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ASYMPTOTIC VALUES OF MIXED GAMES*

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1. INTRODUCTION

Since 1960 attention has focused more and more on games with large masses of players,[1/] i.e., where some of the participants are individually insignificant. Milnor and Shapley (1961), Shapiro and Shapley (1960), Shapley (1961) and Hart (1973), investigated value theories of "oceanic games," i.e., weighted majority games in which a sizable fraction of the total vote is controlled by a few large (major) players, and the rest is distributed among a large number of small (minor) voters. Shapiro and Shapley (1960), Milnor and Shapley (1961), and Shapley (1961), presented asymptotic results for the values of the major players, when the others become smaller and smaller. As for the minor ones, finding the limit of their values, turned out to be a much more difficult task; even in the case where there are no major players, this was an open problem for many years - only recently solved by the author (1979).

The main purpose of this paper is to settle the above question in general - i.e., for games with both major and minor voters. Intuitively, the result is that, for a coalition of small players, the (limit) value does not depend on its composition, but only on the total vote it has. More precisely, we consider two measures on the set of small players: the 'voting' and the 'value'. We prove that, as the largest minor vote tends to zero, the distance between the above two measures (defined as the bounded variation of the difference of their normalizations) also tends to zero.

The problem finds its natural and more general setting in the context of values of games with a continuum of players. The central interest is in those that are obtained as limits of values of finite approximants. The asymptotic value is the "strongest" possible such value in the sense, that if it exists for a

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[1/] Kuhn and Tucker list fourteen outstanding research problems in |10|. The eleventh urged us "to establish significant asymptotic properties of n-person games, for large n" (|10|, p. xii).

particular game v, then any limiting value[2/] will exist for that game and equal the asymptotic one. It should be pointed out that the existence of the asymptotic value for v is essentially a strong statement on the limit of the values of games with finitely many players[3/] (which approximate v).

The asymptotic value of games with a continuum of players has been studied extensively, (|10|, |5|, |7|, |9| and |15|). The set of all games having an asymptotic value is denoted by ASYMP. It has long been known [Kannai (1966), Aumann-Shapley (1974)] that non-atomic games that are "sufficiently differentiable" (i.e., games in pNA) have asymptotic values (i.e., pNA ⊂ ASYMP). Recently [Neyman (1979)] established the existence of an asymptotic value for singular non-atomic games (i.e., proved that bv'NA ⊂ ASYMP). As for mixed games, i.e., games with finitely many large players it has been shown [Fogelman and Quinzii (1975)] that mixed games that are sufficiently differentiable are in ASYMP (i.e., that pFL ⊂ ASYMP). The main result of this paper is that singular mixed games are in ASYMP i.e. to prove that bv'FL ⊂ ASYMP.

2. STATEMENT OF THE MAIN RESULT

A game is a set-function $v: \mathcal{C} \to \mathbb{R}$, where $(I,\mathcal{C})$ is a (standard) measurable space (the player space) with $v(\emptyset) = 0$; it is called finite if $\mathcal{C}$ if finite. The Shapley value of a finite game v is the measure on $\mathcal{C}$ given by

$$\phi v(A) = E(v(P^{\mathcal{R}}_A \cup A) - v(P^{\mathcal{R}}_a)) \tag{2.1}$$

where $P^{\mathcal{R}}_A$ is the set of players (atoms of $\mathcal{C}$) preceding A (an atom of $\mathcal{C}$) in the order $\mathcal{R}$, and E is the expectation operation when each order has equal probability |18|. To define the asymptotic value for a game v that is not necessarily finite, one approximate it by finite games. Specifically, if Π is a finite subfield of $\mathcal{C}$, define a finite game v_Π on Π by $v_\Pi = v|\Pi$. Given an S in $\mathcal{C}$ (a "coalition"), an increasing sequence $\{\Pi_1,\Pi_2,\ldots\}$ of finite subfield of $\mathcal{C}$ is called S-admissible if $S \in \Pi_1$ and $\cup_i \Pi_i$ generates $\mathcal{C}$. An asymptotic value of v is a set function ψv on $\mathcal{C}$ such that for all coalitions S and all S-admissible sequence, we have

$$\lim \phi v_{\Pi_n}(S) = \psi v(S) \quad . \tag{2.2}$$

[2/] Like the μ-value |2|, |3| and the partition value |16|.

