STOCHASTIC GAMES AND NONEXPANSIVE MAPS

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Abstract. This chapter studies asymptotic properties of the orbits of nonexpansive maps defined on a normed space, and relates these properties to properties of the value of two-person zero-sum games and to properties of the minmax of n-person stochastic games.

1. Introduction

Let $(X, \| \quad \|)$ be a Banach space. A map $T : X \to X$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$.

We encountered nonexpansive maps in earlier chapters. In [14] we had commented on the iterates of a Markov matrix P. Given a Markov matrix P, or more generally a linear operator P of norm 1 defined on a normed space X, the map $T : y \mapsto x + Py$ is nonexpansive (for every x). The iterates of T are given by

$$T^n y = (I + P + \ldots + P^{n-1})x + P^n y.$$

Since $\|\frac{P^n y}{n}\| \leq \frac{\|y\|}{n} \to_{n\to\infty} 0$, the limit of $\frac{I+P+\ldots+P^{n-1}}{n}$ exists if and only if the limit of $\frac{T^n y}{n}$ exists, and then both limits coincide.

The Shapley operator Ψ,

$$\Psi f[z] = \sup_x \inf_y \left(r(z,x,y) + \sum_{z'} p\,(z' \mid z,x,y) f(z') \right)$$

(where the sum is replaced with an integral in the case of an uncountable state space), appears, either explicitly or implicitly, in several other chapters of this volume, e.g., [19], [14], [15],[20], [21], [16], [23]. It maps a bounded real-valued function f defined on the state space S to a real-valued function

Ψf defined on the state space S. The map $f \mapsto \Psi f$ is nonexpansive with respect to the supremum norm, i.e., $\|\Psi f - \Psi g\|_\infty \leq \|f - g\|_\infty$.

The minmax value of the (unnormalized) n-stage stochastic game, V_n, is the n-th Ψ-iterate of the vector 0, $\Psi^n 0$. The minmax value of the (unnormalized) λ-discounted game, i.e., the game with discount factor $1-\lambda$, is the unique solution V_λ of the Ψ-λ-discounted equation: $\Psi((1-\lambda)V) = V$.

Similarly, the value of the n-stage (respectively, the λ-discounted) game of many other models of multi-stage games corresponds to the n-th iterate $\Psi^n 0$ (respectively, the solution of the Ψ-λ-discounted equation) where Ψ is a nonexpansive map.

In fact, an auxiliary stochastic game Γ' corresponds to every model of a repeated game Γ so that the value of the n-stage game Γ_n (respectively, the λ-discounted game Γ_λ) coincides with the value of the n-stage game Γ'_n (respectively, the λ-discounted game Γ'_λ) ([12], Chapter IV, Section 3).

Several results have established the existence of the limit of the normalized values of the n-stage games (respectively, of the λ-discounted games, and the equality of both limits). For example, these limiting results have been proved by Aumann and Maschler [1] for repeated games with incomplete information on one side, by Mertens and Zamir [13] for repeated games with incomplete information on both sides, and by Bewley and Kohlberg [2] for stochastic game with finitely many states and actions.

A natural question arises as to whether the limits of $\frac{1}{n}\Psi^n 0$ as $n \to \infty$ and of λV_λ as $\lambda \to 0+$, where Ψ is a nonexpansive operator defined on a normed space and V_λ is the unique solution of the equation $\Psi((1-\lambda)V) = V$, exist (and are equal). This question was the initiator of the investigations leading to [7], [8], [17] and [9], which are summarized in this chapter.

Let us point out right now that the nonexpansiveness of Ψ by itself is not sufficient to guarantee the convergence of $\frac{1}{n}\Psi^n 0$ as $n \to \infty$. Additional properties of either the normed space or the nonexpansive operator are needed.

In Section 2 we state the characterization of the normed spaces $(X, \| \|)$ for which the strong (respectively, weak) limit of $\frac{1}{n}\Psi^n 0$ as $n \to \infty$ exists for every nonexpansive map $\Psi : X \to X$ (Theorems 2 and 3 respectively). The characterization is based on Theorem 1 which states an important property of nonexpansive maps. In addition, the norm convergence of $\frac{1}{n}T^n 0$ is proved whenever in addition to T being nonexpansive the function $\lambda \to \lambda V_\lambda$ is of bounded variation (Theorem 4), and thus also whenever $T : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive and semialgebraic (Theorem 5).

Section 3 is based on [17]. It introduces the generalized orbits of a nonexpansive map. Informally, a generalized iterate is obtained by compositions of weighted averages and classical iterates.

Section 4 presents applications of the results of Section 3 to stochastic games with an uncertain number of stages.

Section 5 is based on [9]. It introduces the stochastic analog of the nonexpansive map: the nonexpansive stochastic process. It is also a generalization of Banach space-valued martingales.

The above-mentioned results of [2] also follow from the existence of a (uniform) value of stochastic games [11], [16]. The present chapter will include a proof of these results using the fact that in the case of finitely many states and actions the Shapley operator Ψ is nonexpansive and semi-algebraic. It follows therefore that the same conclusion holds also for the λ-discounted minmax (of player i) v^i_λ and the n-stage minmax (of player i) v^i_n of n-person stochastic games with finitely many states and actions.

2. Nonexpansive Maps

This section provides conditions on a Banach space $(X, \| \ \|)$ and a nonexpansive map $T : X \to X$ that imply the convergence of the the sequence $\frac{1}{n}T^n 0$.

Obviously, if T is nonexpansive, so is each iterate T^n of T. Therefore, $\|T^n x - T^n 0\| \leq \|x\|$ for every $x \in X$, and thus $\frac{1}{n}T^n x$ (respectively, $\|\frac{1}{n}T^n x\|$) converges if and only if $\frac{1}{n}T^n 0$ (respectively, $\|\frac{1}{n}T^n 0\|$) converges. In addition, for every $k, \ell \geq 0$ we have

$$\|T^{kn+\ell}x - x\| \leq k\|T^n x - x\| + \ell\|Tx - x\|.$$

Therefore, $\limsup_{m\to\infty} \|\frac{T^m x}{m}\| \leq \inf_{n\geq 1} \|T^n x - x\|/n \leq \liminf_{m\to\infty} \|\frac{T^m x}{m}\|$, which proves that the limit of $\|\frac{T^m x}{m}\|$ as $m \to \infty$ exists and that the limit equals $\inf_{n\geq 1} \|T^n x - x\|/n$ (and is independent of x). Moreover, Theorem 1 implies in particular that the limit of $\|\frac{1}{n}T^n x\|$ equals $\inf_{x\in X} \|Tx - x\|$.

