical and their strategies can be switched without impacting any payoff. This is an example of interchangeable players.

that the value is an assessment of the cooperative outcome, where all players agree on the side payments.) Example 3 is a case in point: The value is $(2\frac{2}{9},\ 2\frac{2}{9},\ 1\frac{5}{9})$, but when player 3 (who is a strategic dummy) is dropped, the game becomes

$$\begin{bmatrix} 2,2 & 0,0 \\ 0,0 & 1,1 \end{bmatrix},$$

and the value is $(2, 2)$. One can interpret the side payments of $\frac{2}{9}$ that player 3 makes to each of players 1 and 2 as incentives to not deploy the threat strategies, "bottom" and "right."[23]

4.2. **The value of Bayesian games.** The uniqueness theorem and formula (3) are also valid for Bayesian games. In a *Bayesian game*, each player has a finite set, C^i, of possible actions; the players do not know the "true" payoff functions, $u^i\colon \prod_{i\in N} C^i \to \mathbb{R}$; however, each player receives a signal, y^i, which is correlated with $u = (u^i)_{i\in N}$; specifically, the players know the "prior" probability distribution, μ, over $U \times Y$, where U and Y are the finite sets of possible payoff functions and signals, respectively.

A pure strategy for player i is now a mapping, $a^i\colon Y^i \to C^i$, from signals to actions, and A^i is the set of pure strategies; the payoff function, $g^i\colon \prod_{i\in N} A^i \to \mathbb{R}$ is the expectation $g^i(a) := E_\mu u^i(a(y))$; and a correlated strategy for a subset S is a probability distribution over mappings from $\prod_{i\in S} Y^i$ to $\prod_{i\in S} C^i$. Note that in a correlated strategy the players in S not only coordinate their strategic choices, but they also pool their information.

Denote by $\mathbb{B}(N)$ the set of all n-player Bayesian games, and let $B \in \mathbb{B}(N)$. We generalize formula (2), defining the power of threat of a coalition S as follows:

$$(\delta_B G)(S) := \max_{x\in\hat{X}^S} \min_{y\in\hat{X}^{N\setminus S}} \left(\sum_{i\in S} g^i(x,y) - \sum_{i\notin S} g^i(x,y) \right), \tag{4}$$

where $\hat{X}^S$ denotes the set of correlated strategies of S.

We can now define the axiom of balanced threats for Bayesian games in analogy with the definition for strategic games. Similarly, we define the axioms of efficiency

[23]Note, however, that if the others' choices cannot impact the dummy player's payoff, then this consideration becomes moot. Indeed, the "small worlds axiom" (Appendix C.2) says that if a player has no strategic options *and* her payoffs are unaffected by the choices of the other players, then when the player is dropped from the game the value of the remaining players remains the same.

and of individual rationality in analogy with the definitions for strategic games, replacing $a \in A^N$ by $x \in \hat{X}^N$, and $x \in X^{N\setminus i}$ by $x \in \hat{X}^{N\setminus i}$, respectively. Finally, we define the symmetry and the null-player axioms in analogy with the definitions for strategic games, adding to the definition of exchangeable players the requirement that their signals be identical, and to the definition of a null player the requirement that the player receive no signals.

Theorem 2. *There is a unique map from $\mathbb{B}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of efficiency, balanced threats, symmetry, additivity, and null player. It is described by formula (3), modified by replacing δG with $\delta_B G$. Furthermore, this map satisfies the axiom of individual rationality.*

Remark 9. The axiomatic characterization of the value in strategic games does not automatically follow from the characterization in Bayesian games, i.e., Theorem 1 is not a special case of Theorem 2: in general, it is not true that if a list of axioms uniquely determines a function on a certain domain then the specialization of the same axioms to a sub-domain will uniquely determine the function there.

4.3. **The random-order approach.** An alternative formula for computing the value is based on the random-order approach. It is analogous to Shapley's [16] random-order formula for the value of a cooperative game. In some applications it is more convenient[24] to use than formula (3).

Proposition 1. *The value of a strategic game*[25] *G may be described as follows:*

$$\gamma_i G = \frac{1}{n!} \sum_{\mathcal{R}} (\delta G)(S_i^{\mathcal{R}}), \tag{5}$$

where the summation is over the $n!$ possible orderings of the set N and where $S_i^{\mathcal{R}}$ denotes the subset consisting of i and those $j \in N$ that precede i in the ordering $\mathcal{R}$.

The equivalence of formulas (3) and (5) is easy to verify.[26]

[24] See Example 10, where we compute the value in a Counrot oligopoly.
[25] The proposition is valid for Bayesian games as well, provided δG is replaced by $\delta_B G$.
[26] This equivalence is established in Proposition 3 of [8].

4.4. **Proofs of the theorems.** The proofs of Theorems 1 and 2 are given in section 7.4. These proofs require the notion of games of threats [8]. We provide the relevant definitions and results in Section 7.1.

5. Applications of the value solution

5.1. **The economic worth of a public official.** The examples below are highly simplified models of a public official who has the authority to make decisions in matters of financial importance to private individuals or companies. In each of the examples, we apply formula (3) to determine the economic value of the official's position. The value then provides a measure of the official's potential gain from side payments, i.e., bribes. This measure may be useful in designing systems of incentives and penalties intended to deter bribery.

5.1.1. *Authority to select a winner.*

Example 4. Player 1 decides which one of $n-1$ other players will receive a contract worth 1. Clearly, $\delta(1) = 0$ while $\delta(1 \cup S) = 1$ for any non-empty subset, S, of players other than 1. (Player 1 awards the contract to a member of S, thereby guaranteeing payoff 1 to $(1 \cup S)$. Thus $\delta_{1,1} = 0$ and $\delta_{1,k} = 1$ for $k = 2, \ldots, n$, and therefore $\gamma_1 = \frac{1}{n} \sum_{k=1}^{n} \delta_{i,k} = \frac{n-1}{n}$. We see, then, that the authority to choose the winner of a contract is worth $\frac{n-1}{n}$ of the value of the contract.

Example 5. There are n players – a public official (player 1) and $n-1$ builders (players $2, \ldots, n$) competing for the development of a site. The official can grant approval to one of the builders or not grant any approval; the builders have no strategic choices. The payoff to the official is 0 regardless of his choice; the payoff to developer i is w_i if approved and 0 otherwise.

For simplicity, assume that $w_2 = W$ and $w_3 = w_4 = \ldots = w_n = w < W$. The formula for the value then shows that the economic worth of the official is $.5W + .5(\frac{n-2}{n})w$, which converges to $.5(W + w)$ when n is large.[27] Thus, when there is only

[27] One way to see this is as follows: For any non-trivial partition S_1, S_2 of $\{2, \ldots, n\}$ we have: $\delta(1 \cup S_1) + \delta(1 \cup S_2) = W + w$, since one of these sets includes 1 and 2 (and therefore its threat power is W), while the other does not include 2 but includes 1 and one of $3, \ldots, n$ (and therefore its threat power is w). Taking the average of this equation over all sets S_1 with $k-1$ elements (and sets S_2 with $n-k$ elements), we have $\delta_{1,k} + \delta_{1,n-k+1} = W + w$. This equation is valid for $k = 2, \ldots, n-1$, but we also have $\delta_{1,1} + \delta_{1,n} = W$ (since $\delta_{1,1} = \delta(1) = 0$ and $\delta_{1,n} = \delta(1, \ldots, n) = W$).

one builder then the expected side payments to the official constitute half the profit from developing the site. But if there is competition from many other less profitable builders, then the side payments are greater, by one half the profit of one of these builders.

When bribery is illegal, we represent the risk to the official of being penalized as follows: The official's payoff is reduced by an amount c, where $c > 0$, whenever he fails to approve builder 2, i.e., when he approves another builder or when he does not approve any builder; the idea being that the development with the highest potential profit is the one that ought to be approved. In this version of the game, the value solution shows that the economic worth of the official is $.5W + .5(\frac{n-2}{n})w - .5c$, which converges to $.5(W + w - c)$ when n is large.[28] Thus, the size of the penalty required to make bribery unprofitable is W when there is only one builder but it is $W + w$ when there is competition from many other less profitable builders.

5.1.2. *Authority to approve.*

Example 6. There are n-players. Player 1 can approve or disapprove. If she approves, each player receives 1. If she disapproves, each player receives 0. The other players have no choices. It is easy to see that $\delta_{1,k} = \max(0,\ k - (n - k))$. By formula (3), $\gamma_1 = \frac{1}{n}\sum_{k=1}^{n} \delta_{i,k} = \frac{1}{n}\sum_{k=1}^{n} \max(0,\ 2k - n)$ and, by symmetry and efficiency, $\gamma_i = \frac{n-\gamma_1}{n-1}$ for $i = 2, \ldots, n$.

Plugging in $n = 3$ *and* $= 4$ we see that in a three-player version $\gamma = \frac{1}{6}(8, 5, 5)$ and in a four-player version $\gamma = \frac{1}{6}(9, 5, 5, 5)$, and it is straightforward to verify that as the number of players, n, becomes large, γ is approximately $\frac{1}{4}(n, 3, \ldots, 3)$; thus the value of player 1 is about one-fourth of the total feasible output. In effect, each of the remaining players concedes one-fourth of their equal share to player 1. Note that here, as in Example 1, the threat is effective despite the fact that carrying it out would damage the player herself.

