ISRAEL JOURNAL OF MATHEMATICS, Vol. 38, No. 4, 1981

ASYMPTOTIC BEHAVIOR OF NONEXPANSIVE MAPPINGS IN NORMED LINEAR SPACES

BY

ELON KOHLBERG AND ABRAHAM NEYMAN[†]

ABSTRACT

Let T be a nonexpansive mapping on a normed linear space X. We show that there exists a linear functional f, $\|f\| = 1$, such that, for all $x \in X$, $\lim_{n\to\infty} f(T^n x/n) = \lim_{n\to\infty} \|T^n x/n\| = \alpha$, where $\alpha = \inf_{y \in C} \|Ty - y\|$. This means, if X is reflexive, that there is a face F of the ball of radius α to which $T^n x/n$ converges weakly for all x ($\inf_{z \in F} g(T^n x/n - z) \to 0$ for every linear functional g); if X is strictly convex as well as reflexive, the convergence is to a point; and if X satisfies the stronger condition that its dual has Frechet differentiable norm then the convergence is strong. Furthermore, we show that each of the foregoing conditions on X is satisfied if and only if the associated convergence property holds for all nonexpansive T.

1. Introduction and statement of main results

A mapping $T: C \to C$ on a subset C of a normed linear space is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Let $S(X^*) = \{f \in X^*: \|f\| = 1\}$. Our main result is:

1.1. THEOREM. *Let C be a convex subset of a normed space X and let $T: C \to C$ be nonexpansive. Then there exists an $f \in S(X^*)$ such that for every $x \in C$,*

$$\lim_{n\to\infty} f\left(\frac{T^n x}{n}\right) = \lim_{n\to\infty} \left\|\frac{T^n x}{n}\right\| = \inf_{y \in C} \|Ty - y\|.$$

Two immediate consequences are:

1.2. COROLLARY. *$T^n x/n$ converges for all $x \in C$ if X has the following property:*

every sequence $\{x^n\}$ in X satisfying $\|x^n\| = 1$ and

$f(x^n) \to 1$ for some $f \in S(X^*)$ must converge.

[†] Supported by National Science Foundation Grant MCS-79-066.
Received March 2, 1980 and in revised form May 8, 1980

1.3. COROLLARY. *$T^n x/n$ converges weakly for all $x \in C$ if X has the following property*:

$$(**) \qquad \text{every sequence } \{x_n\} \text{ in } X \text{ satisfying } \|x_n\| = 1 \text{ and } f(x_n) \to 1 \text{ for some } f \in S(X^*) \text{ must converge weakly.}$$

It is easily verified (see [2]) that (*) holds if and only if X is a Banach space whose dual has Fréchet differentiable norm and that (**) holds if and only if X is a strictly convex and reflexive Banach space. We observe in addition that, since $\|T^n x - T^n y\| \leqq \|x - y\|$, if $T^n y/n$ converges for some $y \in C$ then $T^n x/n$ converges to the same limit for all $x \in C$.

We shall also prove converses of Corollaries 1.2 and 1.3. The direct and converse statements are summarized in the two theorems below.

1.4. THEOREM. *The following conditions on a Banach space X are equivalent*:

(i) *X^* has Fréchet differentiable norm.*

(ii) *If C is a closed convex subset of X and $T: C \to C$ is nonexpansive, then there exists an $x_0 \in C$ such that $T^n x/n \to x_0$ for all $x \in C$.*

1.5. THEOREM. *The following conditions on a Banach space X are equivalent*:

(i) *X is strictly convex and reflexive.*

(ii) *If C is a closed convex subset of X and $T: C \to C$ is nonexpansive, then there exists an $x_0 \in C$ such that $T^n x/n$ converges weakly to x_0, for all $x \in C$.*

The implication (i) $\Rightarrow$ (ii) in Theorem 1.4 generalizes the results of Pazy [4], Reich [5, 6, 7] and Kohlberg and Neyman [3]. Pazy first proved (ii) with the assumption that X is a Hilbert space; Reich [5, 6] extended Pazy's result to a wider class of Banach spaces, (namely, spaces X whose norm is uniformly Gâteaux differentiable and whose dual has Fréchet differentiable norm), but with additional restrictions on the set C; Kohlberg and Neyman [3] gave a simple geometric proof of (ii) in uniformly convex spaces and Reich [7], using a variant of that proof, was then able to drop the previously mentioned restrictions on the set C.

