STOCHASTIC GAMES IN ECONOMICS: THE LATTICE-THEORETIC APPROACH

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Abstract. This chapter considers a recent trend in the application of stochastic games to economics characterized by the use of the lattice-theoretic approach to capture the monotonic properties of Markovian equilibria. The topics covered are: (i) a general framework for discounted stochastic games with Lipschitz-continuous and monotone equilibrium strategies and values, (ii) a model of capital accumulation, and (iii) two classes of games with perfect information: strategic bequests and oligopoly with commitment.

In view of the restriction to pure-strategy equilibria and of the natural monotonicity property of strategies and value functions in most economic applications, this approach appears most promising.

1. Introduction

This chapter reviews applications of stochastic games in economics, where lattice-theoretic arguments have played a central role (at least implicitly). Some areas of application are the same as in the general overview given in the next chapter. However, the studies contained in this survey are generally discussed in more, but still incomplete, detail. We begin with a general approach to stochastic games with continuous equilibrium strategy and value functions [10]. Then we consider games of capital accumulation [1] and two classes of sequential-move games: strategic bequests and oligopoly competition with alternating moves. All the new notions and results from lattice programming invoked here are presented (without proof) in the appendix.

From a methodological perspective, it is hoped that this review will convey a sense that the lattice-theoretic approach is well-suited for analyzing dynamic games in economics, as it provides a natural framework for turning natural economic structure into appealing monotonic relationships that survive the dynamic programming recursion while satisfying the desired restriction to pure strategies.

2. Existence of Pure-Strategy Markov Equilibrium

Consider an n-player discounted stochastic game described by the tuple $\{S, A_i, \lambda_i, r_i, p\}$ with the standard meaning as in earlier chapters. The state space S and action spaces A_i are all *compact Euclidean intervals*, with $S \subset R^k$ and $A_i \subset R^{k_i}$. Denote the joint action set $A = A_1 \times ... \times A_n$ and a typical element $a = (a_1, ..., a_n) = (a_i, a_{-i})$, for any i. The previous definitions of Markov and Markov-stationary strategies and expected discounted payoffs are also easily adapted to the case of *pure strategies* considered here. With p denoting the transition probability from $S \times A$ to S, let F be its associated cumulative distribution function. The following assumptions are in effect throughout this chapter (see the appendix for definitions).

(A1) The state distribution function F and the reward functions r_i are all twice continuously differentiable, for all $i = 1, ..., n$.
(A2) F and r_i are supermodular in a_i and have increasing differences in $(a_i; a_{-i}, z)$.
(A3) F satisfies a dominant diagonal condition in $(a_i; a_{-i})$, and r_i satisfies a strong dominant diagonal condition in $(a_i; a_{-i})$, for all i.
(A4) F is increasing in (z, a) in the sense of first-order stochastic dominance, and r_i is increasing in (z, a_{-i}), for all i.

Let $C(S, R)$ be the Banach space of continuous functions from S to R with the sup norm, to be denoted $\|\cdot\|$. By Assumption (A1) and the compactness of S and A_i, there exists $K > 0$ such that $r_i(z, a) \leq K$, $\forall$ i, z, a. Hence, all feasible payoffs in this game are also $\leq K$. Denote by $CM_K(S, R)$ the subset of the ball of radius K in $C(S, R)$ consisting of nondecreasing functions. The main results in this section are in

Theorem 1 *Under Assumptions (A1)-(A4), we have:*
(a)The infinite-horizon discounted stochastic game has a pure-strategy Markov-stationary equilibrium, with strategies and corresponding value functions that are nondecreasing and Lipschitz-continuous in the state vector.
(b)For any finite horizon T, there exists a unique pure-strategy Markov equilibrium, with strategy components and corresponding value functions that are nondecreasing and Lipschitz-continuous in z. Moreover, this is also the unique Markov equilibrium in mixed and correlated strategies, and the game is dominance-solvable.

Curtat [10] developed the above framework and established Part (a). The elaboration given in Part (b) is due to Amir [4]. Curtat also proved a comparative dynamics result: the first-period equilibrium actions in the infinite-horizon problem are higher than the equilibrium actions of the one-stage game. He then concludes with several applications to economic models.