[3/] For example, in (1977) Aumann-Kurz used the μ-value as the underlying value concept, which restricted the conclusions of their model to democratic societies. Later, when their games were shown to have asymptotic value (cf. |15|), it led to more general results.

The set of all games having asymptotic values is denoted by ASYMP.

Let $(I,\mathcal{C})$ be a measurable space isomorphic to $([0,1],\mathcal{B})$ where $\mathcal{B}$ is the σ-field of Borel sets in $[0,1]$. We denote by FL the set of all measures on $(I,\mathcal{C})$ with finitely many atoms. The space of all real-valued functions f of bounded variation on $[0,1]$ that obey $f(0) = 0$ and are continuous at 0 and 1 is denoted bv'. The closed subspace of BV spanned by the set functions of the form $f \circ \mu$, where $\mu \in$ FL is a probability measure and $f \in$ bv' is denoted bv'FL.

Main Theorem: bv'FL ⊂ ASYMP.

3. VALUES FOR FINITE GAMES - PREPARATIONS FOR THE PROOF

We begin by recalling that a finite game in coalitional (or characteristic function) form (a finite game for short) is usually represented as a pair (N,v), where N is a finite set and v is a real-valued function on the family 2^N of all subsets of N, with $v(\emptyset) = 0$ (clearly, this is equivalent to our definitions in the previous section; N is the set of atoms of the finite field $\mathcal{C}$). We may consider ϕv as a measure on N.

The formula (2.1) for the Shapley value uses the finite probability space of the orders on N. It turns out that it is much more convenient and powerful to replace that discrete probability space by a continuous one.

This is done as follows: Let (Ω,Σ,P) be a probability space such that to every $i \in N$ corresponds a real-valued random variable X_i, defined on (Ω,Σ,P), having uniform distribution on $(0,1)$; furthermore, let the random variables X_i be mutually independent. This "continuous embedding" induces, for almost all $\omega \in \Omega$, an order $\mathcal{R}(\omega)$ on N by $i\mathcal{R}(\omega)j$ iff $X_i(\omega) < X_j(\omega)$, and for every order $\mathcal{R}$ on Ω Prob $(\mathcal{R}(\omega) = \mathcal{R}) = 1/|N|!$. In what follows we will use P^ω_i or P_i instead of $P^{\mathcal{R}(\omega)}_i$. Observe that the stochastic process $N^t: [0,1] \to 2^N$ defined by $N^t(\omega) = \{i: i \in N, X_i(\omega) \leq t\}$, is nondecreasing, has stationary increments, which are sums (unions) of independent random variables, $N^0 = \emptyset$, and $N^1 = N$. In particular, if w is a measure on N, then $w(N^t)$ is a sum of independent (real-valued) random variables. For $\omega \in \Omega$, $i \in N$ and a game v on N we define

$$\Delta(i,\omega) = v(N^{X_i(\omega)}) - v(N^{X_i(\omega)} \setminus \{i\}) \quad .$$

Obviously $P_i^{\omega} = N^{X_i(\omega)} \setminus \{i\}$ and therefore,

$$(3.1) \qquad \phi v(i) = E(\Delta(i,\omega)) = \int_0^1 E(\Delta(i,\omega) \mid X_i = t) \cdot dt \quad .$$

A weighted majority game is one of the form $f_g \circ w$ where w is a non-negative measure on the players' set N (called: the voting measure), $0 < q < w(N)$, and $f_q(x) = 0$ or 1, according to $x < q$ or $x \geq q$. It is <u>normalized</u> if w is a probability measure, i.e., $w(N) = 1$. In the case of a finite (or countable) N, we shall sometimes use a more explicit symbol for the game, namely $[q; w_1, w_2, \ldots]$. Here $N = \{1,2,\ldots\}$ and w_i stands for $w(\{i\})$. We will also use M_i for $M \setminus \{i\}$; in particular $N_i^t = N^t \setminus \{i\}$.