Example 7. This is a variant of Example 6. Now there is a further cost, $c > 0$, to the spoiler: if player 1 approves then each player receives 1; if player 1 disapproves then

Adding these equations over $k = 1, \ldots, n$ yields $2\sum_{k=1}^{n} \delta_{1,k} = nW + (n-2)w$. By equation (3), $\gamma_1 = \frac{1}{n}\sum_{k=1}^{n} \delta_{i,k} = .5W + .5(\frac{n-2}{n})w$.

[28]To see this, modify the above argument slightly, noting that now $\delta(1 \cup S_1) + \delta(1 \cup S_2) = W + w - c$, because the threat power of a set that includes 1 but not 2 is reduced by c.

each of the other players receives 0 but player 1 receives $-c$. Consider the 4-player version. If $c \geq 2$ then $\gamma = (1,1,1,1)$; but if $c \leq 2$ then $\gamma_1 = \frac{1}{4} \times (-c) + \frac{1}{4} \times 0 + \frac{1}{4} \times 2 + \frac{1}{4} \times 4 = \frac{3}{2} - \frac{c}{4}$; thus $\gamma = (\frac{3}{2} - \frac{c}{4}, \frac{5}{6} + \frac{c}{12}, \frac{5}{6} + \frac{c}{12}, \frac{5}{6} + \frac{c}{12})$.

Note that the value of player 1 in this game, $\frac{3}{2} - \frac{c}{4}$, is decreasing in c. Thus, the economic worth of the ability to spoil is diminished when the cost to the spoiler is increased. It is straightforward to verify that as the number of players, n, becomes large, the value of player 1 becomes approximately one-fourth of the total payoff, the same as in Example 6. However, if the cost to the spoiler is proportional to the damage imposed on the others, say $c = c_0 n$, where $c_0 < 1$, then as the number of players becomes large, the value of player 1 becomes approximately $\frac{(1-c_0)^2}{4}$ of the total feasible output.

Example 8. This is a variant of Example 6, where there is more than one distinguished player whose approval is required for all players to receive 1. If one of these players disapproves then all players receive zero.

It is easy to compute the asymptotic behavior as $n \to \infty$. In the case of two distinguished players, the payoff to each of them divided by n – the total feasible output – converges, as $n \to \infty$, to $\int_{1/2}^{1}(2x-1)x\,dx = \frac{5}{24}$, which is about 21%. Thus the two spoilers receive 42% of the total feasible output, compared with 25% in the case of a single spoiler.

In the case of k distinguished players the payoff to each of them divided by n converges, as $n \to \infty$, to $\int_{1/2}^{1}(2x-1)x^{k-1}dx$. Since $k\int_{1/2}^{1}(2x-1)x^{k-1}dx$ converges, as $k \to \infty$, to 1, we see that when there are many spoilers – each with the power to reduce everyone's payoff to zero – essentially all of the economic output goes to them.

Example 9. This is a variant of Example 6, where there is more than one player whose approval is sufficient for all players to receive 1. If none of the distinguished players approves then all players receive zero.

Assume there are k distinguished players. The asymptotic behavior as $n \to \infty$ is as follows. The payoff to each distinguished player divided by n – the total feasible output – converges, as $n \to \infty$, to $\frac{2^{-k}}{k(k+1)}$. Thus the combined payoff to all the distinguished players is $\frac{2^{-k}}{(k+1)}$.

When $k = 1$ this amounts to $\frac{1}{4}$, as we have seen in Example 6. When $k = 2$ this amounts to $\frac{1}{12}$. Thus, when there are two distinguished players, only one of whose approvals is required, the fraction of the total value that they command is about 8.5%; this is in contrast to 42% in the case where both approvals are required, as in Example 8.

5.2. **The value of a Cournot oligopolist.** The example below is a formal version of the Cournot game described in section 1.3.

Example 10. Consider a Cournot oligopoly with inverse demand function $1 - \sum_1^n q_i$, where q_i is the quantity of firm i, and with constant unit costs $c_1 < c_2 < \ldots < c_n$. Denote by $\pi_i = \pi_i(q_1, \ldots, q_n)$ the profit of firm i, and denote by $M_i = \max q_i(1 - q_i - c_i)q_i$ the monopoly profit of firm i. First consider the case of a duopoly, $n = 2$. Note that

$$\pi_1 - \pi_2 = (1 - q_1 - q_2 - c_1)q_1 - (1 - q_1 - q_2 - c_2)q_2 = (1 - q_1 - c_1)q_1 - (1 - q_2 - c_2)q_2.$$

Thus $\pi_1 - \pi_2$ is the difference of a function that depends only on q_1 and a function that depends only on q_2, and therefore:

$$\delta(1) = \max_{q_1} \min_{q_2} (\pi_1 - \pi_2) = \max_{q_1} (1 - q_1 - c_1)q_1 - \max_{q_2} (1 - q_2 - c_2)q_2 = M_1 - M_2.$$

Since $c_1 < c_2$, the maximal total profit is obtained when all the production is done by firm 1, Thus, $\delta(1, 2) = M_1$. Applying formula (3), the value is $\gamma = (M_1 - \frac{1}{2}M_2,\ \frac{1}{2}M_2)$.

Next consider a general n. For each $S \subseteq N$, denote by $m(S)$ the firm in S with the lowest unit cost, i.e., $m(S) \in S$ and $c_m \leq c_j$ for every $j \in S$. If the members of S wish to maximize their total payoff then they will assign all the production to firm mS). Thus, in the same way as above, we have:

$$\delta(S) = M_{m(S)} - M_{m(N \setminus S)}, \quad \text{for any } S \subseteq N. \tag{6}$$

The value of firm i, γ_i, is a weighted average of $\delta(S)$, where S ranges over all subsets of players S (the averaging weight of a subset that does not include i is 0). Therefore, γ_i is a linear combination of M_j, $j \in N$, i.e.,

$$\gamma_i = \sum_{j=1}^{n} \alpha_{ij} M_j \ \text{ for all } \ i = 1, \ldots, n.$$

In order to determine the coefficients α_{ij} we use the random order formula of the value. According to the random order formula (5), γ_i is the arithmetic average, over all orders $\mathcal{R}$, of $\delta(S_i^{\mathcal{R}})$, where $S_i^{\mathcal{R}}$ consists of player i and all players that precede i

in the order $\mathcal{R}$. Equivalently, γ_i is the expectation of $\delta(S_i^{\mathcal{R}})$, where all orders $\mathcal{R}$ are equally likely.

From equation (6), we see that the coefficient, α_{ij}, of M_i is the probability that in a random order $m(S_i^{\mathcal{R}}) = j$ minus the probability that in a random order $m(N \backslash S_i^{\mathcal{R}}) = j$.

Thus, α_{ii} is the probability that $m(S_i^{\mathcal{R}}) = i$, i.e., that none of $1, \ldots, i-1$ appear before i. But this is just the probability that i appears first among $1, \ldots, i$, namely $\frac{1}{i}$. Hence, $\alpha_{ii} = 1/i$.

If $j > i$ then $m(S_i^{\mathcal{R}}) \neq j$ and the probability that $m(N \setminus S_i^{\mathcal{R}}) = j)$ is just the probability that j is last among $\{1, \ldots, j\}$ and i is second to last among $\{1, \ldots, j\}$, which equals $\frac{1}{j(j-1)}$. Hence, if $j > i$ then $\alpha_{ij} = -\frac{1}{j(j-1)} = (\frac{1}{j} - \frac{1}{j-1})$.

If $j < i$ then $m(S_i^{\mathcal{R}}) = j$ iff $m(N \setminus S_i^{\bar{\mathcal{R}}}) = j$, where $\bar{\mathcal{R}}$ is the reverse order of $\mathcal{R}$. Therefore, the probability that $m(S_i^{\mathcal{R}}) = j$ equals the probability that $m(N \backslash S_i^{\bar{\mathcal{R}}}) = j$, which equals the probability that $m(N \setminus S_i^{\mathcal{R}}) = j$. Hence, if $j < i$ then $\alpha_{ij} = 0$.

We conclude that for every $i = 1, \ldots, n$,

$$\gamma_i = \sum_{j=1}^{n} \alpha_{ij} M_j,$$

where

$$\alpha_{ij} = \begin{cases} 0 & \text{when } j < i, \\ \frac{1}{i} & \text{when } j = i, \\ \frac{1}{j} - \frac{1}{j-1} & \text{when } j > i. \end{cases}$$

As mentioned in the introduction, this formula may be interpreted as follows: First M_n, the monopoly profit of the least efficient firm, is divided equally among all the firms. Next, $M_{n-1} - M_n$ is shared equally among firms $1, \ldots, n-1$. And so on, until finally $M_1 - M_2$ is received only by firm 1.

5.3. **The value of information.** The example below is a formal version of the Bayesian game described in section 1.4.

Example 11. There are three players. Player 1 chooses the row, player 2 chooses the column, and player 3 has only a single strategy. The payoff matrix is either

$$\begin{bmatrix} 4,0,0 & 0,0,0 \\ 0,0,0 & 0,0,0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0,0,0 & 0,0,0 \\ 0,0,0 & 4,0,0 \end{bmatrix} \quad \text{with equal probabilities.}$$

Player 3 knows the true matrix, but players 1 and 2 do not.