The sequence $\frac{1}{n}T^n x$ need not converge, even if X is finite dimensional. Indeed, for every norm $\| \ \|$ on $\mathbb{R}^n$ such that the unit ball is not strictly convex there is a map $T : \mathbb{R}^n \to \mathbb{R}^n$ which is nonexpansive with respect to the norm $\| \ \|$ and such that the limit of $\frac{T^n y}{n}$ as $n \to \infty$ does not exist [7]. However, if the unit ball of $(\mathbb{R}^n, \| \ \|)$ is strictly convex then the limit does exist for every nonexpansive map T. More generally, if $(X, \| \ \|)$ is a uniformly convex normed space, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| > \varepsilon$ we have $\|(x + y)/2\| < 1 - \delta$, then for every nonexpansive map $T : X \to X$ the limit of $\frac{T^n 0}{n}$ exists [8].

If $P : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator of norm 1 w.r.t. a norm $\| \, \|$, then, as can be seen from the Jordan decomposition of P, P is also of norm one with respect to a Hilbertian norm. The unit ball of a Hilbert space is uniformly convex. Therefore, the result on the convergence of $\frac{T^n}{n}$ in a uniformly convex space implies the convergence of $\frac{I+P+...+P^{n-1}}{n}$ as $n \to \infty$.

The Shapley operator Ψ is nonexpansive w.r.t. the supremum norm which is not uniformly convex, and therefore additional information concerning the Shapley operator is used in deriving the convergence of $\frac{1}{n}\Psi^n 0$.

The next result is essential for several results that follow. In particular, it enables us to characterize all the Banach spaces for which the limit of $\frac{1}{n}T^n x$ exists for every nonexpansive map T.

Given a Banach (or normed) space X we denote by $S(X)$ the set of all vectors $x \in X$ with $\|x\| = 1$, and X^* denotes the dual of X.

Theorem 1 (Kohlberg and Neyman [7]) *Let $(X, \| \ \|)$ be a normed space. Assume that $T : X \to X$ is nonexpansive and that $\rho := \inf_x \|Tx - x\| > 0$. Then for every $x \in X$ there is $f_x \in S(X^*)$ such that*

$$f_x(T^n x) \geq f_x(x) + n\rho.$$

Moreover,[1] *we could find such a continuous linear functional f_y in the w^* closure of the extreme points of the unit ball of X^*.*

The reader is referred to [7] for the proof. An immediate corollary of Theorem 1 is that for all $x \in X$ we have $\|T^m x - x\| \geq m \inf_y \|Ty - y\|$ and therefore $\lim_{m\to\infty} \|\frac{T^m x}{m}\| = \rho$.

The value of the (unnormalized) n-stage repeated game with incomplete information on one side is the n-iterate of a nonexpansive map Φ defined on the space of continuous functions over the simplex $\Delta(K)$ where K is the state space. An additional property that follows is that Φ is covariant with respect to the addition of linear functions. Exercise 5, p. 298 of [12] derives the convergence of the normalized values $\frac{1}{n}\Phi^n 0$ $(= v_n(p))$ as $n \to \infty$ and the formula of the limit using Theorem 1.

We now turn to the characterization of all Banach spaces for which the limit of $\frac{1}{n}T^n(0)$ as $n \to \infty$ exists for every nonexpansive map $T : X \to X$.

The norm of a Banach space X is *Fréchet differentiable* (away from zero) whenever for every $x \in X$ with $x \neq 0$, $\lim_{\lambda\to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$ exists uniformly in $y \in S(X) \equiv \{x \in X : \|x\| = 1\}$. A Banach space X is *strictly convex* if $\|x + y\| < 2 \ \forall x, y \in S(X)$ with $x \neq y$.

Theorem 2 (Kohlberg and Neyman [7]) *The following two conditions on a Banach space X are equivalent.*

$$\textit{For every nonexpansive map } T : X \to X, \ \ T^n(0)/n \textit{ converges.} \tag{2}$$

$$\textit{The norm of } X^* \textit{ is differentialble.} \tag{3}$$

[1]This conclusion is not mentioned in [7] but is easily derived from the proof there.

Theorem 3 (Kohlberg and Neyman [7]) *The following two conditions on a Banach space X are equivalent.*

$$\text{For every nonexpansive map } T : X \to X,\ T^n(0)/n \text{ converges weakly.} \tag{4}$$

$$X \text{ is reflexive and the norm of } X \text{ is strictly convex.} \tag{5}$$

Given a nonexpansive map $T : X \to X$, we will denote (in what follows) by $x(r)$, $r > 0$, the unique solution of the equation $\frac{1}{1+r}T(x(r)) = x(r)$. The equality $rx(r) = Tx(r) - x(r)$ implies that $\|rx(r)\| \geq \inf\{\|Tz - z\| : z \in X\}$. For every $z \in X$ and $r > 0$ we have $\|x(r) - z\| \geq \|(1 + r)x(r) - Tz\| \geq (1 + r)\|x(r) - z\| - r\|z\| - \|Tz - z\|$; hence $r\|x(r) - z\| \leq r\|z\| + \|Tz - z\|$, so that $\|rx(r)\| \leq 2r\|z\| + \|Tz - z\| \to_{r\to 0+} \|Tz - z\|$ (in particular $\|r(x(r))\| \leq \|T(0)\|$). Therefore, $\limsup_{r\to 0+} \|rx(r)\| \leq \inf_{z\in X} \|Tz - z\|$. Therefore, $\lim_{r\to 0+} \|rx(r)\| = \inf_{z\in X} \|Tz - z\|$ [8].

If T is the Shapley operator associated with a two-person zero-sum stochastic game then $(1 + r)x(r)$ corresponds to the unnormalized value of the discounted stochastic game with discount factor $\frac{1}{1+r}$.

Condition (3) (respectively, (5)) is equivalent to the strong (respectively, weak) convergence of $rx(r)$ as $r \to 0+$ for every nonexpansive map $T : X \to X$, and under these conditions the strong (repectively, weak) limits $\lim_{r\to 0+} rx(r)$ and $\lim_{n\to\infty} \frac{1}{n}T^n 0$ coincide [7].