Due to space constraints, we provide a self-contained outline of the proof of Theorem 1 but omit some lengthy details of a technical nature. The argument proceeds in several steps, via the analysis of auxiliary games defined here as follows. Let $v = (v_1, ..., v_n) \in CM_K(S, R)^n$ be an n-vector of continuation values, and consider an n-person one-shot game G_v parameterized by the state variable, where Player i has action set A_i and payoff function

$$\Pi_i(v, z, a_i, a_{-i}) \stackrel{\triangle}{=} (1 - \lambda_i) r_i(z, a_i, a_{-i}) + \lambda_i \int v_i(z') dF(z'/z, a_i, a_{-i}). \quad (2.1)$$

With z fixed, let the above game be denoted by G_v^z.

Lemma 1 *For any $v = (v_1, ..., v_n) \in CM_K(S, R)^n$, the game G_v has a unique Nash equilibrium $a^v(z) = (a_1^v(z), ..., a_n^v(z))$. Furthermore, each $a_i^v(z)$ is nondecreasing and Lipschitz-continuous in z uniformly in v.*

Proof of Lemma 1. By Theorem 3 and Assumption (A.2), since v is nondecreasing, $\int v_i(z') dF(z'/z, a_i, a_{-i})$ is supermodular in a_i and has nondecreasing differences in (a_i, a_{-i}). From Assumption (A.3), it also satisfies a dominant diagonal condition in (a_i, a_{-i}). Since supermodularity, increasing differences and dominant diagonals are preserved under addition, it follows from Assumptions (A2)-(A3) that Π_i is supermodular in a_i and has increasing differences (and dominant diagonals) in $(a_i; a_{-i})$. Then, since the A_i's are compact, it follows in particular that G_v^z is a supermodular game for each z. Existence of a pure-strategy equilibrium $a^v(z) = (a_1^v(z), ..., a_n^v(z))$ is a consequence of Theorem 6. Uniqueness of the Nash equilibrium $a^v(z)$ then follows in a standard way from Π_i satisfying the dominant diagonal condition (see [20] or [16]).

Π_i also has increasing differences in (z, a_i). Hence, by Theorem 7, each $a_i^v(z)$ is nondecreasing in z (due to uniqueness, the maximal and minimal equilibria clearly coincide). The fact that each $a_i^v(z)$ is Lipschitz-continuous in z uniformly in v (i.e., the associated Lipschitz constant D can be chosen independently of v) follows from the compactness of S and A_i, Assumptions (A1) and (A3), Theorem 5 (some omitted lengthy details can be found in [10], p. 188). ∎

Lemma 2 *Given $v = (v_1, ..., v_n) \in CM_K(S, R)^n$, the (unique) equilibrium payoff for Player i, $\Pi_i^*(v, z) \stackrel{\triangle}{=} \Pi_i(v, z, a^v)$ is in $CM_K(S, R)$ and is Lipschitz-continuous in z uniformly in v.*

Proof of Lemma 2. Continuity of $\Pi_i^*(v,z)$ in z follows directly from Lemma 1 and the structure of the payoffs in (2.1). Monotonicity of $\Pi_i^*(v,z)$ in z follows from Assumption (A4). To show the uniform Lipschitz continuity, consider

$$\Pi_i^*(v,z) = (1-\lambda_i)r_i(z, a^v(z)) + \lambda_i \int v_i(z')dF(z'/z, a^v(z)).$$

Hence, by Taylor's theorem, for any z_1, z_2 in S, there are constants C_1, C_2, C_3, C_4 such that

$$|\Pi_i^*(v,z_1) - \Pi_i^*(v,z_2)| \leq (1-\lambda_i)(C_1 + D.C_2)\left\|z_1 - z_2\right\| +$$

$$\lambda_i(C_3 + D.C_4)\left\{\int_S |v_i(t)dt|\right\}\left\|z_1 - z_2\right\|$$

where use is made of Assumption (A1), the compactness of S and A_i, the Lipschitz continuity of $a^v(z)$ from Lemma 1, and standard facts about composition of functions and integrals. With

$$M \stackrel{\triangle}{=} (1-\lambda_i)(C_1 + kC_2) + \lambda_i(C_3 + D.C_4)K\int_S dt \tag{2.2}$$

being independent of v, it follows that

$$\left\|\Pi_i^*(v,z_1) - \Pi_i^*(v,z_2)\right\| \leq M\left\|z_1 - z_2\right\|,$$

which concludes the proof. ■

Let $\Pi^*(v,z) \stackrel{\triangle}{=} (\Pi_1^*(v,z), ..., \Pi_n^*(v,z))$. We now define a single-valued operator mapping continuation values to equilibrium payoffs as follows.