We apply formula (3) to compute the value. $\delta(1)$ is the minmax value of the zero-sum game that is either

$$\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{with equal probabilities,}$$

where player 2 (who is in a coalition with player 3) knows the true matrix but player 1 does not. It is easy to see that $\delta(1) = 0$. (Player 2 chooses Right if it's the first matrix, Left if it's the second matrix.)

$\delta(1,3)$ is the minmax value of the same zero-sum game, except that now player 1 knows the true matrix but player 2 does not. It is easy to see that $\delta(1,3) = 2$. (Player 1 chooses Top if it's the first matrix, Bottom if it's the second matrix; and player 2 puts weight .5 on each column.)

$\delta(1,2)$ is the maximal sum of payoffs over the four entries of the average payoff matrix, i.e., $\delta(1,2) = 2$.

$\delta(1,2,3)$ is the average of the maximal sum of payoffs in each one of the two payoff matrices, i.e., $\delta(1,2,3) = 4$.

We have: $\gamma_1 = \frac{1}{3} \times 0 + \frac{1}{3} \times 2 + \frac{1}{3} \times 4 = 2$; $\gamma_2 = \frac{1}{3} \times (-2) + \frac{1}{3} \times \frac{2+0}{2} + \frac{1}{3} \times 4 = 1$ (note that $\delta(2) = -\delta(1,3) = -2$); and $\gamma_3 = 4 - (2+1) = 1$. Thus $\gamma = (2,1,1)$.

It may be interesting to compare the values of players 1 and 2, namely $(2,1)$, with their values in a variant of the game where player 3 is not present. Here, $\delta(1) = 1$ and $\delta(1,2) = 2$, so $\gamma = (1.5, 0.5)$.

It may also interesting to consider a variant of the game with a fourth player who, like player 3, knows the true state but is a strategic dummy. Here, $\delta(1) = 0$, $\delta(1,2) = 2$, $\delta(1,3) = \delta(1,4) = 0$, $\delta(1,2,3) = \delta(1,2,4) = 4$, $\delta(1,3,4) = 2$, and $\delta(1,2,3,4) = 4$. Applying formula (3), the value is $\gamma = (2, \frac{4}{3}, \frac{1}{3}, \frac{1}{3})$: competition between the informed players has reduced their total value, to the benefit of player 2.

6. The vNM-Shapley value of strategic games

In Section 2 we argued that the (von Neumann-Morgenstern-) Shapley value is a less convincing solution concept than is the (Nash-Harsanyi-Shapley) value. Here we present a characterization of the vNM-Shapley value that parallels the characterization the value and clarifies the relationship between the two concepts; and we indicate conditions under which the concepts concide.[29]

6.1. **Axiomatization of the vNM-Shapley value.** The Shapley value of a strategic game G is the Shapley value of the vNM-coalitional game vG that is defined by

$$(vG)(S) := \max_{x \in X^S} \min_{y \in X^{N \setminus S}} \sum_{i \in S} g^i(x, y). \tag{7}$$

Let $G \in \mathbb{G}(N)$. Define

$$(\hat{\delta}G)(S) := (vG)(S) - (vG)(N \setminus S). \tag{8}$$

We introduce the following axiom. For all $G \in \mathbb{G}(N)$

Balanced Security Levels If $(\hat{\delta}G)(S) = 0$ for every $S \subseteq N$, then $\hat{\gamma}_i G = 0$ for all $i \in N$.

Proposition 2. *The vNM–Shapley value is the unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of efficiency, balanced security levels, symmetry, null player, and additivity. It may be described as follows:*

$$\hat{\gamma}_i G = \frac{1}{n} \sum_{k=1}^{n} \hat{\delta}_{i,k}, \tag{9}$$

where $\hat{\delta}_{i,k}$ denotes the average of $(\hat{\delta}G)(S)$ over all k-player coalitions that include i. Furthermore, this map satisfies the axiom of individual rationality.

6.2. **Games where the Shapley value and the value coincide.** A pure-exchange economy is a model of strategic interaction between n agents, each having an initial endowment, where each agent is free to trade with any other agent and the payoff to each agent is a function of his final allocation. Note that

Proposition 3. *In constant-sum games and in pure-exchange economies the Shapley value and the value coincide.*

[29]For ease of exposition we restrict attention to strategic games with complete information, but an analogous characterization holds for Bayesian games.

Proof. If the strategic game is constant-sum, then an optimal strategy for S in the problem $\max_{x\in X^S}\min_{y\in X^{N\setminus S}}\sum_{i\in S} g^i(x,y)$ is also an optimal strategy for S in the problem $\max_{y\in X^{N\setminus S}}\min_{x\in X^S}\sum_{i\in N\setminus S} g^i(x,y)$. This is also true, trivially, in an exchange economy, where the sum of the payoffs to the agents in any coalition depends only on the strategies of the agents belonging to that coalition, so that any strategy for S is optimal in minimizing the total payoff to $N\setminus S$.

In both these cases, then, the minmax strategies in the two person zero-sum game where the payoff (to player 1) is the total payoff to S, are also minmax strategies in the game where the payoff is the difference between the total payoff to S and the total payoff to $N\setminus S$. Thus the optimal values in (8) and in (2) are the same. It follows that $(\hat{\delta}G)(S) = (\delta G)(S)$, hence $\hat{\delta}_{i.k} = \delta_{i.k}$ for all $1 \leq i,k \leq n$ and therefore, by (9) and (3), $\hat{\gamma}G = \gamma G$. □

Remark 10. Note that the value and the vNM-Shapey value do not coincide in exchange economies with taxes or with voting [1, 2, 3, 4].

7. Proof of the main results

In this section we present the proof of our main results, Theorems 1 and 2. In preparation, we provide background on games of threats and present an alternative definition of the value in terms of such games. And we present preliminary results, some of which are of interest in their own right.

7.1. **Games of Threats.** A *coalitional game of threats* is a pair (N,d), where

- $N = \{1,\ldots,n\}$ is a finite set of players.
- $d\colon 2^N \to \mathbb{R}$ is a function such that $d(S) = -d(N\setminus S)$ for all $S \subseteq N$.

Remark 11. A game of threats need not be a coalitional game as $d(\emptyset) = -d(N)$ may be non-zero.

Remark 12. If d is a game of threats then so is $-d$.

Denote by $\mathbb{D}(N)$ the set of all coalitional games of threats.

Let $\psi\colon \mathbb{D}(N) \to \mathbb{R}^n$. This may be viewed as a map that associates with any game of threats an allocation of payoffs to the players. Following Shapley [16] we consider the following axioms.

For all games of threats $(N,d_1),(N,d_2)$, and for all players i,j,

- *Efficiency* $\sum_{i\in N} \psi_i d = d(N)$.
- *Symmetry* $\psi_i d = \psi_j d$ if i and j are interchangeable in d (i.e., if $d(S \cup i) = d(S \cup j)\ \forall S \subseteq N\backslash\{i,j\}$).
- *Null player* $\psi_i d = 0$ if i is a null player in d (i.e., if $d(S\cup i) = d(S)\ \ \forall S \subseteq N$).
- *Additivity* $\psi(d_1 + d_2) = \psi d_1 + \psi d_2$.

Below are two results from [8] that will be needed in the sequel.

Proposition 4. *There exists a unique map $\psi\colon \mathbb{D}(N) \to \mathbb{R}^n$ satisfying the axioms of efficiency, symmetry, null player, and additivity. It may be described as follows:*

$$\psi_i d = \frac{1}{n}\sum_{k=1}^{n} d_{i,k}, \tag{10}$$

where $d_{i,k}$ denotes the average of $d(S)$ over all k-player coalitions that include i.

We refer to this map as the *Shapley value for games of threats.*

Definition 2. *Let $T \subseteq N$, $T \neq \emptyset$. The unanimity game of threats, $u_T \in \mathbb{D}(N)$, is defined by*

$$u_T(S) = \begin{cases} |T| & \textit{if } S \supseteq T, \\ -|T| & \textit{if } S \subseteq N\backslash T, \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 5. *Every game of threats is a linear combination of the unanimity games of threats u_T.*

7.2. **Rephrasing the Main Result.** Using the notion of games of threats we can provide an alternative definition of the value:

Proposition 6. *The value of a strategic game G is the Shapley value of the game of threats associated with G, i.e., $\gamma = \psi \circ \delta$, where $\gamma\colon \mathbb{G}(N) \to \mathbb{R}^n$, $\psi\colon \mathbb{D}(N) \to \mathbb{R}^n$, and $\delta\colon \mathbb{G}(N) \to \mathbb{D}(N)$ are as in (3), (10), and (2), respectively.*

Proof. Formula (3) is the same as formula (10), applied to the game of threats $d = \delta G$. ∎

Thus, Theorem 1 can be rephrased as follows: *$\gamma = \psi \circ \delta$ is the unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of efficiency, balanced threats, symmetry, null player, and additivity*

7.3. **Preliminary results.** In this section we present properties of the mapping $\delta\colon \mathbb{G}(N) \to \mathbb{D}(N)$ that are needed for the proof of the main result.

Let $G \in \mathbb{G}(N)$. For any $S \subseteq N$, let $(\delta G)(S)$ be as in (2).