Theorem 4 *Let X be a Banach space and $T : X \to X$ a nonexpansive map for which $rx(r)$, $0 < r \leq 1$, is of bounded variation. Then the limit $\lim_{n\to\infty} \frac{1}{n}T^n(0)$ exists and equals $\lim_{r\to 0+} rx(r)$.*

Proof. As $rx(r)$, $0 < r \leq 1$, is of bounded variation, the limit of $rx(r)$ as $r \to 0+$ exists. Letting $x_n = \frac{1}{n}x(\frac{1}{n})$ we deduce in particular that $\lim_{n\to\infty} x_n = \lim_{r\to 0+} rx(r)$.

By the triangle inequality and the nonexpansiveness of T,

$$\begin{aligned}
\|T^{n+1}(0) - x(\frac{1}{n+1})\| &\leq \|T^{n+1}(0) - Tx(\frac{1}{n})\| + \|T(x(\frac{1}{n})) - x(\frac{1}{n+1})\| \\
&\leq \|T^n(0) - x(\frac{1}{n})\| + \|T(x(\frac{1}{n})) - x(\frac{1}{n+1})\| \\
&= \|T^n(0) - x(\frac{1}{n})\| + (n + 1)\|x_n - x_{n+1}\|.
\end{aligned}$$

Summing the above inequalities over $n = 1, \ldots, m$, we deduce that

$$\|T^{m+1}(0) - x(\frac{1}{m+1})\| \leq \|T(0) - x(1)\| + \sum_{i=1}^{m}(i + 1)\|x_i - x_{i+1}\|.$$

As the sequence x_i, $i = 1, 2, \ldots$, is of bounded variation, $\frac{1}{m}\sum_{i=1}^{m}(i+1)\|x_i - x_{i+1}\| \to 0$ as $m \to \infty$, and therefore $\lim_{n\to\infty}\frac{1}{n}T^n(0)$ exists and equals $\lim_{n\to\infty} x_n = \lim_{r\to 0+} rx(r)$. ∎

Theorem 5 *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a semialgebraic map which is nonexpansive with respect to some norm. Then the limits $\lim_{r\to 0+} rx(r)$ and $\lim_{n\to\infty}\frac{1}{n}T^n(0)$ exist and are equal.*

Proof. The set $V = \{(r,x) \in \mathbb{R} \times \mathbb{R}^n \mid r > 0 \text{ and } \frac{1}{1+r}Tx = x\}$ is semialgebraic. For every $r > 0$ there is a unique point $x \in \mathbb{R}^n$ with $(r,x) \in V$ and $\|rx(r)\| \leq \|T0\|$. Thus the function $x : (0,1] \to \mathbb{R}^n$ with $(r, x(r)) \in V$ is semialgebraic and bounded, and thus of bounded variation. ∎

3. Generalized Orbits of Nonexpansive Maps

A classical iterate of a map T from a set X to itself is the composition of T with itself several times. The present section introduces a generalized iterate of a nonexpansive map $T : X \to X$ when X is a Banach space.

Informally, a generalized iteration is obtained by compositions of weighted averages and classical iterates. An example of a generalized iterate is

$$\Phi = \frac{1}{3}T^5 + \frac{2}{3}T(\frac{2}{5}T^3 + \frac{3}{5}T^2(\sum_{n=0}^{\infty} a_n T^n))$$

where $a_n \geq 0$ with $\sum_n a_n n < \infty$ and T^0 is the identity. The map $\Phi : X \to X$ can be derived from the following sequence of maps.

$$\begin{array}{lllllll}
\Phi_1 & = & T^3 & \qquad & \Phi_2 & = & \sum_{n=0}^{\infty} a_n T^n \\
\Phi_3 & = & T^2 \circ \Phi_2 & & \Phi_4 & = & \frac{2}{5}\Phi_1 + \frac{3}{5}\Phi_3 \\
\Phi_5 & = & T \circ \Phi_4 & & \Phi_6 & = & T^5 \\
\Phi_7 & = & \frac{1}{3}\Phi_6 + \frac{2}{3}\Phi_5 & & \Phi_7 & = & \Phi.
\end{array}$$

Note that Φ_1 is a classical iterate. Φ_2 is a weighted average of classical iterates. Φ_3 is a composition of Φ_2 with a classical iterate T^2. Φ_4 is a weighted average of the previously defined Φ_1 and Φ_3. Φ_5 is the composition of T and Φ_4. Φ_6 is a classical iterate and Φ_7 is a weighted average of Φ_4 and Φ_5.

The nonexpansiveness of T and the triangle inequality imply that $\|\Phi_i 0\| \leq t(\Phi_i)\|T0\|$ and more generally

$$\|\Phi_i y - y\| \leq t(\Phi_i)\|Ty - y\| \quad \forall y \in X$$

where $t(\Phi_1) = 3$, $t(\Phi_2) = \sum_{n=0}^{\infty} na_n$, $t(\Phi_3) = 2 + t(\Phi_2)$, $t(\Phi_4) = \frac{2}{5}t(\Phi_1) + \frac{3}{5}t(\Phi_3)$, $t(\Phi_5) = 2 + t(\Phi_4)$, $t(\Phi_6) = 5$ and $t(\Phi_7) = \frac{1}{3}t(\Phi_6) + \frac{2}{3}t(\Phi_5)$.

The main result of the present section (Proposition 2 or Theorem 6) will show in particular that for every $y \in X$ there is a linear functional $f_y \in S(X^*)$ such that

$$f(\Phi_i y - y) \geq t(\Phi_i)\rho$$

where $\rho := \inf_{x \in X} \|Tx - x\|$.

We now turn to the formal definition of a generalized iteration. A *generalized iteration* Θ consists of

– a probability space $\langle \Omega, \mathcal{F}, P \rangle$
– an increasing sequence of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}$
– a vector-valued random variable $N : \Omega \to \mathbb{N}_0$ with finite expectation

where the σ-algebra generated by $\cup_i \mathcal{F}_i$ and the events $N = k$, $k \geq 0$, span $\mathcal{F}$.

The generalized iterate T^Θ of the nonexpansive map $T : X \to X$ is a map defined on a space of bounded X-valued $\mathcal{F}$-measurable functions. It is defined in particular on X (where $x \in X$ is identified with the constant-valued function) with values in X. T^Θ is defined as follows.