$$\begin{array}{cccc} T: & CM_K(S,R)^n & \rightarrow & CM_K(S,R)^n \\ & v(\cdot) & \rightarrow & \Pi^*(v,\cdot). \end{array}$$

The rest of the proof consists of showing that the operator T has a fixed point $\overline{v} = T\overline{v}$, in which case the associated equilibrium strategies $(a_1^{\overline{v}}(z), ..., a_n^{\overline{v}}(z))$ clearly constitute a Markov-stationary equilibrium of the infinite-horizon discounted stochastic game.

Lemma 3 *The operator T is continuous in the topology of uniform convergence.*

Proof of Lemma ??. Let $\Rightarrow$ denote uniform convergence. We have to show that if $v_i^k(\cdot) \Rightarrow v_i(\cdot)$ for all i, then $\Pi_i^*(v^k,\cdot) \Rightarrow \Pi_i^*(v,\cdot)$ for all i.

With $v_i^k(\cdot) \Rightarrow v_i(\cdot)$, it follows from the well-known property of upper hemi-continuity of the equilibrium correspondence in the game G_v^z that, for each fixed z and each i, $a_i^{v^k}(z) \to a_i^v(z)$ in R. In other words, we have pointwise convergence of the functions $a_i^{v^k}(z)$ to the limit $a_i^v(z)$. Since these functions are all Lipschitz-continuous (Lemma 1), the convergence is actually uniform. The pointwise, and thus uniform, convergence of $\Pi^*(v^k, \cdot)$ to $\Pi^*(v, \cdot)$ in view of Lemma 2, follows from standard results on the composition of continuous functions. ∎

We are now ready to conclude the overall proof.

Proof of Theorem 1. (a) In order to invoke Shauder's fixed-point theorem for T, we need to show that there exists a convex and norm-compact subset Φ of $CM_K(S, R)^n$ such that $T(\Phi) \subset \Phi$. To this end, define the following subset of $CM_K(S, R)^n$:

$$\Phi \stackrel{\triangle}{=} \{v \in CM_K(S, R)^n : \|v_i(z_1) - v_i(z_2)\| \leq M \|z_1 - z_2\| \text{ for all } i, z_1, z_2\}$$

where M is defined as in (2.2). It follows from that Lemma 2 that $Tv \in \Phi$ whenever $v \in \Phi$. Since all the functions in Φ are uniformly Lipschitz-continuous, Φ is an equi-continuous set of functions, so that its compactness in the sup-norm follows from the Arzela-Ascoli theorem. Hence, by Shauder's fixed-point theorem, T has a fixed point $\overline{v} = T\overline{v}$ in Φ. Then, from standard results in discounted dynamic programming, the associated equilibrium strategies $(a_1^{\overline{v}}(z), ..., a_n^{\overline{v}}(z))$ in the game $G_{\overline{v}}$ clearly constitute a Markov-stationary equilibrium.

(b) Uniqueness of a pure-strategy Markov equilibrium for every finite horizon T follows simply by iterating $v_n = T(v_{n-1})$ starting from $v_0 \equiv 0$, for $n = 1, 2, ..., T$, and invoking Lemma 1 at every iteration. The rest then follows directly from [16], Theorem 5, applied to the games G_v^z for each z. ∎