Lemma 1. *δG is a game of threats.*

Proof. By the minmax theorem, $(\delta G)(S) = -(\delta G)(N\backslash S)$ for any $S \subseteq N$. □

We refer to δG as the game of threats associated with G.

Lemma 2. *$\delta : \mathbb{G}(N) \to \mathbb{D}(N)$ satisfies:*

- *$\delta(G_1 \oplus G_2) = \delta G_1 + \delta G_2$ for any $G_1, G_2 \in \mathbb{G}(N)$.*
- *$\delta(\alpha G) = \alpha \delta G$ for any $G \in \mathbb{G}(N)$ and $\alpha \geq 0$.*

Proof. Let $\mathrm{val}(G)$ denote the minmax value of the two-person zero-sum strategic game G. Then $\mathrm{val}(G_1 \oplus G_2) = \mathrm{val}(G_1) + \mathrm{val}(G_2)$.

To see this, note that by playing an optimal strategy in G_1 as well as an optimal strategy in G_2, each player guarantees the payoff $\mathrm{val}(G_1) + \mathrm{val}(G_2)$.

Now apply the above to all two-person zero-sum games played between a coalition S and its complement $N\backslash S$, as indicated in (2). □

The next lemma is an immediate consequence of the definition of δ.

Lemma 3. *$\delta : \mathbb{G}(\mathbb{N}) \to \mathbb{D}(\mathbb{N})$ satisfies:*

- $(\delta G)(N) = \max_{a \in A^N}(\sum_{i \in N} g^i(a))$.
- *If i and j are interchangeable in G then i and j are interchangeable in δG.*
- *If i is a null player in G then i is a null player in δG.*

Denote by $1_T \in \mathbb{R}^n$ the indicator vector of a subset $T \subseteq N$, i.e., $(1_T)_i = 1$ or 0 according to whether $i \in T$ or $i \notin T$.

Definition 3. *Let $T \subseteq N$, $T \neq \emptyset$. The unanimity strategic game on T, henceforth the unanimity game on T, is $U_T = (N, A, g_T)$, where*

$A^i = \{0, 1\}$ *for all* $i \in N$,

$g_T(a) = 1_T$ *if* $a^i = 1$ *for all* $i \in T$, *and* $g_T(a) = 0$ *otherwise.*

That is, if all the members of T consent then they each receive 1; however, if even one member dissents, then all receive zero; the players outside T always receive zero.

Lemma 4. *Let $T \neq \emptyset$, and let $U_T \in \mathbb{G}(N)$ be the unanimity game on T and $u_T \in \mathbb{D}(N)$ be the unanimity game of threats on T. Then $\delta U_T = u_T$.*

Proof. Consider the two-person zero-sum game between S and $N \backslash S$.

If $S \cap T$ is neither $\emptyset$ nor T, then both S and $N \backslash S$ include a player in T. If these players dissent then all players receive 0. Thus the minmax value, $(\delta U_T)(S)$, is 0.

If $S \cap T = T$ then, by consenting, the players in S can guarantee a payoff of 1 to each player in T and 0 to all the others. Thus $(\delta U_T)(S) = |T|$.

If $S \cap T = \emptyset$ then, by consenting, the players in $N \backslash S$ can guarantee a payoff of 1 to each player in $T \subset N \backslash S$ and 0 to all the others. Thus $(\delta U_T)(S) = -|T|$.

By Definition 2 , $\delta U_T = u_T$. □

Definition 4. *The anti-unanimity game on T is $V_T = (N, A, g)$, where*

$A^i = \{S \subseteq T : \ S \neq \emptyset\}$,

$g(S_1, \ldots, S_n) = \sum_{i \in T} -1_{S_i}$.

That is, each player in T chooses a non-empty subset of T where each member loses 1. Players outside T also choose such subsets, but their choices have no impact. Thus the payoff to any player, i, is minus the number of players in T whose chosen set includes i.

Lemma 5. $\delta V_T = -u_T$.

Proof. Let S be a subset of N such that $T \subseteq S$. In the zero-sum game between S and its complement, each player in S chooses a subset of T of size 1. Thus $(\delta V_T)(S) = -|T|$.

Let S be a subset of N such that $T \cap S \neq \emptyset$ and $T \setminus S \neq \emptyset$. In the zero-sum game between S and its complement, the minmax strategies are for the players in S to choose $T \setminus S$ and for the players in $N \setminus S$ to choose $T \cap S$. The resulting payoff is $-t_1 t_2 - (-t_2 t_1) = 0$, where t_1 and t_2 are the number of elements of $T \cap S$ and $T \setminus S$, respectively. Thus $(\delta V_T)(S) = 0$.

Therefore, $\delta V_T = -u_T$. □

Lemma 6. *For every game of threats $d \in \mathbb{D}(N)$ there exists a strategic game $U \in \mathbb{G}(N)$ such that $\delta U = d$. Moreover, there exists such a game that can be expressed as a direct sum of non-negative multiples of the unanimity games $\{U_T\}_{T \subseteq N}$ and the anti-unanimity games $\{V_T\}_{T \subseteq N}$.*

Proof. By Proposition 5, d is a linear combination of the unanimity games of threats u_T.

$$d = \sum_T \alpha_T u_T - \sum_T \beta_T u_T \text{ where } \alpha_T, \beta_T \geq 0 \text{ for all } T.$$

By Lemmas 4 and 5,

$$d = \sum_T \delta(\alpha_T U_T) + \sum_T \delta(\beta_T V_T),$$

and, by Lemma 2,

$$d = \delta((\bigoplus_{T \subseteq N} \alpha_T U_T) \oplus (\bigoplus_{T \subseteq N} \beta_T V_T)),$$

where $\oplus_T$ stands for the direct sum of the games parameterized by T. □

Remark 13. In particular, Lemma 6 establishes that the mapping $\delta \colon \mathbb{G}(N) \to \mathbb{D}(N)$ is onto.

As was pointed out earlier, the operation $\oplus$ does not have a natural inverse. However, we have the following:

Lemma 7. *For every $G \in \mathbb{G}(N)$ there exists a $\delta-$inverse, i.e., $U \in \mathbb{G}(N)$ such that $\delta(G \oplus U) = 0$. Moreover, if $G' \in \mathbb{G}(N)$ is such that $\delta G' = \delta G$ then there exists $U \in \mathbb{G}(N)$ that is a $\delta-$inverse of both G and G'.*

Proof. Consider $-\delta G \in \mathbb{D}(N)$. By Lemma 6, there exists $U \in \mathbb{G}(N)$ such that $-\delta G = \delta U$. By Lemma 2, $\delta(G \oplus U) = 0$. And if G' is such that $\delta G' = \delta G$ then, by the same argument, $\delta(G' \oplus U) = 0$. □

Proposition 7. *If* $\gamma \colon \mathbb{G}(N) \to \mathbb{R}^n$ *satisfies the axioms of balanced threats, efficiency, and additivity then* γG *is a function of* δG.

Proof. Let $G, G' \in \mathbb{G}(N)$ be such that $\delta G = \delta G'$. We must show that $\gamma G = \gamma G'$. By Lemma 7, there exists $U \in \mathbb{G}(N)$ such that $\delta(G \oplus U) = 0 = \delta(G' \oplus U)$. By the axiom of balanced threats, $\gamma(G \oplus U) = 0 = \gamma(G' \oplus U)$. Thus, by the additivity axiom, $\gamma G = -\gamma U = \gamma G'$. □

Lemma 8. *For any* $T \neq \emptyset$ *and* $\alpha \geq 0$, *the axioms of symmetry, null player, and efficiency determine* γ *on the game* αU_T. *Specifically,* $\gamma(\alpha U_T) = \alpha 1_T$.

Proof. Any $i \notin T$ is a null player in U_T, and so $\gamma_i = 0$. Any $i, j \in T$ are interchangeable in U_T, and so $\gamma_i = \gamma_j$. By efficiency, the sum of the γ_i is the maximum total payoff, which, since $\alpha > 0$, is $\alpha|T|$. Thus each of the $|T|$ non-zero γ_i is equal to α. □

Lemma 9. *For any* $\alpha \geq 0$, *the axioms (of symmetry, null player, additivity, balanced threats, and efficiency) determine* γ *on the game* αV_T. *Specifically,* $\gamma(\alpha V_T) = -\alpha 1_T$.

Proof. By Lemma 8 the axioms determine $\gamma(\alpha U_T) = \alpha 1_T$. By Lemmas 4 and 5, $\delta(\alpha V_T \oplus \alpha U_T) = 0$. Therefore, by the axiom of balanced threats, $\gamma(\alpha V_T \oplus \alpha U_N) = 0$. Thus, by additivity, $\gamma(\alpha V_T) = -\gamma(\alpha U_T) = -\alpha 1_T$. □

Remark 14. We cannot rely on the same proof as that of Lemma 8, by appealing to symmetry and efficiency. In the game V_T, it is not true that any two players, $i, j \in T$, are interchangeable, because the payoff functions are not identical. If we had adopted a more restrictive version of the symmetry axiom – that the names of the players do not matter – then any $i, j \in T$ would be interchangeable and the direct proof would be valid. But this more restrictive version of the axiom would lead to a weaker uniqueness theorem.