Let $\mathcal{G}_k$ be the σ-algebra spanned by $\mathcal{F}_k$ and the event $N > k$. Define the maps φ_k from the space of X-valued integrable functions f for which the conditional expectation $E(f \mid \mathcal{G}_{k-1})$ are well defined to the space of X-valued $\mathcal{G}_{k-1}$-measurable functions by

$$\begin{aligned} \varphi_k f &= T^{I(N \geq k)} E(f \mid \mathcal{G}_{k-1}) \\ &:= I(N < k) E(f \mid \mathcal{G}_{k-1}) + I(N \geq k) T(E(f \mid \mathcal{G}_{k-1})). \end{aligned}$$

Note that for every two X-valued integrable functions f and g we have

$$\int \|\varphi_k f - \varphi_k g\| \, dP \leq \int \|f - g\| \, dP. \tag{6}$$

It follows by induction on m that

$$\int \|\varphi_k \circ \ldots \circ \varphi_{k+m} g - E(g \mid \mathcal{G}_{k-1})\| \, dP$$

$$\leq E((N - k + 1)^+)(\|T0\| + \|g\|_\infty) \to_{k \to \infty} 0 \tag{7}$$

where x^+ stands for $\max(x, 0)$ and $\|g\|_\infty := \sup_\omega \|g(\omega)\|$. Indeed, on $N \geq k$ $\|\varphi_k h - E(h \mid \mathcal{G}_{k-1}))\| \leq \|T0\| + \|E(h \mid \mathcal{G}_{k-1}))\|$, and on $N < k$ the two functions $\varphi_k h$ and $E(h \mid \mathcal{G}_{k-1})$ coincide. Therefore,

$$\int \|\varphi_k h - E(h \mid \mathcal{G}_{k-1})\| \, dP \leq P(N \geq k)(\|T0\| \|h\|) \tag{8}$$

which proves inequality (7) for $m = 0$. For $m > 0$, set $f = \varphi_{k+1} \circ \ldots \circ \varphi_{k+m} g$. We have

$$\int \|\varphi_k f - \varphi_k E(g \mid \mathcal{G}_k)\| \, dP \leq \int \|f - E(g \mid \mathcal{G}_k)\| \, dP \tag{9}$$

which by the induction hypothesis is

$$\leq E((N-k)^+)(\|T0\| + \|g\|_\infty). \tag{10}$$

As $E((N-k+1)^+) = P(N \geq k) + E((N-k)^+)$, summing inequalities (8) with $h = E(g \mid \mathcal{G}_k)$ and (10) proves (7).

Fix a bounded $\mathcal{F}$-measurable function $f : \Omega \to X$ (for which the functions $\varphi_0 \circ \ldots \circ \varphi_k E(f \mid \mathcal{F}_{k+1})$ are well defined). It follows from (6) and (7) that the sequence of X-vectors $E(\varphi_0 \circ \ldots \circ \varphi_k E(f \mid \mathcal{F}_{k+1}))$ is a Cauchy sequence and thus converges. Its limit is defined as $T^\Theta f$. Every element $x \in X$ is also identified with the constant function $\omega \mapsto x$, and thus $T^\Theta x$ is defined.

For every generalized iterate $\Theta = \langle (\Omega, \mathcal{F}, P), (\mathcal{F}_k)_{k \geq 0}, N \rangle$ and a positive integer n we denote by $\Theta \wedge n$ the generalized iterate $\langle (\Omega, \mathcal{F}, P), (\mathcal{F}_k)_{k \geq 0}, N \wedge n \rangle$ where $N \wedge n := \min(N, n)$. It follows that for every bounded measurable function $f : \Omega \to X$ we have

$$T^{\Theta \wedge n} f = E(\varphi_0 \circ \ldots \circ \varphi_{n-1} E(f \mid \mathcal{F}_n))$$

$$\text{and} \qquad T^{\Theta \wedge n} f \to_{n \to \infty} T^\Theta f.$$

The next proposition generalizes the inequality $\|T^n y - y\| \leq n \|Ty - y\|$ to the generalized orbit of a nonexpansive map.

Proposition 1 *Let X be a Banach space and $T : X \to X$ a nonexpansive map. Then, for every generalized iterate $\Theta = \langle (\Omega, \mathcal{F}, P), (\mathcal{F}_i)_{i \geq 0}, N \rangle$ and every $y \in X$ we have*

$$\|T^\Theta y - y\| \leq E(N) \|Ty - y\|.$$

The next proposition generalizes Theorem 1 to the generalized orbit of a nonexpansive map.

Proposition 2 *Let X be a normed space. Assume that $T : X \to X$ is nonexpansive and $\rho := \inf_x \|Tx - x\| > 0$. Then for every $y \in X$ there exists a linear functional $f_y \in S(X^*)$ such that for every generalized iterate $\Theta = \langle (\Omega, \mathcal{F}, P), (\mathcal{F}_i)_{i \geq 0}, N \rangle$ we have*

$$f_y(T^\Theta y - y) \geq E(N) \rho.$$

Moreover, we could find such a continuous linear functional f_y in the w^* closure of the extreme points of the unit ball of X^*.

A generalized iterate Θ is called finite if Ω is finite. If $f : \Omega \to X$ has finite range and is measurable w.r.t. $\mathcal{G}_n$, e.g., if f is the constant function y, then the X-valued functions $\varphi_k \circ \ldots \circ \varphi_n f$ have finite range and are measurable w.r.t. $\mathcal{G}_{k-1}$. Therefore, if in addition $N \leq n$ everywhere, the generalized iterate $T^{\Theta} f$ equals $T^{\Theta'} f$ where Θ' is a finite generalized iterate.

Therefore, in view of the approximation of $T^{\Theta} y$ by $T^{\Theta \wedge n} y$, it is sufficient to prove Propositions 1 and 2 for finite generalized iterates.

For every $y \in X$ the set of all $T^{\Theta} y$ where Θ is a finite generalized iterate is denoted by C_y. It follows that C_y is the smallest convex subset of X that contains y and is invariant under T, i.e., $T(C_y) \subset C_y$.

Let γ_y^+ (γ_y^-) be the smallest (largest) concave (convex) extended real-valued function on C_y, i.e., $\gamma_y^+, \gamma_y^- : C^y \to R \cup \{-\infty, \infty\}$ such that

$$(a) \quad \gamma_y^+(y) \geq 0, \quad (a') \quad \gamma_y^-(y) \leq 0,$$

and for every $x \in C_y$

$$(b) \quad \gamma_y^+(Tx) \geq \gamma_y^+(x) + 1 \quad (b') \quad \gamma_y^-(Tx) \leq \gamma_y^-(x) + 1.$$

Note that the pointwise infimum of all concave extended real-valued functions that obey (a) and (b) is concave and obeys (a) and (b) and therefore γ_y^+ is well defined. Similarly, the supremum of all convex extended real-valued functions that obey (a′) and (b′) is convex and obeys (a′) and (b′) and therefore γ_y^- is well defined.