<u>Lemma 1</u>: Let $v \equiv [q; w_1, \ldots, w_n]$ be a finite weighted majority game, with $0 < q < w(N)$. Then

$$\phi v(i) = \int_0^1 \text{Prob } (w(N_i^t) \in [q - w_i, q)) \cdot dt$$

$$= E(\int_0^1 \chi_{[q-w_i,q)}(w(N_i^t)) \cdot dt) \quad .$$

<u>Proof</u>:
$$\phi v(i) = E(\Delta(i,\omega)) = \text{Prob } (w(P_i) \in [q - w_i, q))$$

$$= \int_0^1 \text{Prob } (w(P_i) \in [q - w_i, q) \mid X_i = t) \cdot dt$$

$$= \int_0^1 \text{Prob } (w(N_i^t) \in [q - w_i, q)) \cdot dt \quad ,$$

and apply Fubini's theorem.

<u>Lemma 2</u>: Let $v = [q; w_1, \ldots, w_n]$ be a weighted majority game, and let $T = \inf \{t: w(N^t) \geq q\}$. Then

$$\phi v(i) = \text{Prob } (T = X_i) \quad .$$

<u>Proof</u>: Follows easily from the finiteness of N.

We turn now to the "key" lemma, which is a reformulation, in terms of weighted majority games, of the main result of |14|.

<u>Lemma 3</u>: For every $\varepsilon > 0$ there exist $K > 0$ and $\delta > 0$ such that if $v = [q; w_1, \ldots, w_n]$ is a normalized weighted majority game with $w_i < \delta$ and $K \cdot w_i < q < 1 - K \cdot w_i$ for every $i \in N$, then

$$\sum_{i=1}^{n} |\phi v(\{i\}) - w_i| < \varepsilon \quad .$$

Consider a sequence of $(m + n_k)$-person normalized weighted majority games

$$v_k = [q; w_1^k, \ldots, w_m^k, w_{m+1}^k, \ldots, w_{m+n_k}^k]$$

such that

$$(3.2) \qquad \sum_{j=1}^{n_k} w_{m+j}^k \to \alpha > 0 \quad \text{as} \quad k \to \infty$$

$$(3.3) \qquad w_i^k \to w_i \qquad \text{as} \quad k \to \infty$$

and

$$(3.4) \qquad \max_j w_{m+j}^k = w_{\max}^k \to 0 \quad \text{as} \quad k \to \infty \quad .$$

We shall use the following notations: M for the set $\{1,\ldots,m\}$ of "major" players; N_k for the set $\{m + 1, \ldots, m + n_k\}$ of "minor" players; w^k for the voting measure on $M \cup N_k$; and w for the 'limiting' voting measure on M. (Note that the players do not retain their identities from game to game in the sequence, nevertheless, in our continuous embedding, $X_1, \ldots, X_m$ will be independent of k.) We associate to that sequence of games the "ideal" stochastic process Z^t defined by $Z^t = w(M^t) + \alpha t$, and we define the <u>hitting time</u> of $[q,1]$ by $t_q = \inf \{t: Z^t \geq q\}$. The stochastic process Z^t and the hitting time t_q are "limits" of the "actual" stochastic processes $Z_k^t = w^k(M^t) + w^k(N_k^t)$, and hitting times $t_q^k = \inf \{t: Z_k^t \geq q\}$.