Proposition 8. *The map* γ *of formula (3) satisfies the axiom of individual rationality.*

Proof. Let $G = (N, A, g)$ be a strategic game. By symmetry, it is sufficient to prove individual rationality for player 1, i.e., that $\gamma_1 G \geq \pi^1$, where π^1 denotes player 1's security level.

Let S_1, S_2 be a partition of $N\backslash 1$. We claim that

$$(\delta G)(S_1 \cup 1) + (\delta G)(S_2 \cup 1) \geq 2\pi^1. \tag{11}$$

To see this, let $\bar{x}^1$ be a strategy that guarantees player 1 her security level, i.e.,

$$\min_{x^{N\backslash 1} \in X^{N\backslash 1}} g^1(\bar{x}, x^{N\backslash 1}) = \max_{x^1 \in X^1} \min_{x^{N\backslash 1} \in X^{N\backslash 1}} g^1(x^1, x^{N\backslash 1}) = \pi^1. \tag{12}$$

We have:

$$\begin{aligned}
&(\delta G)(S_1 \cup 1) \\
= \;& \max_{x \in X^{S_1 \cup 1}} \min_{y \in X^{S_2}} \left(\sum_{i \in S_1 \cup 1} g^i(x, y) - \sum_{i \in S_2} g^i(x, y) \right) \\
\geq \;& \max_{x \in X^{S_1}} \min_{y \in X^{S_2}} \left(\sum_{i \in S_1 \cup 1} g^i(\bar{x}^1, x, y) - \sum_{i \in S_2} g^i(\bar{x}^1, x, y) \right) \\
\geq \;& \max_{x \in X^{S_1}} \min_{y \in X^{S_2}} \left(\pi^1 + \sum_{i \in S_1} g^i(\bar{x}^1, x, y) - \sum_{i \in S_2} g^i(\bar{x}^1, x, y) \right) \\
= \;& \pi^1 + \max_{x \in X^{S_1}} \min_{y \in X^{S_2}} \left(\sum_{i \in S_1} g^i(\bar{x}^1, x, y) - \sum_{i \in S_2} g^i(\bar{x}^1, x, y) \right). \qquad (13)
\end{aligned}$$

The first inequality follows since restricting the set of available strategies cannot increase the maximum of a function, and the second inequality follows from (12) and the fact that the maxmin of a function is monotonic in that function.

Similarly, we have

$$(\delta G)(S_2 \cup 1) \geq \pi^1 + \max_{x \in X^{S_2}} \min_{y \in X^{S_1}} \left(\sum_{i \in S_2} g^i(\bar{x}^1, x, y) - \sum_{i \in S_1} g^i(\bar{x}^1, x, y) \right). \tag{14}$$

By the minmax theorem, the sum of the right-hand sides of (13) and (14) is $2\pi^1$; therefore, adding these two inequalities implies (11).

Now, as S_1 ranges over all the sets of size $k - 1$ that do not include 1, S_2 ranges over all the sets of size $n - k$ that do not include 1; thus $S_1 \cup 1$ ranges over all the sets of size k that include 1 and $S_2 \cup 1$ ranges over all the sets of size $n-k+1$ that include 1. Taking the average of inequality (11) over all these sets we have

$$\delta_{1,k} + \delta_{1,n-k+1} \geq 2\pi^1,$$

where $\delta_{1,k}$ denotes the average of $(\delta G)(S)$ over all k-player coalitions that include 1.

Taking the average over $k = 1, ..., n$ we obtain

$$2 \times \frac{1}{n} \sum_{k=1}^{n} \delta_{1,k} \geq 2\pi^1.$$

Thus, by formula (3), $\gamma_1 G \geq \pi^1$. □

7.4. **Proof of Theorems 1 and 2.**

Proof. of Theorem 1.

We first prove uniqueness. Let $G \in \mathbb{G}(N)$. Consider $\delta G \in \mathbb{D}(N)$; by Lemma 6 there exists a game $U \in \mathbb{G}(N)$ that is a direct sum of non-negative multiples of the unanimity games $\{U_T\}_{T \subseteq N}$ and the anti-unanimity games $\{V_T\}_{T \subseteq N}$, such that $\delta G = \delta U$.

By Proposition 7, $\gamma G = \gamma U$ and so it suffices to show that γU is determined by the axioms.

Now, by Lemmas 8 and 9 , γ is determined on non-negative multiples of the unanimity games $\{U_T\}_{T \subseteq N}$ and the anti-unanimity games $\{V_T\}_{T \subseteq N}$. It then follows from the axiom of additivity that γ is determined on U.

To prove existence we show that the value, $\gamma = \psi \circ \delta$, satisfies the axioms.

Efficiency, symmetry, and the null player axiom follow from Lemma 3 and the corresponding properties of the Shapley value ψ.

Additivity follows from Lemma 2 and the linearity of the Shapley value.

The axiom of balanced threats follows from formula (15). If $(\delta G)(S) = 0$ for all $S \subseteq N$ then $\gamma_i G = 0$ for all $i \in N$.

Finally, Proposition 8 establishes that γ satisfies the axiom of individual rationality. □

The proof of Theorem 2 proceeds along the same lines as the proof of Theorem 1, but with δ_B replacing δ.

7.5. **Proof of Proposition 2.** Uniqueness can be proved in the same way as in Theorem 1. It is straightforward to verify that all the lemmas that involve δ remain valid when δ is replaced by $\hat{\delta}$. In particular, note that $\hat{\delta}U_T = \delta U_T = u_T$ and $\hat{\delta}V_T = \delta V_T = -u_T$.

Recall that ψ denotes the Shapley value for games of threats. The proof that $\psi \circ \hat{\delta}$ satisfies the axioms is similar to the proof in Theorem 1 that $\psi \circ \delta$ satisfies the axioms of that theorem. The proof that $\hat{\gamma} = \psi \circ \hat{\delta}$ is similar to the proof of Proposition 6.

Appendix A. The coco value

Kalai and Kalai [7] introduced the "coco value," which coincides with the value of two-person Bayesian games. The axioms that they consider are the following: efficiency, shift invariance (if G_α is a modification of G obtained by adding to the payoff of one player, say player 1, an amount α everywhere, then $\gamma G_\alpha = \gamma G + (\alpha, 0)$), invariance to redundant strategies (removing a duplicate row or column in the payoff matrix does not affect the value), monotonicity in actions (removing a pure strategy of a player cannot increase the player's value), monotonicity in information (coalescing two signals cannot increase the player's value), and payoff dominance (if player 1's payoff is everywhere strictly greater than player 2's payoff then $\gamma_1 \geq \gamma_2$). They prove that there is a unique map from two-person strategic games to $\mathbb{R}^2$ that satisfies these axioms.

Since Kalai and Kalai characterize the same concept for two-person games as we do, their axioms are equivalent to ours in the two-person case. To see a direct connection between the two sets of axioms, it may be helpful to note that in two-person games the general additivity axiom can be replaced by the requirement that the solution be additive over the direct sum of a game and a trivial game, which amounts to shift invariance.

In games with more than two players the value still satisfies all the Kalai and Kalai axioms other than payoff dominance. This follows from Remarks 4 and C.3 and Proposition 11. However, the value does not satisfy payoff dominance. This is a reflection of the more complex considerations in games with more than two players. In Example 7 (with $n = 4$ and $c = 1$), player 1's payoff is everywhere smaller than player 2's, but 1's value is greater. This is so because of player 1's ability to play off some of her opponents against each other.

Appendix B. n-player Bargaining with variable threats

B.1. **Harsanyi's cooperative solution.** Harsanyi's [6] original definition of the value of a strategic game G and Myerson's [12] (related) cooperative solution of a strategic game G are based on a scheme that generalizes Nash's [13] two-person model of bargaining with variable threats.

Recall (Section 2) that Nash defines an auxiliary game, as follows: Each one of the two players determines a "threat strategy;" the outcome resulting from deploying these strategies is the "disagreement point;" the payoff to the players in the auxiliary game is determined by moving along the 45-degree line to the Pareto frontier. The Nash solution is then defined as the unique equilibrium point of the auxiliary game.

Harsanyi's scheme consists of several steps. The first step assigns to any n-player strategic game $G = (N, A, g)$ an auxiliary game form (CL, B), whose set of players CL is the set of the $2^n - 1$ nonempty coalitions S, $CL = \{S : \emptyset \neq S \subseteq N\}$, and the set of strategies of player S, B^S, consists of all correlated strategies b^S of the members of S.

The second step defines a payoff function $h : B \to \mathbb{R}^{CL}$, where B is the set of all strategy profiles $b = (b^S)_{\emptyset \neq S \subseteq N}$. The payoff function h depends on a "settlement function" $f : B \to \mathbb{R}^n$, and is defined by $h^S(b) := \sum_{i \in S} f_i(b)$.

The solution of a strategic game G is then defined as the set of all payoff vectors $f(b)$ where b ranges over all Nash equilibria of the strategic game (CL, B, h).

Harsanyi's solution takes, as a settlement function f^H, the Shapley value φ, of the coalitional game w_b, where $w_b(S) = \sum_{i \in S} g^i(b^S, b^{N \setminus S})$.

Note that the strategies b are the n-person analogs of Nash's threat strategies, the set-valued function w_b is the analog of the disagreement point, and the settlement function is the analog of the rule for moving from the disagreement point to the Pareto frontier.