2. Proof of the main results

If T is nonexpansive then, for every $x, y \in C$, $\|T^n x/n - T^n y/n\| \to 0$ and $\limsup_{n\to\infty} \|T^n y/n\| \leqq \|Ty - y\|$. Therefore, if $f \in S(X^*)$ and if α denotes $\inf_{y \in C} \|Ty - y\|$, then

$$\limsup_{n\to\infty} f\left(\frac{T^n x}{n}\right) \leqq \limsup_{n\to\infty} \left\| \frac{T^n x}{n} \right\| \leqq \alpha. \tag{2.1}$$

Thus, to prove Theorem 1.1 it is sufficient to show that there exists an $f \in S(X^*)$ such that, for some $y \in C$, $\liminf_{n\to\infty} f(T^n y/n) \geqq \alpha$. Assuming, without loss of generality, that $0 \in C$, it is therefore sufficient to show that there is an $f \in S(X^*)$ such that

$$f\left(\frac{T^n 0}{n}\right) \geqq \alpha \qquad \text{for all } n = 1, 2, \cdots. \tag{2.2}$$

The mapping $T: C \to C$ has an obvious extension to a nonexpansive mapping on a closed convex subset of the completion of X. There is therefore no loss of generality in assuming that X is a Banach space and that C is closed. Since $0 \in C$, if $r > 0$ then $T/(1+r)$ is a contraction mapping that maps C into C, and therefore has a unique fixed point, $x(r)$, satisfying $Tx(r) = (1+r)x(r)$. Clearly, $\|rx(r)\| = \|Tx(r) - x(r)\| \geqq \alpha$ for all $r > 0$.

For $\alpha = 0$, Theorem 1.1 follows trivially from (2.1). The essential geometric idea of our proof for $\alpha > 0$ rests on the fact that, for small $r > 0$, $x(r)$ is long compared to Tx and hence $x(r)$ and $x(r) - Tx$ are nearly parallel. It follows that $\|Tx(r) - Tx\| = \|rx(r) + x(r) - Tx\|$ is approximately $\|rx(r)\| + \|x(r) - Tx\|$. On the other hand, by nonexpansiveness, $\|Tx(r) - Tx\| \leqq \|x(r) - x\|$. Therefore, except for a small error, $\|x(r) - Tx\| \leqq \|x(r) - x\| - \|rx(r)\|$, that is, application of T reduces the distance from $x(r)$ by at least $\|rx(r)\| \geqq \alpha$. A convenient algebraic statement corresponding to this geometry is the following.

2.3. Lemma. *For all $r > 0$ and $x \in C$,*

$$\|Tx - x(r)\| \leqq \|x - x(r)\| - \alpha + 2r\|Tx\|.$$

Proof.

$$\begin{aligned} \|Tx - x(r)\| &= (1+r)\|Tx - x(r)\| - r\|Tx - x(r)\| \\ &\leqq \|Tx - (1+r)x(r)\| - \|rx(r)\| + 2r\|Tx\| \\ &\leqq \|x - x(r)\| - \alpha + 2r\|Tx\|. \end{aligned}$$

Q.E.D.

In what follows, for $x \neq 0$, f_x denotes a linear functional of norm 1 satisfying $f_x(x) = \|x\|$. Clearly

$$\|x - y\| \leqq \|x\| - \beta \quad \text{implies} \quad f_x(y) \geqq \beta \tag{2.4}$$

since $\|x\| - f_x(y) = f_x(x - y) \leqq \|x - y\|$.

Applying the lemma n times, for $x = 0$, $x = T0, \cdots, x = T^{n-1}0$ and adding the resulting inequalities, we obtain

$$\|x(r) - T^n 0\| \leqq \|x(r)\| - n\alpha + 2r \sum_{k=1}^{n} \|T^k 0\|.$$

By (2.4), therefore, $f_{x(r)}(T^n 0) \geqq n\alpha + O(r)$ and hence $f_{x(r)}(T^n 0/n) \geqq \alpha + O(r)$. Let $f \in X^*$, $\|f\| \leqq 1$, be an accumulation point of the $f_{x(r)}$ in the w^*-topology. (The existence of such an f is guaranteed by the Banach–Alaoglu theorem.) Then $f(T^n 0/n) \geqq \alpha$ for all n, so f satisfies (2.2). Obviously $f/\|f\|$, which is in $S(X^*)$, also satisfies (2.2) and the proof of Theorem 1.1 is complete.