3. Stochastic Games of Capital Accumulation

The model is well known in economic dynamics, as the one-player version of this game is the standard optimal growth model under uncertainty [9]. Consider two agents who jointly own a productive asset (or natural resource) and who consume some amount of the available stock at each stage in order to maximize their (individual) discounted sum of utilities. The payoff and feasible set of (say) Agent 1 and the state transition law are given by

$$\sum_{t=0}^{T} (1 - \lambda_1)\lambda_1^t r_1(a_t^1),\ 0 \leq\ a_t^1 \leq K_1(z_t) \quad \text{and} \quad z_{t+1} \sim p(\cdot/z_t - a_t - b_t)$$

where z_t is the asset stock level; a_t, b_t are the consumption levels of Agents 1 and 2 at time t, bounded by the $K_i's$ as exogenous extraction capacities; r_i is

Agent i's one-period utility function with λ_i being his discount factor; and p is a transition probability mapping "savings" into (probability distributions on) the next stock. Let F denote the cumulative distribution of p. The following assumptions are made throughout.

(B1) r_i is strictly increasing and strictly concave, with $r_i(\cdot) \leq B$, $B > 0$.
(B2) (i) $F(z^{'}/\cdot)$ is weakly continuous, with $F(0/0) = 1$.
(ii) $F(z'/\cdot)$ is strictly decreasing for every z'.
(iii) $F(z'/\cdot)$ is strictly convex for every z'.
(B3) $K_i(\cdot)$ is continuous and uniformly bounded, with $K_i(0) = 0$, $0 \leq K_i(z_1) - K_i(z_2) \leq z_1 - z_2$ for all $z_1 > z_2$, and $K_1(z) + K_2(z) < z$, for all z.

(B1) is standard. (B2)(ii-iii) can be naturally interpreted as saying that the probability that the next state exceeds a given level is continuous and increasing at a decreasing rate in the savings, and is thus a stochastic version of the standard assumption of decreasing marginal returns. Nonetheless, the convexity of $F(z'/\cdot)$, a key assumption here, is fairly restrictive in that it rules out the deterministic case and forces the effective state space to be all of $[0, \infty)$: see [1], [2] for details. Finally, (B3) is natural in many contexts, and serves to rule out trivial equilibria with stock exhaustion in the first stage.

Although this problem cannot be formally viewed as a special case of the framework of the previous section, it has essentially the same mathematical structure (characterized by strategic complementarity and diagonal dominance), and can thus be analyzed along a very similar line of reasoning. The effective space of consumption strategies and value functions are respectively

$$\Lambda_i \stackrel{\triangle}{=} \{v : [0, \infty) \to [0, B] : v \text{ is continuous and nondecreasing } \} \text{ and}$$

$$\Sigma_i \stackrel{\triangle}{=} \{\alpha : S \to A_i : \alpha(0) = 0 \text{ and } 0 \leq \alpha(z_1) - \alpha(z_2) \leq z_1 - z_2, \forall z_1 \geq z_2 \geq 0\}.$$

The main result here is:

Theorem 2 *Under Assumptions (B1)-(B3), we have:*
(a) The infinite-horizon discounted stochastic game has a Markov-stationary equilibrium, with strategies in Σ_i and corresponding value functions in Λ_i.
(b) For every finite horizon T $(t = 0, 1, ..., T - 1)$, there exists a unique Markov equilibrium in Σ_i^T and corresponding value functions in Λ_i^T.

An important difference exists between this model and the framework of [10], though: in order not to rule out utility functions r with $r^{'}(0) = \infty$, e.g., the log and the constant-risk-aversion utilities that are frequently used in economic dynamics, one cannot work here with the auxiliary games of the previous sections, since the resulting value functions need not be uniformly Lipschitz-continuous. Rather, in order to avoid the second fixed-point argument (in value function space) above that requires compactness, hence equi-continuity, of the resulting value functions, Amir [1] defined a standard best-response mapping in strategy space. Exploiting supermodularity and diagonal dominance arguments in ways similar to the proof above, one can show that the best response to a strategy in Σ_i is unique and lies in Σ_j, so that the best response mapping, from $\Sigma_i \times \Sigma_j$ (with the topology of uniform convergence) to itself, has a fixed point. The details are not presented here. The proof of Part (b) is analogous to Theorem 1 (b). We close by noting that with the extra (restrictive) assumption $r^{'}(0) < \infty$, the approach of the previous section could easily be followed here.

4. Games of Perfect Information

This section summarizes two strands of economic literature dealing with dynamic games of perfect information that are closely related to stochastic games. Details and proofs may be found in the references.