<u>Lemma 4</u>: For every $\varepsilon > 0$, $0 \leq t \leq 1$, and $0 < q < 1$

$$\text{Prob } (|Z_k^t - Z^t| > \varepsilon) \xrightarrow[k\to\infty]{} 0$$

and

$$\text{Prob } (|t_q - t_q^k| > \varepsilon) \xrightarrow[k\to\infty]{} 0 \quad .$$

<u>Proof</u>: The first part follows from (3.3) and Chebyshev's inequality (the weak law of large numbers) applied to $w^k(N^t)$ (which is a sum of independent random variables). The second part is then implied by the observation that $Z^{t_2} - Z^{t_1} \geq \alpha(t_2 - t_1)$ whenever $0 \leq t_1 < t_2 \leq 1$. $+\alpha X_i$

Let $A_i(\omega) = (w(M_i^{X_i}), w(M^{X_i})]$, and let $\phi_i = \text{Prob } (q \in A_i(\omega)) = \int_0^1 \text{Prob } (q \in A_i(\omega) \mid X_i = t) dt$.

<u>Lemma 5</u>: Let ϕ_i^k denote the value of the game v_k to the i-th major player. Then

$$\phi_i^k \to \phi_i \quad \text{as} \quad k \to \infty \quad .$$

Proof: This is Theorem 1 in Milnor-Shapley (1961) (alternatively, Theorem 1 in Shapiro-Shapley (1960)). For completeness we present here a short proof, based on our continuous embedding. Let $B_q = \{t\colon t\alpha + w(S) = q \text{ for some } S \subset M\}$. The set B_q is finite. For every $t \notin B_q$, $\min_S |w(S) + \alpha t - q| > 0$, and thus (by Chebyshev's inequality) we deduce that, for every $1 \le i \le m$

$$f_k(t) = \text{Prob}\,(w(M_i^t) + w(N_k^t) \in (q - w_i, q]\,|\,X_i = t)$$
$$\underset{k\to\infty}{\longrightarrow} \text{Prob}\,(q \in A_i(\omega)\,|\,X_i = t) \quad .$$

Using Lebesque dominated convergence theorem, we finally conclude that

$$\phi_i^k = \int_0^1 f_k(t)\cdot dt \underset{k\to\infty}{\longrightarrow} \int_0^1 \text{Prob}\,(q \in A_i(\omega)\,|\,X_i = t)\cdot dt = \phi_i \quad .$$

4. PROOF OF THE MAIN THEOREM

The notations are as in the previous sections.

Lemma 6: Let ϕ_i^k, for $1 \le i \le m + n_k$, denote the value of the game v_k to the i-th player, and let $\eta = 1 - \sum_{i=1}^{m} \phi_i$. Then,

(4.1) $$\sum_{i=m+1}^{m+n_k} |\alpha\cdot\phi_i^k - \eta\cdot w_i^k| \to 0 \quad \text{as} \quad k \to \infty \quad .$$

Proof: Let $t^k = \inf\{t\colon w^k(M^t \cup N_k^t) \ge q\}$; by Lemma 2, $\phi_i^k = \text{Prob}\,(X_i = t^k)$. In order to prove (4.1) it is enough to prove that for every $S_k \subset N_k$,

(4.2) $$\lim_{k\to\infty} \alpha(\phi v_k)(S_k) - \eta w^k(S_k) = 0 \quad .$$

By Lemma 5, and the efficiency of the Shapley value, $\phi v_k(N_k) \to \eta$, as $k \to \infty$, and therefore it will be enough to prove that for every $S_k \subset N_k$,

(4.3) $$\liminf_{k\to\infty} (\alpha\phi v_k(S_k) - \eta w^k(S_k)) \ge 0 \quad .$$

Let $S_k \subset N_k$ be given. Thus,

(4.4) $$\phi v_k(S_k) = \sum_{i\in S_k} \text{Prob}\,(t^k = X_i) = \text{Prob}\,\{t^k \in X(S_k)\}$$

where $X(S_k)$ is the random finite set $\{X_i\}_{i\in S_k}$. The idealization of t^k, namely t_q, is a function of $X_1,\ldots,X_m$ only. Let

$$\bar{t}^k = \inf\{t\colon w^k(N_k^t) + w^k(M^{t_q}) \ge q\} = \inf\{t\colon w^k(N_k^t) \ge q_k\} \quad ,$$
$$\text{where} \quad q_k = q - w^k(M^{t_q}) \quad .$$

Consider the weighted majority games (which depend on $X_1,\ldots,X_m$),

$$u_k = [q_k;\, w_{m+1}^k,\ldots,w_{m+n_k}^k]$$

on the set of players N_k. Let $\bar{\phi}^k$ be its value, and let ϕ^k be the restriction to N_k of the value of v_k. We would like to approximate ϕ^k in terms of $\bar{\phi}^k$ (as a "sub average"), and then use Lemma 3.