It remains to prove the implication (ii) $\Rightarrow$ (i) in Theorems 1.4 and 1.5. We give only the proof for Theorem 1.4, the other being essentially the same. What we prove, in fact, is that if X does not have property (i), then (ii) fails even for $C = X$, that is, there exists a nonexpansive mapping $T : X \to X$ such that $T^n 0/n$ does not converge.

Suppose that (i) of Theorem 1.4 is not satisfied, i.e. (*) is not satisfied. Then there exist an $f \in S(X^*)$ and a nonconvergent sequence $\{z_m\}$ such that $\|z_m\| = 1$ and $f(z_m) \to 1$. Let γ be a piecewise linear curve starting at 0 with successive segments $(t_m - t_{m-1})z_m$ where $\{t_m\}$ is an increasing sequence of real numbers such that $t_0 = 0$ and $\lim t_{m-1}/t_m = 0$. Let $\gamma(t)$ be the point on this curve at arclength t from 0. Specifically

$$\gamma(t) = \gamma(t_{m-1}) + (t - t_{m-1})z_m \qquad \text{if } t_{m-1} \leqq t \leqq t_m, \quad m = 1, 2, \cdots.$$

Then $\gamma(t_m) = (t_m - t_{m-1})z_m + O(t_{m-1}) = t_m z_m + O(t_{m-1})$, so

$$\frac{\gamma(t_m)}{t_m} - z_m \to 0 \qquad \text{as } m \to \infty. \tag{2.5}$$

Define $T : X \to X$ by $Tx = \gamma(|f(x)| + 1)$. Then, for every x and y, Tx and Ty lie on γ and the arclength between them is $\big| |f(x)| - |f(y)| \big| \leqq \|x - y\|$. Hence T is nonexpansive.

For all $t \geqq 0$, we have $f(\gamma(t)) \leqq \|\gamma(t)\| \leqq t$. By choosing a subsequence of the z_m if necessary, we can assume without loss of generality that $\sum_{m=1}^{\infty} (t_m - t_{m-1}) \times (1 - f(z_m)) \leqq \frac{1}{2}$, which implies that $f(\gamma(t)) \geqq t - \frac{1}{2}$.

On γ, therefore, T moves each point further from 0 by an arclength between $\frac{1}{2}$ and 1. It follows that, given any m, there is an $n = n(m)$ such that $\|\gamma(t_m) - T^n 0\| \leqq 1$ and $n \leqq 2t_m$. Hence, by (2.5), convergence of $T^n 0/n$ would imply convergence of $\{t_m z_m/n\}$. Since $t_m/n \geqq \frac{1}{2}$ and $\|z_m\| = 1$, the z_m themselves must converge, a contradiction. Q.E.D.

3. Extensions and remarks

As in section 2, let $\alpha = \inf_{y\in C} \|Ty - y\|$.

3.1. REMARK. The f in Theorem 1.1 can be chosen, depending on x, so that $f(T^n x - x) \geqq n\alpha$ for $n = 1, 2, \cdots$.

For $x = 0$, this is (2.2). It holds for all x by translation, but Example 3.2 below shows that f may have to depend on x.

3.2. EXAMPLE. Let X be R^2 with $\|(x_1, x_2)\| \equiv |x_1| + |x_2|$, and let Tx be the position of x after one unit of time in a flow of constant speed that moves every point first toward the vertical axis and then upwards along it. Here a different f is needed to satisfy $f(T^n x - x) \geqq n\alpha$ depending on whether x lies to the left or the right of the vertical axis. (This example is due to J.F. Mertens.)

3.3. REMARK. In Theorem 1.1 and its corollaries, C need not be convex but only star-shaped, i.e.:

(3.4) There exists z such that $z + \lambda(x - z) \in C$ for all $x \in C$ and $0 \leqq \lambda \leqq 1$.

This suffices because, without loss of generality, $z = 0$, and the only use of convexity was to insure that $Tx/(1 + r) \in C$ for all $x \in C$ and $r > 0$.

In many dynamic programming problems, e.g. [1], a nonexpansive mapping T naturally arises such that $T^n 0/n$ is the average value over the first n periods, while $rx(r)$ is the average value when the nth-period return has weight $r(1 + r)^{-n}$, corresponding to discounting at interest rate r. It is therefore interesting to compare the behavior of $T^n 0/n$ and $rx(r)$, as in the following theorem and corollary.