4.1. GAMES OF STRATEGIC BEQUESTS

Consider an infinite sequence of identical generations in a one-good economy, each of whom decides on a consumption level c out of the capital stock x inherited from the previous generation, with the residual $x - c$ forming the bequest to the next generation. With stochastic production, the next stock is determined according to the c.d.f. $F(\cdot/x - c)$, and the payoff to a generation is then

$$\int U[c, h(t)]dF(t/x - c) \ , \ c \in [0, x] \ ,$$

where U is the (common) utility function, and h is the next generation's consumption strategy. Here, the Markov assumption takes the form that each generation is interested only in the welfare of their immediate offspring (in addition to their own). The first main question is: when is the existence of a stationary equilibrium consumption strategy guaranteed?

Proposition 1 *Let $U(c_1, c_2)$ be strictly increasing and supermodular in (c_1, c_2) and strictly concave in c_1. Then a Markov-stationary equilibrium exists, with strategies in*

(a) Σ_i if F satisfies Assumptions (B2)(i-iii), and in

(b) $\widetilde{\Sigma}_i \stackrel{\triangle}{=} \{\alpha : S \to A_i : \alpha(0) = 0$ and $\alpha(z_1) - \alpha(z_2) \leq z_1 - z_2\}$ if F satisfies (B2)(i-ii).

Leininger [13] and Bernheim and Ray [7] independently proved Part (b) in the deterministic production case, while Amir [2] proved Part (a). The key assumption in moving from Part (a) to Part (b) is clearly (A2)(iii) and the mechanism is the same as in the model of Section 2. Lane and Leininger [12] and Bernheim and Ray [8] study the properties of Markov equilibria. In addition, Amir [2] shows that if U and F are twice continuously differentiable, the equilibrium consumption strategy will be continuously differentiable.

4.2. A CLASS OF GAMES WITH ALTERNATING MOVES

Consider the following alternating-move dynamic game from Maskin and Tirole [14], [15]. Firm 1 (Firm 2) chooses an action in odd-numbered (even-numbered) periods, each firm remaining committed to its action for 2 periods (so for all k, $a^1_{2k+2} = a^1_{2k+1}$ for firm 1, and $a^2_{2k+1} = a^2_{2k}$ for firm 2). Firm i's payoff is

$$\sum_{t=0}^{\infty}(1-\lambda)\lambda^t \Pi^i(a^1_t, a^2_t), a^1_t, a^2_t \in A,$$

where Π^i is a reduced form for a per-period (static equilibrium) payoff in price, quantity or other type of competition. A pair of "reaction" functions (R^1, R^2) forms a Markov-stationary equilibrium if $a^2_{2k} = R^2(a^1_{2k-1})$ maximizes firm 2's payoff at any time $2k$ given a^1_{2k-1} and assuming that, henceforth, firm 1 will follow $R^i, i = 1, 2$, with an analogous condition for firm 2. Thus, an equilibrium can be described by a triplet (R^i, V^i, W^i) for firm 1, such that (say) for firm 1:

$$\begin{aligned} V^1(a^2) &= \max\left\{(1-\lambda)\Pi^i(a^1, a^2) + \lambda W^1(a^1) : a^1 \in A\right\} \\ &= (1-\lambda)\Pi^i(R^1(a^2), a^2) + \lambda W^1(R^1(a^2)), \mathit{and} \end{aligned}$$

$$W^1(a^1) = \Pi^i(a^1, R^2(a^1)) + \lambda V^1(R^2(a^1)).$$

Maskin and Tirole [14], [15] prove that at a Markov equilibrium, each of the R^is is nonincreasing (nondecreasing) if the Π^i have decreasing (increasing) differences. (This can be obtained as an application of Topkis's monotonicity theorem to the above functional equations.) Then they use this framework to provide a new look at various well-known key issues in quantity and price competition, including natural monopoly, kinked demand curve, and strategic excess capacity. They conclude that this new framework is more suitable than the traditional approaches to analyzing some of the issues at hand.

5. Appendix

A brief summary of the lattice-theoretic notions and results is presented here.