A priori, Harsanyi's solution is set valued, as Nash equilibrium is not unique. However, as Harsanyi notes, the Shapley value of the coalitional game w_b is independent of the equilibrium strategy profile b.

It is well known that the Shapley value of a player in a coalitional game, (N, v), is a weighted average of the differences, $v(T) - v(N \setminus T)$, where T ranges over the subsets of N that include the player. Thus, for every $i \in S$, $\varphi_i w_b$ is a linear function of the differences $w_b(T) - w_b(N \setminus T)$, $T \subseteq N$, and it is monotonic increasing in $w_b(S) - w_b(N \setminus S)$. It follows that the same is true for the payoff, $h^S(b) = \sum_{i \in S} \varphi_i w_b$, of player $S \in CL$ in the game (CL, B, h). .

Note that, for any coalition T, the difference $w_b(T) - w_b(N \setminus T)$ is a function of the strategies b^T and $b^{N \setminus T}$ alone. Therefore, $h^S(b)$ can be written as a sum of two terms: $h_1^S(b)$, which does not depend on the strategies b^S and $b^{N \setminus S}$ of players S and $N \setminus S$, and $h_2^S(b)$, which depends only on the strategies b^S and $b^{N \setminus S}$ and is monotonic increasing in $w_b(S) - w_b(N \setminus S)$.

Similarly, $h^{N \setminus S}(b)$ can be written as a sum of two terms: $h_1^{N \setminus S}(b)$, which does not depend on the strategies b^S and $b^{N \setminus S}$ of players S and $N \setminus S$, and $h_2^{N \setminus S}(b)$, which depends only on the strategies b^S and $b^{N \setminus S}$ and is monotonic decreasing in $w_b(S) - w_b(N \setminus S)$.

Let b be an equilibrium of the strategic game (CL, B, h). Then the strategy b^S of player S maximizes, given the strategies of the other players, the difference $w_b(S) - w_b(N \setminus S)$, and the strategy $b^{N \setminus S}$ of player $N \setminus S$ minimizes, given the strategies of the other players, this difference. Hence,

$$w_b(S) - w_b(N \setminus S) = \max_{b^S} \min_{b^{N \setminus S}} \left(\sum_{i \in S} g^i(b^S, b^{N \setminus S}) - \sum_{i \in N \setminus S} g^i(b^S, b^{N \setminus S}) \right) = (\delta G)(S).$$

As the Shapley value of the coalitional game w_b equals the Shapley value of the game of threats δG (Proposition 6), the above-described solution of Harsanyi coincides with the concept that we call "the value."

B.2. **Myerson's cooperative solution.** Myerson [12] takes issue with Harsanyi's settlement function. He defines two axioms, which a reasonable settlement function might satisfy, and shows that there is a unique settlement function f^M that satisfies both axiom. This settlement functions is based on the natural generalization of the Shapley value to partition function games, as defined in [11].

Myerson's solution of a strategic game G is the set of all payoff vectors $f^M(b)$, where b ranges over all Nash equilibria of the strategic game (CL, B, h), where $h^S(b) = \sum_{i \in S} f_i^M(b)$.

B.3. **Our alternative approach.** As mentioned in the introduction, this paper takes a different route to generalizing Nash's "bargaining with variable threats" to

n-player games. Instead of trying to generalize the process, we attempt to generalize a key property. In a two-player game, if the minmax value of the game of differences is zero, then the Nash variable-threats solution allocates the same payoff to both players. This can be interpreted as saying that if neither player has threat power, then each receives the same payoff. Now, in a two-person game the only proper subsets are the two singletons, therefore the only threats to consider are those by one player against the other. But in n-person games there are many proper subsets, each of which can threaten its complement. If we wish to generalize the Nash variable-threats solution to n-person games, then it seems reasonable to require that if no proper coalition has threat power, i.e., $\delta(S) = 0$ for all proper subsets of N, then all players receive the same payoff. This is a variant of our axiom of balanced threats. (See Proposition 13.)

B.4. **Myerson's "individual rationality".** Myerson considers the equilibrium threat strategies, $\hat{b}$, to be meaningful in their own right. Thus, he requires that the solution allocate to each player, i, at least the amount that the player can guarantee vs. the threat strategy, $\hat{b}^{N\setminus i}$, of the opponents. He refers to this requirement as "individual rationality".

To illustrate this concept, consider the following three-player game: Player 3 has no choice. Players 1 and 2 can each choose to be active or passive; the payoffs are: $g(P, P) = (0, 0, 0)$, $g(A, P = (-1, 3, 2 - \varepsilon)$, $g(P, A) = (3, -1, 2 - \varepsilon)$, $g(A, A) = (2, 2, 4 - 2\varepsilon)$, where ε is a parameter satisfying $0 \leq \varepsilon \leq .5$.

It is easy to verify that $\delta(1) = 0$, $\delta(1, 2) = 2\varepsilon$, $\delta(1, 2, 3) = 8 - 2\varepsilon$. By formula (3), the value solution is $(\frac{8}{3} - \frac{\varepsilon}{3}, \frac{8}{3} - \frac{\varepsilon}{3}, \frac{8}{3} + \frac{2\varepsilon}{3})$. In contrast, the Myerson solution is $(2, 2, 4 - 2\varepsilon)$.

We note that, in the case $\varepsilon > 0$, the value solution does not satisfy Myerson's requirement of individual rationality. In this case, the equilibrium threat strategies, $\hat{b}$, of the zero-sum games between complementary coalitions, are uniquely defined. (In the case $\varepsilon = 0$, they are not.) In particular, $\hat{b}^{\{1,2\}} = (A, A)$. Thus player 3 can guarantee $4 - 2\varepsilon$ vs $\hat{b}^{\{1,2\}}$; but this is more than the amount that player 3 receives according to the value solution.

Some have argued that the failure to satisfy "individual rationality" is a serious flaw of the value solution. We beg to differ: in our approach, it is immaterial whether the threat strategies are uniquely defined, or what these strategies guarantee to each player. Indeed, an essential point of our solution is that it is does not require any auxiliary game - the axioms are defined directly on the strategic form.

In the case $\varepsilon = 0$ the Myerson solution, $(2, 2, 4)$, is *consistent* in the sense that the allocation to players 1 and 2 is the same as it would be when player 3, who is a strategic dummy, is dropped from the game. In contrast, the value solution, $(\frac{8}{3}, \frac{8}{3}, \frac{8}{3})$, is not consistent. One could argue that the failure to satisfy consistency is a flaw of the value solution; however, we disagree. (Remark 8.)

Appendix C. Additional properties of the value

In this appendix we show that the value satisfies some additional desirable properties. For ease of exposition, we restrict attention to games with complete information; but analogous results hold for Bayesian games.

C.1. **Another formula for the value.**

Proposition 9. *The value of a strategic game G may be described as follows:*

$$\gamma_i G = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\binom{n-1}{k-1}} \sum_{\substack{S: i \in S \\ |S|=k}} (\delta G)(S). \tag{15}$$

This is merely a rearrangement of formula (3) or formula(5); it expresses the value of player i as a weighted average of $(\delta G)(S)$ over the coalitions S that include i, where for each $k = 1, \ldots, n$, the weight $\frac{1}{n}$ is equally divided among the $\binom{n-1}{k-1}$ coalitions of size k that include i.

C.2. **Small worlds.** If the set of players is the union of two disjoint subsets such that the payoffs to the players in each subset are unaffected by the actions of the players in the other subset, then the value of each player is the same as it would be in the game restricted to the subset that includes the player.

Proposition 10. *The value satisfies the small-worlds axiom.*

Proof. Let $G = (N, A, g)$, where $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, and where the actions of players in N_1 do not affect the payoffs to players in N_2, and vice versa.

Assume, w.l.o.g., that $1 \in A_2^i$ for all $i \in N$. Define $G_1 \in \mathbb{G}(N)$ by modifying G as follows. Restrict the set of pure strategies of each player in N_2 to $\{1\}$ and define $g_1^i = g^i$ for $i \in N_1$ and $g_1^i = 0$ for $i \in N_2$; and define G_2 in a similar way.

By the definition (2) of δ,

$$\delta G = \delta G_1 + \delta G_2.$$

Recall that $\gamma = \psi \circ \delta$, where ψ is the Shapley value for games of threats (Proposition 6). Since ψ is additive,

$$\gamma G = \psi \circ \delta G = \psi \circ \delta(G_1 + G_2) = \psi \circ \delta G_1 + \psi \circ \delta G_2 = \gamma G_1 + \gamma G_2.$$

Since any $i \in N_1$ is a null player in G_2, it follows from the null-player axiom that $\gamma_i G_2 = 0$. Thus

$$\gamma_i G = \gamma_i G_1 + \gamma_i G_2 = \gamma_i G_1 \text{ for all } i \in N_1.$$

Similarly, $\gamma_i G = \gamma_i G_2$ for all $i \in N_2$.

Thus, for $i \in N_1$, the value of G is the same as the value of G_1, which may be viewed as the restriction of G to N_1; and similarly for $i \in N_2$. □

Remark 15. It is insufficient to assume that the payoffs to the players in N_1 are unaffected by the actions of the players in N_2. In Example 3, player 3 has only one strategy and so she obviously cannot affect the payoffs of the other players. Yet when player 3 is dropped, the values for players 1 and 2 change.