Let H^ε denote the event that for all $1 \le i \le m$, $|t_q - X_i| > \varepsilon$; the event $q \notin A_i(\omega)$ for all $1 \le i \le m$ will be denoted by H. Obviously, $\eta = \text{Prob}\,(H)$. Now we claim that

(4.5) $$\text{Prob}\,(H^\varepsilon|H) \underset{\varepsilon\to 0}{\longrightarrow} 1 \quad .$$

Indeed, if $|t_q - X_i| < \varepsilon$ and $q \notin A_i(\omega)$ then $X_i \in B_q + (-\varepsilon,\varepsilon)$ where $B_q = \{t\colon q = t\alpha + w(S) \text{ for some } S \subset M\}$. Since B_q is finite, the Lebesque measure of $B_q + (-\varepsilon,\varepsilon)$ tends to zero as $\varepsilon \to 0$, which completes the proof of (4.5).

For every fixed $\varepsilon > 0$, if q^k satisfies

(4.6) $$\alpha(\tfrac{\varepsilon}{2}) < q_k < \alpha(1 - \tfrac{\varepsilon}{2}) \quad ,$$

then by Lemma 3,

(4.7) $$\lim_{k\to\infty} \alpha\phi u_k(S_k) - w^k(S_k) = 0 \quad .$$

Lemma 4 now implies $\text{Prob}\,(|t_q - t^k| > \varepsilon/2) \underset{k\to\infty}{\longrightarrow} 0$, hence,

(4.8) $$\text{Prob}\,(\bar{t}^k = t^k|H^\varepsilon) \underset{k\to\infty}{\longrightarrow} 1 \quad .$$

Now,

(4.9) $$\phi v_k(S_k) = \text{Prob}\,(t^k \in X(S_k)) \ge \text{Prob}\,(t^k = \bar{t}^k \wedge \bar{t}^k \in X(S_k) \wedge H) \quad .$$

Observe that for (sufficiently) small $\varepsilon > 0$, there is K (large enough), such that for every $k \ge K$, (4.6) is implied by H^ε. As $\phi u_k(S_k) = \text{Prob}\,(\bar{t}^k \in X(S_k))$, we can combine (4.5), (4.7), (4.8) and (4.9), using standard probability rules, to get (4.2) - which completes the proof of this lemma.

We proceed by proving that the "jump functions" are in ASYMP. We identify a finite subfield of $\mathcal{C}$ with the partition it induces. If v is any set function on $(I,\mathcal{C})$, its dual v^* is defined by $v^*(S) = v(I) - v(I\setminus S)$. Observe that $v \in$ ASYMP iff $v^* \in$ ASYMP (easily shown by reversing order, see (|5|, p, 140)). If $0 < q < 1$ then $\bar{f}_q(x) = 0$ or 1 according to $x \le q$ or $x > q$.

Lemma 7: Let λ be a probability measure in FL, and $0 < q < 1$. Then $f_q \circ \lambda \in$ ASYMP and $\bar{f}_q \circ \lambda \in$ ASYMP.