In what follows we state (without proof) two properties of the functions γ_y^+ and γ_y^-. Let A_y be the smallest convex subset of $X \times \mathbb{R}$ such that $(y, 0) \in A_y$ and such that $(Tx, t+1) \in A_y$ whenever $(x, t) \in A_y$. It turns out that $\gamma_y^+(x) = \sup\{t \mid (x, t) \in A_y\}$ and that $\gamma_y^-(x) = \inf\{t \mid (x, t) \in A_y\}$. The other property is that $\gamma_y^+(x)$ equals the supremum of all $E(N)$ where $\Theta = \langle \ldots, N \rangle$ is a finite generalized iterate such that $T^{\Theta} y = x$.

Propositions 1 and 2 follow from the above comments and the following theorem which is stated without proof.

Theorem 6 *Let X be a normed space, $T : X \to X$ a nonexpansive map and $y \in X$. Then, $\gamma_y^+(x) \geq \gamma_y^-(x)$ for every $x \in C_y$, and*

$$\|x - y\| \leq \gamma_y^-(x) \|Ty - y\|.$$

If $\rho := \inf_x \|Tx - x\| > 0$, there exists a continuous linear functional $f_y \in S(X^*)$ such that, for every $x \in C_y$,

$$f_y(x) - f_y(y) \quad \geq \quad \gamma_y^+(x) \rho. \tag{11}$$

Moreover, we could find such a continuous linear functional f_y in the w^* closure of the extreme points of the unit ball of X^*.

4. Stochastic Games with Uncertain Duration

This section includes an application of the previous section to stochastic games where the number of stages is unknown. The information on the number of stages is identical for all players and the information gets refined as the game progresses.

Before presenting the general model we start with several examples that illustrate that the values of the stochastic games with uncertain duration reduces to the computation of a generalized Ψ-iterate of the vector 0.

4.1. EXAMPLE 1

Fix a two-person zero-sum stochastic game Γ. Consider a decreasing sequence $\alpha = (\alpha_t)_{t=1}^\infty$, $\alpha_t \downarrow 0$, with $\sum_{t=1}^\infty \alpha_t < \infty$. The α-weighted stochastic game, Γ_α, is the game where the evaluation of a stream (x_t) of stage payoffs is $\sum_t \alpha_t x_t$. Assume that $\alpha_1 \leq 1$ and set $\alpha_0 = 1$. The game is equivalent to the model of the stochastic game Γ_N where the number of stages (the duration) is a random variable N such that $\Pr(N \geq t) = \alpha_t$ and the players are informed about the distribution of N but do not receive any information about the value of N. The two models are equivalent in the following sense. The set of strategies in both game models are identical, and the payoff to each strategy pair, $E_{\sigma,\tau}(\sum_{t=1}^\infty \alpha_t r(z_t, a_t))$ and $E_{\sigma,\tau,N}(\sum_{t=1}^N r(z_t, a_t))$, coincide. Therefore, the value of Γ_α exists if and only if the value of Γ_N, $\operatorname{val}\Gamma_N$, exists and then both values coincide. If N' is another random variable with nonnegative integer values and $\alpha'_t := P(N' \geq t)$ then for every strategy pair σ and τ we have $|E_{\sigma,\tau}(\sum_{t=1}^\infty \alpha_t r(z_t, a_t)) - E_{\sigma,\tau}(\sum_{t=1}^\infty \alpha'_t r(z_t, a_t))| \leq \sum_{t=1}^\infty |\alpha_t - \alpha'_t| K$ where K bounds all absolute values $|r(z, a)|$ of stage payoffs. Therefore, $|\operatorname{val}\Gamma_N - \operatorname{val}\Gamma_{N'}| \leq \sum_{t=1}^\infty |\alpha_t - \alpha'_t| K$. In particular, $|\operatorname{val}\Gamma_N - \operatorname{val}\Gamma_{N\wedge n}| \leq \sum_{t>n} \alpha_t K$, where $N \wedge n$ is the nonnegative-integer-valued random variable whose distribution given by $P(N \wedge n \geq k)$ equals α_k if $k \leq n$ and equals 0 if $k > n$.

The common value $\operatorname{val}\Gamma_N$ can thus be approximated by the values $\operatorname{val}\Gamma_{N\wedge n}$ of the truncated games $\Gamma_{N\wedge n}$. Val $\Gamma_{N\wedge n}$ can be expressed by means of the Shapley operator Ψ. For every n let $V_{N,n}$ be defined by

$$V_{N,n} = (\Psi_1 \circ \Psi_2 \circ \ldots \circ \Psi_n)(0)$$

where $\Psi_t V = \Psi(\frac{\alpha_t}{\alpha_{t-1}} V)$ and $\alpha_0 = 1$. It follows by induction on n that $V_{N,n} = \operatorname{val}\Gamma_{N\wedge n}$.

4.2. EXAMPLE 2

The general model of a game with a symmetric uncertain duration process includes public incremental information about the uncertain duration that the players receive as the game evolves. First we illustrate how to express the unnormalized value of the corresponding game in three examples. In each one the uncertain number of stages N is either n or m, each equally likely. The three examples differ in the structure of the information about N.

Fix a zero-sum two-person stochastic game Γ and let Ψ denote the corresponding Shapley operator. Assume that before the start of play the players are informed about the number of stages. The value of the unnormalized game is thus the average of the value of the n-stage game and the m-stage game; i.e., it is represented by $\frac{1}{2}\Psi^n 0 + \frac{1}{2}\Psi^m 0$.

If the players do not receive any information about N and $n > m$ the value of the corresponding game is $\Psi^m(\frac{1}{2}\Psi^{n-m}0)$. Indeed, following the play at stage m, the players can assume without loss of generality that there are additional $n - m$ stages. Therefore, the expected future payoff in stages $m+1, \ldots, n$ equals $\frac{1}{2}\Psi^{n-m}0$. Therefore, by backward induction it follows that the value of the entire game equals $\Psi^m(\frac{1}{2}\Psi^{n-m}0)$.