Remark 16. The small-worlds axiom may be viewed as an instance of the more general statement, that the additivity of the value extends to games over two different sets of players. Let $G_1 \in \mathbb{G}(N_1)$ and $G_2 \in \mathbb{G}(N_2)$. By adding the members of $N_2 \setminus N_1$ as dummy players in G_1, and the members of $N_1 \setminus N_2$ as dummy players in G_2, we may view both G_1 and G_2 as games in $\mathbb{G}(N_1 \cup N_2)$. Thus $\gamma(G_1 \oplus G_2) = \gamma G_1 + \gamma G_2$. Since the value of the existing players is unaffected by the addition of dummy players, $\gamma_i(G_1 \oplus G_2) = \gamma_i G_1$ for all $i \in N_1 \setminus N_2$ and $\gamma_i(G_1 \oplus G_2) = \gamma_i G_2$ for all $i \in N_2 \setminus N_1$. The small-worlds axiom corresponds to the case where N_1 and N_2 are disjoint.

C.3. **Monotonicity in Information.** The value is *monotonic in information.* i.e., if a player can no longer distinguish between some signals that she previously was able to distinguish, then her value has not increased. To see this, note that in a correlated strategy of a subset S each player is assumed to have the aggregate information of all members of S. Therefore, for any S that includes i, the set of correlated strategies (mixtures of mappings from the aggregate information of S to actions by members of S) is now a subset of the original set of correlated strategies, and consequently the minmax value, $\delta(S)$, of the game between S and its complement is less than or equal to what it previously was. By formula (3), the same is true for the value of player i.

Remark 17. Every Bayesian game can be associated with a strategic game. Specifically, the Bayesian game B defined by the actions, C^i, the payoff functions, U, the signals, Y^i, and the prior distribution, μ, is associated with the strategic game G with the pure strategies $a^i\colon Y^i \to C^i$ and the payoff functions $g^i(a) := E_\mu u^i(a(y))$. However, unlike the case for solution concepts such as Nash equilibrium, the value of the Bayesian game B is *not the same* as the value of the strategic game G. The reason is that in G, correlated strategies only involve coordination of the strategic choices, but not aggregation of the information. Example 11 is a case in point: in the associated strategic game, no coalition can make use of player 3's information; so the value is $(1.5, 0.5, 0)$ rather than $(2, 1, 1)$.

C.4. **Shift Invariance.** Given a game $G = (N, A, g)$ and $\alpha \in \mathbb{R}^n$, let $G + \alpha$ be the game obtained from G by adding α to each payoff entry, namely, $G+\alpha = (N, A, g+\alpha)$.

Axiom of shift invariance $\gamma(G + \alpha) = \gamma G + \alpha$.

Proposition 11. *The value satisfies the axiom of shift invariance.*

Proof. The definition (2) of δ implies that

$$(\delta(G + \alpha))(S) = (\delta G)(S) + \sum_{j\in S} \alpha_j - \sum_{j\in N\setminus S} \alpha_j.$$

Therefore, if $i \in S$ then $(\delta(G + \alpha))(S) + (\delta(G + \alpha))(i \cup (N \setminus S)) = (\delta G)(S) + \sum_{j\in S} \alpha_j - \sum_{j\in N\setminus S} \alpha_j + (\delta G)(i \cup (N \setminus S)) + \sum_{j\in i\cup N\setminus S} \alpha_j - \sum_{i\neq j\in S} \alpha_j = (\delta G)(S) + (\delta G)(i \cup (N \setminus S)) + 2\alpha_i$. As the map from subsets S of size k that contain player i, defined by $S \mapsto i \cup (N \setminus S)$, is 1-1 and onto the subsets of size $n - k + 1$ that contain player i, we deduce that $(\delta(G+\alpha))_{i,k} + (\delta(G+\alpha))_{i,n-k+1} = (\delta G)_{i,k} + (\delta G)_{i,n-k+1} + 2\alpha_i$ Therefore, by formula (3) for the value, $\gamma(G + \alpha) = \gamma G + \alpha$. □

Remark 18. Let I_α be a game where the payoff is the constant α. The game $G + \alpha$ is strategically equivalent to $G \oplus I_\alpha$. As the values of two strategically equivalent games coincide, it would have been sufficient to prove that $\gamma(G \oplus I_\alpha) = \gamma G + \alpha$. For this equality one need not rely on the axiom of balanced threats.

Proposition 12. *A map $\gamma : \mathbb{G}(N) \to \mathbb{R}^n$ that satisfies the additivity, efficiency, and null player axioms satisfies $\gamma(G \oplus I_\alpha) = \gamma G + \alpha$.*

Proof. We prove that $\gamma I_\alpha = \alpha$.

Note that a player i is a null player in I_α if and only if $\alpha_i = 0$. If all the players in I_α are null players then $\alpha = 0$ and $\gamma(I_\alpha) = 0$ by the null-player axiom. Assume that there is one non-null player in I_α, say player i. Then, $\alpha_i \neq 0$ and $\forall j \neq i$, $\alpha_j = 0$, and by the null-player axiom $\gamma_j(I_\alpha) = 0$, and by the efficiency axiom $\gamma_i(I_\alpha) = \alpha_i$. Therefore, $\gamma(I_\alpha) = \alpha$.

We continue by induction on the number of non-null players in I_α. If there are $k > 1$ non-null players in I_α, then let $\alpha(1)$ and $\alpha(2)$ be such that $\alpha = \alpha(1)+\alpha(2)$ and in each game $I_{\alpha(1)}$ and $I_{-\alpha(2)}$ there are fewer than k non-null players. By the additivity axiom, $\gamma(I_\alpha \oplus I_{-\alpha(2)}) = \gamma I_\alpha + \gamma I_{-\alpha(2)}$, and by the induction hypothesis $\gamma(I_\alpha \oplus I_{-\alpha(2)}) = \alpha(1)$ and $\gamma(I_{-\alpha(2)}) = -\alpha(2)$. We conclude that $\gamma(I_\alpha) = \alpha(1) + \alpha(2) = \alpha$.

Therefore, $\gamma(G \oplus I_\alpha) = \gamma G + \gamma I_\alpha = \gamma G + \alpha$, where the first equality follows from the axiom of additivity and the second equality from the previously proved $\gamma I_\alpha = \alpha$. □

Appendix D. Alternative version of the axioms

D.1. **Variants of the axiom of balanced threats.** In this appendix we show that the value satisfies several additional properties, each of which can replace the axiom of balanced threats in the characterization of the value.

We say that the function δG is symmetric if $(\delta G)(S)$ is the same for all coalitions S with the same number of players.

We say that players i and j are interchangeable in δG if $(\delta G)(S \cup i) = (\delta G)(S \cup j)$ for all $S \subset N \backslash \{i, j\}$.

Proposition 13. *The map $\gamma : \mathbb{G}(N) \to \mathbb{R}^N$, defined by formula (3), satisfies the following properties.*

(BT1) *If $(\delta G)(S) = 0$ for all $S \subseteq N$, then $\gamma_i G = \gamma_j G$ for all $i, j \in N$.*

(BT2) *If $(\delta G)(S) = 0$ for all proper subsets of N, then $\gamma_i G = \gamma_j G$ for all $i, j \in N$.*

(BT3) *If the function δG is symmetric, then $\gamma_i G = \gamma_j G$ for all $i, j \in N$.*

(BT4) *If i and j are interchangeable in δG, then $\gamma_i G = \gamma_j G$.*

Proof. Moving down the list, the properties become more restrictive as they require that $\gamma_i G = \gamma_j G$ on a wider class of games G. Letting $\Gamma(BTm)$ be the set of all maps γ from n-person strategic games to $\mathbb{R}^n$ that satisfy property (BTm), this means that

$$\Gamma(BT4) \subset \Gamma(BT3) \subset \Gamma(BT2) \subset \Gamma(BT1).$$

Thus, it is sufficient to prove that the map γ of formula (3) satisfies $\gamma \in \Gamma(BT4)$.

Assume, then, that $(\delta G)(S \cup i) = (\delta G)(S \cup j)$ for all $S \subset N \setminus \{i, j\}$. This implies that $\delta_{i,k} = \delta_{j,k}$ (where we suppress the dependence of $\delta_{i,k}$ on G) for every $k \leq n$. and therefore, by formula (3) $\gamma_i G = \gamma_j G$. Thus (BT4) holds, i.e., $\gamma \in \Gamma(BT4)$. □

Next, we note that each one of the properties (BTm), along with the efficiency axiom, (E), implies the axiom of balanced threats, (BT). That is,

$$\Gamma(BTm) \cap \Gamma(E) \subset \Gamma(BT) \text{ for all } m = 1, 2, 3, 4.$$

Let G be a strategic game such that $(\delta G)(S) = 0$ for every $S \subseteq N$. Then each one of the conditions in the properties (BTm) holds. Therefore, if γ satisfies any one of these properties, then $\gamma_i G = \gamma_j G$ for all $i, j \in N$. By efficiency, $\sum_{i \in N} \gamma_i G = (\delta G)(N) = 0$ and therefore $\gamma_i G = 0$ for every $i \in N$.