3.5. THEOREM. *Let X be a Banach space and C a closed subset of X satisfying* (3.4). *For $r > 0$, let $x(r)$ be the solution of the equation $T(x - z) = (1 + r)(x - z)$. Then there exists an f satisfying Theorem* 1.1 *and in addition,*

$$\lim_{r\to 0} f(rx(r)) = \lim_{r\to 0} \|rx(r)\| = \inf_{y\in C} \|Ty - y\|.$$

PROOF. Assume without loss of generality that $z = 0$. We saw in the proof of Lemma 2.3 that

$$\|Tx - x(r)\| \leqq \|x - x(r)\| - \|rx(r)\| + 2r\|Tx\|.$$

It follows that $\|rx(r)\| \leqq \|Tx - x\| + 2r\|Tx\|$ and hence, for $\|f\| \leqq 1$,

$$\limsup_{r\to 0} f(rx(r)) \leqq \limsup_{r\to 0} \|rx(r)\| \leqq \inf_{x\in C} \|Tx - x\| = \alpha.$$

To complete the proof, it suffices to show that f in Theorem 1.1 can be chosen so that

$$f(rx(r)) \geqq \alpha \qquad \text{for all } r > 0. \tag{3.6}$$

We first show that

$$\|x(r)-x(s)\| \leqq \frac{|r-s|}{r}\|x(s)\| \qquad \text{for all } r>0 \quad \text{and} \quad s>0. \tag{3.7}$$

Indeed, since $T/(1+r)$ is a contraction mapping, $(T/(1+r))^n x$ converges to its fixed point $x(r)$ for all $x \in C$, in particular, for $x = x(s)$. Hence

$$\begin{aligned}
\|x(r)-x(s)\| &\leqq \sum_{n=1}^{\infty}\left\|\left(\frac{T}{1+r}\right)^n x(s)-\left(\frac{T}{1+r}\right)^{n-1} x(s)\right\| \\
&\leqq \sum_{n=1}^{\infty}\left(\frac{1}{1+r}\right)^{n-1}\left\|\frac{T}{1+r}x(s)-x(s)\right\| \\
&= \frac{1+r}{r}\left|\frac{1+s}{1+r}-1\right|\|x(s)\| \\
&= \frac{|s-r|}{r}\|x(s)\|.
\end{aligned}$$

From (3.7), recalling that $\|sx(s)\| = \|Tx(s)-x(s)\| \geqq \alpha$ for all $s>0$, we obtain, for all $r>s>0$, $\|x(s)-x(r)\| \leqq \|x(s)\| - \alpha/r$, and therefore, by (2.4), $f_{x(s)}(x(r)) \geqq \alpha/r$. Letting $s \to 0$, we obtain (3.6) for the accumulation point f.

Q.E.D.

The previous theorem together with the characterizations (*) and (**) immediately imply the following:

3.8. Corollary. *If X is a strictly convex and reflexive Banach space then both $\{T^n x/n\}_{n=1}^{\infty}$ and $\{rx(r)\}_{r>0}$ converge weakly to the same limit. If X^* has Fréchet differentiable norm, then the convergence is strong.*

Acknowledgement

We would like to thank John Pratt for many helpful suggestions.

References

1. T. Bewley and E. Kohlberg, *The asymptotic theory of stochastic games*, Math. Oper. Res. **1** (1976), 197–208.

2. J. Diestel, *Geometry of Banach Spaces — Selected Topics*, Springer Verlag, 1975.

3. E. Kohlberg and A. Neyman, *Asymptotic behavior of nonexpansive mappings in uniformly convex Banach spaces*, Report No. 5/80, Institute for Advanced Studies, The Hebrew University of Jerusalem, Israel.

4. A. Pazy, *Asymptotic behavior of contractions in Hilbert space*, Israel J. Math. **9** (1971), 235–240.

5. S. Reich, *Asymptotic behavior of contractions in Banach spaces*, J. Math. Anal. Appl. **44** (1973), 57–70.

6. S. Reich, *Asymptotic behavior of semigroups of nonlinear contractions in Banach spaces*, J. Math. Anal. Appl. **53** (1976), 277–290.

7. S. Reich, *On the asymptotic behavior of nonlinear semigroups and the range of accretive operators*, J. Math. Anal. Appl., to appear.

Institute for Advanced Studies
The Hebrew University of Jerusalem
Jerusalem, Israel

and

Department of Mathematics
University of California
Berkeley, California, USA