However, if $m, n > 0$ and the players are informed of the value of N only following the play at stage 1 the value of the unnormalized game is $\Psi(\frac{1}{2}\Psi^{n-1}0 + \frac{1}{2}\Psi^{m-1}0)$. Indeed, following the play at stage 1 the expected total payoff in stages $t \geq 1$, as a function of the state in stage 2, is $\frac{1}{2}\Psi^{n-1}0 + \frac{1}{2}\Psi^{m-1}0$. Therefore, the value of the game equals $\Psi(\frac{1}{2}\Psi^{n-1}0 + \frac{1}{2}\Psi^{m-1}0)$. The expected number of stages in each of the three above-mentioned game models is $(m+n)/2$. Let us however compute this expected number in two additional ways that correspond to the final formula of V_Θ. First, $E(N) = m + (n-m)/2$, and second, $E(N) = 1 + (m - 1 + n - 1)/2$.

4.3. EXAMPLE 3

Consider for example a zero-sum two-person stochastic game Γ where the uncertain number of stages N is either 6 or 7 or 8 or 9, each equally likely. Assume that following the play at stage 3 the players are informed as to whether the number of stages equals 6 or not. Following the play at stage 5 the players are informed whether or not the number of stages is 9. The value V_N of the unnormalized game can be expressed by generalized iterates of the Shapley operator as follows.

$$V_N = \Psi^3\left(\frac{1}{4}\Psi^3 0 + \frac{3}{4}\Psi^2\left(\frac{2}{3}\Psi^2\left(\frac{1}{2}\Psi 0\right) + \frac{1}{3}\Psi^4 0\right)\right).$$

4.4. SYMMETRIC UNCERTAIN DURATION PROCESS

The uncertain number of stages is an integer-valued random variable N defined on a probability space $(\Omega, \mathcal{F}, \mu)$ with finite expectation $E_\mu(N)$.

The players receive partial information about the value of N as the game proceeds. Formally, the incremental information regarding N is modelled as an increasing sequence of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$, $t \geq 0$. The interpretation is that $\mathcal{F}_t$ represents the information on N of the players prior to the play at stage $t+1$. Equivalently, letting $d_t : \Omega \to D_t$ be measurable ($(D_t, \mathcal{D}_t)$ is a measurable space) random variables such that the σ-algebra generated by $d_1, \ldots, d_t$, $\sigma(d_1, \ldots, d_t)$, equals $\mathcal{F}_t$, the players receive information about N via a sequence of public signals $d_0, d_1, \ldots$. Each signal d_t is a measurable function defined on the probability space $(\Omega, \mathcal{B}, \mu)$ with finite range D_t.

A *public (symmetric) uncertain duration process*[2] Θ is a generalized iterate $\Theta = \langle (\Omega, \mathcal{F}, P), (\mathcal{F}_k)_{k \geq 0}, N \rangle$.

It is called *finite* (*countable*) if Ω is finite (countable). If Ω is finite, then N is bounded and the sequence $\mathcal{F}_k$ is eventually constant; i.e., there is m such that for every $k \geq m$, $\mathcal{F}_k = \mathcal{F}_m$.

Public uncertain duration processes model symmetric incomplete information about the (active) number of stages, where the information is revealed (to the players) gradually over time but is independent of past history.

The interpretation is as follows. The number of (active) stages is N, and it depends on $\omega \in Y$. The information of the players on the random duration of the game, prior to the selection of actions at stage t, is given by $\mathcal{F}_{t-1}$: given $\theta \in Y$, the atom of $\mathcal{F}_{t-1}$ that includes θ is revealed to the players before the play at stage t. Thus, in a stochastic game with a finite public uncertain duration process, the action of a player at stage t may depend on the past play $z_1, a_1, \ldots, z_t$ and the atom of[3] $\mathcal{F}_{t-1}$. The resulting set of strategies is called $\mathcal{F}_{k\geq 0}$-adapted strategies.

4.5. THE RESULTS

In the following theorem we assume a fixed two-person constrained stochastic game with state space S (see [15], Section 7). $B(S)$ stands for the normed space of the bounded measurable functions $f : S \to \mathbb{R}$. The map $\Psi : B(S) \to B(S)$ is defined by

$$\Psi f(z) = \sup_{\nu \in X^1(z)} \inf_{\mu \in X^2(z)} r(z, \nu, \mu) + \int f(z') dp(z' \mid z, \nu, \mu).$$

We use the notations of Section 3: the subsets C_y and A_y of $B(S)$ and $B(S) \times \mathbb{R}$ respectively and the function $\gamma_y^+ : C_y \to \mathbb{R}$ are associated to Ψ and defined as in Section 3. Given a public uncertain duration process $\Theta =$

[2]The term was introduced in [18].

[3]I.e., the information regarding the uncertain duration which is available prior to the play at stage t.

$\langle(\Omega,\mathcal{F},P),\mathcal{F}_{k\geq 0},N\rangle$, we denote by $\Sigma^i_{X^i,N}$ all $\mathcal{G}_{k\geq 0}$-adapted X^i-constrained behavioral strategies (where $\mathcal{G}_k$ is the σ-algebra generated by $\mathcal{F}_k$ and the event $N>k$).

Theorem 7 *a) For every public uncertain duration process* $\Theta=\langle *,*,N\rangle$, *the point*

$$x=\sup_{\sigma^1\in\Sigma^1_{X^1,N}}\ \inf_{\sigma^2\in\Sigma^2_{X^2,N}} E_{\sigma^1,\,\sigma^2}\left(\sum_{t=1}^{N} r(z_t,a_t)\right)$$

equals $\Psi^\Theta 0$.
b) For any point $(x,t)\in A_0$, *there is a finite public uncertain duration process* $\Theta\langle *,*,N\rangle$ *with* $E(N)=t$ *and*

$$x=\sup_{\sigma^1\in\Sigma^1_{X^1,N}}\ \inf_{\sigma^2\in\Sigma^2_{X^2,N}} E_{\sigma^1,\,\sigma^2}\left(\sum_{t=1}^{N} r(z_t,a_t)\right).$$

Given a real-valued function $v:S\to\mathbb{R}$ we denote $v^+=\sup_{z\in S}v(z)$ and $v^-=\inf_{z\in S}v(z)$. In what follows we consider a two-player zero-sum constrained stochastic game, and we let the functions v_n, $n\geq 1$, and V_Θ, where Θ is an uncertain duration process, stand for either the corresponding maxmin $\underline{v}_n$ and $\underline{V}_\Theta$ respectively or the corresponding minmax $\bar{v}_n$ and $\bar{V}_\Theta$ respectively.