Thus each of the properties (BTm) can replace the axiom of balanced threats in the uniqueness theorem for the value. Proposition 13 says that each of the properties is satisfied by the value. Therefore, we have:

Proposition 14. *The characterization of the value (Theorem 1) remains valid when the axiom of balanced threats (BT) is replaced by any one of the properties (BT1), (BT2), (BT3,) or (BT4).*

Each one of these alternative characterizations has a different flavor, as each one relies on a different property. Properties (BT1), (BT2,) and (BT3) are global. They impose conditions on the threat powers that imply that all players have the same value. In contrast, property (BT4) focuses on a specific pair of players. It imposes conditions on the threat powers that imply that these two players have the same value.

We conclude this discussion with another property, of a somewhat different nature, that can replace the axiom of balanced threats in the characterization of the value:

Sum of flows. The value of player i in the game G is the sum of flows, $x_i^G(S)$, over the subsets $S \subseteq N$, where $x_i^G(S) \geq 0$ whenever $i \in S$ and $(\delta G)(S) \geq 0$.

Formula (15) implies that the value satisfies this property. Indeed, the formula shows that the flows, $x_i^G(S)$, may be expressed as follows: $x_i^G(S) = \frac{1}{\binom{n-1}{|S|-1}}(\delta G)(S)$.

In order to deduce the axiom of balanced threats it is not necessary to know the origin of the flows: If $(\delta G)(S) = 0$ for all S then obviously $(\delta G)(S) \geq 0$ for all S and therefore, for every $i \in N$, $\gamma_i G$ is the sum of non-negative numbers. Since $(\delta G)(N) = 0$, efficiency implies that $\sum_{i \in N} \gamma_i G = (\delta G)(N) = 0$; it follows that $\gamma_i G = 0$ for all $i \in N$.

D.2. **Variant of the Axiom of symmetry.** Recall that our axiom of symmetry says that if two players, i and j, are interchangeable in the game G then $\gamma_i G = \gamma_j G$. By contrast, the classic axiom requires that the value be invariant to permutations of the players' names. Since every permutation of N consists of a sequence of pairwise exchanges, the axiom can be stated as follows.

Axiom of full symmetry Let $G = (N, A, g)$ and let $\hat{G} = (N, \hat{A}, \hat{g})$ be such that $\hat{A}^i = A^j$, $\hat{A}^j = A^i$ and $\hat{g}^i = g^j$, $\hat{g}^j = g^i$; then $\gamma_i \hat{G} = \gamma_j G$, $\gamma_j \hat{G} = \gamma_i G$, and $\gamma_k \hat{G} = \gamma_k G$ for $k \neq i, j$.

Clearly, this axiom is stronger[30] than our symmetry axiom. Still, formula (3) establishes that

Proposition 15. *The value satisfies the axiom of full symmetry.*

Appendix E. The axioms for the value are tight

In this section we show that the axioms for the value are tight; i.e., if any one of them is dropped then the uniqueness theorem is no longer valid. Furthermore, the axioms are tight even if balanced threats and symmetry are replaced by their more restrictive versions ((BT4) and full symmetry, respectively). Again, for ease of

[30]The less demanding the condition of an axiom, the stronger is the axiom; hence the weaker is the uniqueness result that relies on the axiom, but the stronger is the corresponding existence result.

exposition we restrict attention to games with complete information, but analogous results are valid for Bayesian games.

Let, for all $i \in N$,

$$\gamma_i G = \frac{1}{n}(\delta G)(N), \tag{16}$$

i.e., each player receives the equitable allocation. It is easy to verify that

Claim 1. *The mapping $\gamma \colon \mathbb{G}(N) \to \mathbb{R}^n$ defined by (16) satisfies all the axioms except for the null-player axiom.*

Let, for all $i \in N$,

$$\gamma_i G = 0. \tag{17}$$

It is easy to verify that

Claim 2. *The mapping $\gamma \colon \mathbb{G}(N) \to \mathbb{R}^n$ defined by (17) satisfies all the axioms except for efficiency.*

For each integer $1 \leq k \leq n$, let π_k be the order $k, k+1, \ldots, n, 1, \ldots, k-1$, and let, for all $i \in N$,

$$\gamma_i G = \frac{1}{2n} \sum_{k=1}^{n} ((\delta G)(\mathcal{P}_i^{\pi_k} \cup i) - (\delta G)(\mathcal{P}_i^{\pi_k})), \tag{18}$$

where $P_i^{\pi_k}$ consists of all players j that precede i in the order π_k.

Claim 3. *The mapping $\gamma \colon \mathbb{G}(N) \to \mathbb{R}^n$ defined in (18) satisfies all the axioms except for symmetry.*

Proof. It is easy to verify that the axioms of null player, balanced threats, and additivity are satisfied. As for efficiency, it is sufficient to verify it for G such that δG is a unanimity game in $\mathbb{D}(N)$.

Let then δG be the unanimity game on T, i.e., $(\delta G)(S) = |T|$ if $S \supseteq T$, $-|T|$ if $S \subseteq N \backslash T$, and zero otherwise.

For $i \in T$, $(\delta G)(\mathcal{P}_i^{\pi_k} \cup i) = |T|$ if $\mathcal{P}_i^{\pi_k} \cup i \supseteq T$, i.e., if in the order π_k, i is the last among the members of T, and zero otherwise. Thus

$$\sum_{i \in T} \frac{1}{n} \sum_{k=1}^{n} (\delta G)(\mathcal{P}_i^{\pi_k} \cup i) = \frac{1}{n} \sum_{k=1}^{n} \sum_{i \in T} (\delta G)(\mathcal{P}_i^{\pi_k} \cup i) = \frac{1}{n} \sum_{k=1}^{n} |T| = |T|,$$

where the third equality follows from the fact that in each order π_k exactly one $i \in T$ is last among the members of T.

Similarly, for $i \in T$, $(\delta G)(\mathcal{P}_i^{\pi_k}) = -|T|$ if $\mathcal{P}_i^{\pi_k} \subseteq N \backslash T$, i.e., if in the order π_k, i is the first among the members of T, and zero otherwise. Since in each order π_k exactly one $i \in T$ is first among the members of T, we have

$$\sum_{i \in T} \frac{1}{n} \sum_{k=1}^{n} (\delta G)(\mathcal{P}_i^{\pi_k}) = \frac{1}{n} \sum_{k=1}^{n} \sum_{i \in T} (\delta G)(\mathcal{P}_i^{\pi_k}) = \frac{1}{n} \sum_{k=1}^{n} (-|T|) = -|T|.$$

By (18), $\sum_{i \in T} \gamma_i = \frac{1}{2}(|T| + |T|) = |T|$.

For $i \notin T$, $\mathcal{P}_i^{\pi_k} \cup i \subseteq T$ if and only if $\mathcal{P}_i^{\pi_k} \subseteq T$, and $\mathcal{P}_i^{\pi_k} \cup i \subseteq N \backslash T$ if and only if $\mathcal{P}_i^{\pi_k} \subseteq N \backslash T$. By (18) then, $\gamma_i G = 0$.

Thus $\sum_{i=1}^{n} \gamma_i = |T| = (\delta G)(N)$, completing the proof of efficiency.

To see that γ of equation (18) does not satisfy the symmetry axiom, consider the unanimity game on $\{1, 2, 5\}$ in the game with player set $\{1, \ldots, 5\}$.

Player 1 is first in T for the order π_1 and last in T for the order π_2. Thus $\gamma_1 = \frac{1}{10}(|T| - (-|T|)) = \frac{|T|}{5}$.

Player 2 is first in T for the order π_2 and last in T for the orders π_3, π_4 and π_5. Thus $\gamma_2 = \frac{1}{10}(|T| - (-3|T|)) = \frac{2|T|}{5}$.

But 1 and 2 are interchangeable. □

Next, observe that the vNM–Shapley value for strategic games satisfies all the axioms except for the axiom of balanced threats. (See Section 6.)

Finally, consider the following map. All dummy players in G receive the same as in the value formula (3), and the others share equally the remainder relative to $(\delta G)(N)$.

It is easy to verify that this solution satisfies all the axioms except for additivity. (It does not even satisfy consensus-shift invariance.)

We conclude this section by commenting on the axioms required to imply that the value, γG, is a function of δG. The axiom of balanced threats says that if $(\delta G)(S) = 0$ for any subset S then $\gamma G = 0$. It would seem then that this axiom alone would suffice. However, this is not the case.

A solution that obeys the axiom of balanced threats, symmetry, efficiency and the null player axiom but is not a function of δG can be constructed as follows.

Let δ and vG be as defined in (2) and (7), respectively, and fix $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t. $f(0, y) = f(x, 0) = 0$, and $f(x, x) = x \;\; \forall x, y$.

Define $\gamma(G)$ as the Shapley value of the coalitional game u with $u(S) := f((\delta G)(S), (vG)(S))$.

Claim 4. *The mapping $\gamma \colon \mathbb{G}(N) \to \mathbb{R}^n$ defined above satisfies the axioms of balanced threats, symmetry, efficiency, and null player, but it is not a function of δG.*

Remark 19. Theorem 2 can be strengthened. It is sufficient to assume the symmetry, null player, and efficiency axioms on the set of strategic games, and to assume the additivity axiom for the direct sum of a Bayesian game and a strategic game; but the conclusion still applies to all Bayesian games.

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