Throughout, S will denote a partially ordered set and A a lattice, and all cartesian products are endowed with the product order. A function F: $A \to R$ is (strictly) supermodular if $F(a \vee a') + F(a \wedge a') \geq (>) F(a) + F(a')$ for all $a, a' \in A$. If $A \subset R^m$ and F is twice continuously differentiable, F is supermodular if and only if $\frac{\partial^2 F}{\partial a_i \partial a_j} \geq 0$, for all $i \neq j$. A function $G : A \times S \to R$ has (strictly) increasing differences in s and a if for $a_1 (>) \geq a_2$, $G(a_1, s) - G(a_2, s)$ is (strictly) increasing in s . If $A \subset R^m$, $S \subset R^n$ and G is smooth, this is equivalent to $\frac{\partial^2 G}{\partial a_i \partial s_j} \geq 0$, for all $i = 1, ..., m$ and $j = 1, ..., n$.

A set I in R^n is increasing if $x \in I$ and $x \leq y \Rightarrow y \in I$. With $S \subset R^n$ and $A \subset R^m$, a transition probability F from $S \times A$ to S is supermodular in a (respectively, has increasing differences in s and a) if for every increasing set $I \subset R^n$, $\int 1_I(t) dF(t/s, a)$ is supermodular in a (respectively, has increasing differences in s and a) where 1_I is the indicator function of I. A characterization of these properties, using first-order stochastic dominance, follows (see [5], [6] for extensive related work).

Theorem 3 *(Topkis [22]). A transition probability F from $S \times A$ to $S \subset R^n$ is supermodular in s (respectively, has increasing differences in s and a) if and only if for every integrable increasing function $v : S \to R$, $\int v(t) dF(t/s, a)$ is supermodular in s (respectively, has increasing differences in s and a).*

Let $L(A)$ denote the set of all sublattices of A. A set-valued function $H : S \to L(A)$ is ascending if for all $s \leq s'$ in S, $a \in A_s, a' \in A_{s'}$, $a \vee a' \in A_{s'}$ and $a \wedge a' \in A_s$. Topkis's main monotonicity result follows (see also [18]).

Theorem 4 *(Topkis [23]). Let $F : S \times A \to R$ be uppersemicontinuous and supermodular in a for fixed s, and have increasing (strictly increasing) differences in s and a, and let $H : S \to L(A)$ be ascending. Then the maximal and minimal (all) selections of $\arg\max \{F(s, a) : a \in H(s)\}$ are increasing functions of s.*

With $S \subset R^n$ and $A \subset R^m$, a function F: $A \to R$ satisfies (strong) diagonal dominance if $\sum_{j=1}^{m} \frac{\partial^2 F}{\partial a_i \partial a_j} (<) \leq 0$ for each $i \in \{1, 2, ..., m\}$. A transition probability F from A to S satisfies strong diagonal dominance in a if $\int 1_I(t) dF(t/a)$ has the same property, for every increasing set $I \subset R^n$, or equivalently, if for every increasing function $v : S \to R$, $\int v(t) dG(t/a)$ satisfies that same property.

Theorem 5 *(Curtat [10]).*[1] *Assume that $F : S \times A \to R$ is uppersemicontinuous and supermodular in a for fixed s, has increasing differences in s and a, and satisfies SDD in a. Then* $\arg\max\{F(s,a) : a \in A\}$ *is an increasing and Lipschitz-continuous (single-valued) function of s.*

A game with action sets that are compact Euclidean lattices and payoff functions that are u.s.c. and supermodular in own action, and have increasing differences in (own action, rivals' actions) is a supermodular game. By Theorem 4, such games have minimal and maximal best responses that are monotone functions, so that a pure-strategy equilibrium exists by (see also [24]):

Theorem 6 *(Tarski [21]). An increasing function from a complete lattice to itself has a set of fixed points that is itself a nonempty complete lattice.*

The last result deals with comparing equilibria.

Theorem 7 *(Milgrom and Roberts [16]). Consider a parameterized supermodular game where each payoff has increasing differences in the parameter (assumed real) and own action. Then the maximal and minimal equilibria are increasing functions of the parameter.*

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[1] Earlier uses of such a result appear, e.g., in [11], [1].

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