The norm dual of $(\mathbb{R}^k,\|\ \|_\infty)$ is $(\mathbb{R}^k,\|\ \|_1)$, and the extreme points of the unit ball of $(\mathbb{R}^k,\|\ \|_1)$ are the unit vectors $(0,\ldots,0,\pm 1,0,\ldots)$. Therefore, the next theorem is a direct implication of Theorems 7 and 6.

Theorem 8 *Fix a two-person constrained stochastic game with a finite state set* S. *Let* V_Θ *denote either the minmax or the maxmin of the stochastic game with uncertain duration process* Θ. *Assume* $\lim_{n\to\infty}\frac{1}{n}\|V_n\|>0$. *Then, there are states* $z,z'\in S$ *such that for any uncertain duration process* $\Theta=\langle *,*,N\rangle$,

$$V_\Theta(z))\geq E(N)\lim_{n\to\infty} v_n^+$$

and

$$V_\Theta(z'))\leq E(N)\lim_{n\to\infty} v_n^-.$$

The following example illustrates that the conclusion of Theorem 8 no longer holds when the state space is countable. The state space S is the set of all integer pairs (i,j) with $0\leq j\leq i$. The payoff function is independent of the actions and is given by $r(i,j)=1$ if $j<i$ and $=0$ if $j=i$. The transition is deterministic and independent of the actions; $P((i,j+1)\mid (i,j))=1$ if $j<i$ and $P(i,j\mid i,j)$ if $j=i$. Note that $V_n(i,j)=\min(i-$

$j-1,n)$. Thus $v_n^+\|v_n\|=1$. However, for every state z $\lim_{n\to\infty} v_n(z)=0$, and thus there is no state z such that $V_n(z)\geq n$ for every n.

The proof of the previous theorem does not extend to the countable state space for the following reason. The Shapley operator acts here on $\ell_\infty(S)$, the bounded functions on the the countable state space S. The norm dual of $\ell_\infty(S)$ consists of all finitely additive measures on S, whose extreme points are no longer the $(\pm)$ Dirac measures, but rather all $\{-1,1\}$-valued finitely additive measures.

The next theorem extends Theorem 8 to two-person zero-sum stochastic games with an infinite state space S (either a countable space S or a measurable space $(S,\mathcal{S})$) and a uniformly bounded stage payoff function r. The conclusion in Theorem 8 regarding the existence of the states z and z' is replaced with the existence of finitely additive probability measures.

Theorem 9 *a) The limits of v_n^+ and v_n^- exist.*
b) There exists an atomic finitely additive probability measure μ such that for every public uncertain duration $\Theta=\langle(\Omega,\mathcal{F},P),\mathcal{F}_{k\geq 0},N\rangle$,

$$\int V_\Theta(z)d\mu(z)\geq n\lim_{n\to\infty} v_n^+.$$

c) There exists an atomic finitely additive measure μ such that for any uncertain duration $\langle Y,\mathcal{F}_{k\geq 0},N\rangle$,

$$\int V_\Theta(z)d\mu(z)\leq n\lim_{n\to\infty} v_n^-.$$

Proof. W.l.o.g. we may assume that all payoffs $r(z,a)$ are in $[0,1]$. Therefore $T^n0\geq 0$ and thus $v_n^+=\|\frac{1}{n}T^n0\|$. As T is nonexpansive w.r.t. the sup norm, and $\|\frac{1}{n}T^n0\|$ converges for every nonexpansive map T, the limit of v_n^+, as $n\to\infty$, exists. Replacing the payoff function r of the stochastic game with $1-r$ results in a new stochastic game where the value of the normalized n-stage game equals $1-v_n^-$, which completes the proof of a). Let f be a continuous linear functional defined on $B(S,\mathcal{S})$ s.t. f is an extreme point of the unit ball of X^*, the dual of $B(S,\mathcal{S})$, and such that $f(V_\Theta)\geq E(N)\lim_{n\to\infty} v_n^+$. The continuous linear functionals on $B(S,\mathcal{S})$ are bounded finitely additive measures on $(S,\mathcal{S})$ and the extreme points of the unit ball correspond to atomic finitely additive measures with total mass either 1 or -1. The linear functional f is an atomic finitely additive measure on $(S,\mathcal{S})$. ∎

However, if S is a compact Hausdorff space and the nonexpansive operator associated with the maxmin or minmax of the constrained stochastic game maps continuous functions to continuous functions, then the Shapley operator acts on the continuous functions on S, $C(S)$. The norm dual of

$C(S)$ is the space of all countably additive regular measures, and the extreme points of the unit ball are the $\pm$ Dirac measures. Therefore, there are always states $z, z' \in S$ such that for any uncertain duration $\Theta = \langle Y, \mathcal{F}_{k\geq 0}, N\rangle$,

$$V_\Theta(z)) \geq n \lim_{n\to\infty} v_n^+$$

and

$$V_\Theta(z')) \leq n \lim_{n\to\infty} v_n^-.$$

5. Law of Large Numbers for Nonexpansive Stochastic Processes

This section is based on [9].

5.1. BACKGROUND

The Operator Ergodic Theorem (OET) asserts that, if $A : H \to H$ is a linear operator with norm 1 on a Hilbert space, then, for every $x \in H$,

$$\frac{x + Ax + \ldots + A^n x}{n} \text{ converges (strongly).}$$

The Strong Law of Large Numbers (SLLN) for martingales in Hilbert spaces says that if $(\mathbf{x}_n)$ is an H-valued martingale such that

$$\sum_{k=1}^{\infty} k^{-2} E(\|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2) < \infty,$$

then

$$\frac{\mathbf{x}_n}{n} \text{ converges a.e. (to zero).}$$

The proposition below provides a result which generalizes both these classical theorems.

Let (Ω, Σ, P) be a probability space and let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n$ be an increasing chain of σ-fields spanning Σ. A sequence $(\mathbf{x}_n)$ of strongly $\mathcal{F}_n$-measurable and strongly P-integrable functions on Ω taking on values in a (real separable[4]) Banach space X, is called an *X-valued stochastic process.* If in addition, for some map $T : X \to X$,

$$E(\mathbf{x}_{n+1} \mid \mathcal{F}_n) = T(\mathbf{x}_n),\ n = 0, 1, \ldots,$$

[4]The results hold for any Banach space. However, as the values of any sequence $(\mathbf{x}_n)$ of strongly $\mathcal{F}_n$-measurable and strongly P-integrable functions on Ω taking on values in a Banach space are with probability 1 in a separable subspace, we may assume w.l.o.g. that the values are in a separable B-space.

then $(\mathbf{x}_n)$ is called a *T-martingale*.

Of course, if T is the identity, then T-martingales are just ordinary martingales. In general, the class of all T-martingales consists of all sequences $(\mathbf{x}_n)$ of the form $\mathbf{x}_0 = \mathbf{d}_0, \ldots, \mathbf{x}_{n+1} = T(\mathbf{x}_n) + \mathbf{d}_{n+1}$ where $(\mathbf{d}_n)$ is an ordinary martingale-difference sequence, i.e., $E(\mathbf{d}_{n+1} \mid \mathcal{F}_n) = 0$.

Proposition 3 (Kohlberg and Neyman [9]) *Let $T : H \to H$ be a nonexpansive map on a Hilbert space H, and let $(\mathbf{x}_n)$ be a T-martingale taking on values in H. If*

$$\sum_{n=1}^{\infty} n^{-2} E(\|\mathbf{x}_{n+1} - T\mathbf{x}_n\|^2) < \infty,$$

then

$$\frac{\mathbf{x}_n}{n} \quad \textit{converges a.e.}$$

To see that the proposition in fact includes both the SLLN and the OET (for Hilbert spaces), note the following equivalent reformulation of the OET: *If $T : H \to H$ is a nonexpansive affine map on a Hilbert space, then $\frac{T^n x}{n}$ converges $\forall\ x \in H$.*

To verify the equivalence of the formulations note that any map $T : H \to H$ is a nonexpansive affine map if and only if it is of the form $Ty = x + Ay$ where A is a linear operator of norm less than or equal to one; since $T^n y = x + Ax + \ldots + A^{n-1}x + A^n y$, the sequence $\frac{T^n x}{n}$ converges $\forall x \in H$ if and only if the sequence $\frac{x + Ax + \ldots + A^{n-1}x}{n}$ converges $\forall x \in H$.

Thus the OET can be obtained from the proposition by restricting attention to deterministic $(\mathbf{x}_n)$, whereas the SLLN is the special case where T is the identity.

But the proposition also yields results combining the OET and the SLLN. For example, [9] shows that it implies the following.

If $A : H \to H$ is a linear operator of norm 1 on a Hilbert space, and if $B_i : H \to H$ are (random) linear operators of norm at most 1 such that

$$E(B_n \mid B_1, \ldots, B_{n-1}) = A$$

and

$$\sum_{k=1}^{\infty} E(\|B_k - A\|^2) < \infty$$

then, for every $x \in H$, almost everywhere

$$\lim_{n\to\infty} A_n x = \lim_{n\to\infty} \frac{x + Ax + A^2 x + \ldots + A^n x}{n+1}$$

where

$$A_n = \frac{I + B_n + B_n B_{n-1} + \ldots + B_n B_{n-1} \ldots B_1}{n+1}.$$

In the next subsection, we present the general version of Proposition 1, which encompasses more general versions of the SLLN (e.g., [24] and [6]) and of the OET.

5.2. THE RESULT

Before stating our theorem we review some definitions.

The modulus of smoothness of a Banach space X is the function $\rho_X : \mathbb{R}_+ \to \mathbb{R}$ defined by $\rho_X(t) = \sup\{(\|x+y\| + \|x-y\|)/2 - 1 : \|x\| = 1 \text{ and } \|y\| \le t\}$. X is *uniformly smooth* if $\rho_X(t) = o(t)$ as $t \to 0+$; it is *p-uniformly smooth*, $1 < p \le 2$, if $\rho_X(t) = O(t^p)$ as $t \to 0+$.

To simplify the statement below, we define a Banach space to be 1-*uniformly smooth* if it is uniformly smooth.[5]

Theorem 10 (Kohlberg and Neyman [9]) *Let $T : X \to X$ be a nonexpansive map on a p-uniformly smooth Banach space X, $1 \le p \le 2$, and let $(\mathbf{x}_n)$ be a T-martingale (taking on values in X). If*

$$\sum n^{-p} E(\|\mathbf{x}_n - T\mathbf{x}_{n-1}\|^p) < \infty, \tag{12}$$

then there exists a continuous linear functional $f \in S(X^*)$ such that

$$\lim_{n\to\infty} \frac{f(\mathbf{x}_n)}{n} = \lim_{n\to\infty} \frac{\|\mathbf{x}_n\|}{n} = \inf\{\|Tx - x\| : x \in X\} \text{ a.e.} \tag{13}$$

If, in addition, the space X is strictly convex,

$$\mathbf{x}_n/n \text{ converges weakly to a point in } X; \tag{14}$$

and if the norm of X^* is Fréchet differentiable (away from zero),

$$\mathbf{x}_n/n \text{ converges strongly to a point in } X. \tag{15}$$

Proposition 3 is a special case of Theorem 10 because any Hilbert space, H, is 2-uniformly smooth, and the norm of H^* (i.e., H) is Fréchet differentiable. [6] and [24] demonstrate the SLLN for martingales in a p-uniformly smooth Banach space, under condition (12). Thus, Theorem 10 may be viewed as a generalization of both the Hoffmann-Jorgensen and Pisier SLLN for martingales as well as the OET for p-uniformly smooth Banach spaces.

[5]Note that if X is p-uniformly smooth for some $1 \le p \le 2$, then X in particular is uniformly smooth and thus ([4] p.38) reflexive.

When $(\mathbf{x}_n)$ is a deterministic sequence, the conclusions of the theorem already follow from the nonexpansiveness of T and the reflexivity of X (which is weaker than p-uniform smoothness of X; Assumption (12) is obviously redundant). In fact, conclusions (13), (14) and (15) are Theorem 1.1, and Corollaries 1.3 and 1.2 of [7], respectively.

The extension of those results to the stochastic case requires the stronger conditions of Theorem 10. Indeed, weaker conditions do not suffice: if the norm of X is not Fréchet differentiable we can construct a nonexpansive T-martingale $(\mathbf{x}_n)$ satisfying $\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq 1$ everywhere and for which $\liminf \|\mathbf{x}_n\|/n < \limsup \|\mathbf{x}_n\|/n$ [9].

One may wonder whether weaker conditions would guarantee that x_n converge in direction, i.e., that $x_n/\|x_n\|$ converge: an example of a finite-dimensional normed space X which is not smooth and a T-martingale $(\mathbf{x}_n)$ satisfying $\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq 1$ with $\liminf \|\mathbf{x}_n\|/n > 0$, but where $\mathbf{x}_n/\|\mathbf{x}_n\|$ does not converge, is given in [9].

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