Proof: Let $(\Pi_k)_{k=1}^{\infty}$ be an increasing S-admissible sequence of finite subfield of $\mathcal{C}$. Let $\{a_1,\dots,a_m\}$ be the finite set of atoms of λ. As $(\Pi_k)_{k=1}^{\infty}$ is increasing and $\cup\Pi_k$ generates $\mathcal{C}$, there exists K (large enough) such that for all $k \ge K$, each of the a_i's is in a different atom of Π_k. For such sequences $(\Pi_k)_{k=1}^{\infty}$, it is known that, as $k \to \infty$, we have $\lambda(A_k^i) \to \lambda(\{a_i\})$ whenever $a_i \in A_k^i \in \Pi_k$, and $\lambda(A_k) \to 0$ whenever $A_k \subset \Pi_k$ does not contain any of the atoms a_i $(1 \le i \le m)$. Therefore, the sequence of finite games $(f_q \circ \lambda)_{\Pi_k}$ are of the form $[q; w_1^k,\dots,w_m^k,w_{m+1}^k,\dots,w_{m+n_k}^k]$, and satisfy (3.2), (3.3), and (3.4) whenever λ contains a (non-trivial) non-atomic part. Thus by Lemma 6 we conclude that $f_q \circ \lambda \in$ ASYMP. If λ is purely atomic, then obviously $f_q \circ \lambda \in$ ASYMP.

Now observe that $\bar{f}_q \circ \lambda = (f_{1-q} \circ \lambda)^*$ and therefore $\bar{f}_q \circ \lambda \in$ ASYMP.

Lemma 8: Let $f \in bv'$ be right continuous, and let λ be a probability measure in FL. Then $f \circ \lambda \in$ ASYMP.

Proof: Let $S \in \mathcal{C}$, and let $(\Pi_k)_{k=1}^{\infty}$ be an S-admissible sequence. By Lemma 3.4 of |15|,

$$\phi(f \circ \lambda)_{\Pi_k}(S) = \int_0^1 \phi(f_q \circ \lambda)_{\Pi_k}(S_k)\,df(q) \quad .$$

For every $0 < q < 1$, $f_q \in$ ASYMP (Lemma 7); thus as $k \to \infty$, $\phi(f_q \circ \lambda)_{\Pi_k}(S) \to \psi(f_q \circ \lambda)_{\Pi_k}(S)$ where ψ denotes the asymptotic value. Using Lebesque's dominated convergence theorem, we conclude that $\psi(f \circ \lambda)_{\Pi_k}(S) \to \int_0^1 \psi(f_q \circ \lambda)(S)\cdot df(g)$, hence $f \circ \lambda \in$ ASYMP and

$$\psi(f \circ \lambda)(S) = \int_0^1 \psi(f_q \circ \lambda)(S)\,df(q) \quad . \tag{4.10}$$

Theorem A: $bv'FL \subset$ ASYMP.

Proof: Theorem F of |5|, asserts that ASYMP is a closed symmetric linear subspace of BV-the space of all set functions of bounded variation. Thus, by Lemma 5 and Lemma 6, ASYMP contains every game of the form

$$f = g + \sum_{i=1}^{\infty} \alpha_i \cdot \bar{f}_{q_i} \quad , \tag{4.11}$$

with $g \in bv'$ right continuous, $q_i \in (0,1)$, and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$. Since every $f \in bv'$ has such (i.e., (4.11)) a representation, we conclude, by recalling definition of $bv'FL$, and using again the closeness of ASYMP that $bv'FL \subset$ ASYMP.

5. FURTHER RESULTS AND OPEN PROBLEMS

In this section we shall state few additional results, and present some open problems.

Denote by M the space of measures on the underlying mesurable space $(I,\mathcal{C})$, M_a will denote the subspace of M of all purely atomic measures, and $M_b = M\setminus M_a$, i.e., M_b is the set of all measures with a (non-trivial) non-atomic part; M^1 denotes the subset of M of all non-negative measures μ, with $\mu(I) = 1$, and FL^1, M_a^1, M_b^1 are similarly defined. The closed subspace of BV spanned by power of measures in M^1 is denoted by pM, and the one spanned by the set functions of the form $f \circ \mu$ where $f \in bv'$ and $\mu \in M^1$, is denoted by $bv'M$. In the same manner the spaces pFL, pM_a, pM_b, $bv'M_a$, $bv'M_b$ are defined. Let A denote the closed algebra generated by games of the form $f \circ \mu$, where $\mu \in FL^1$ and $f \in bv'$ is continuous. If Q_1 and Q_2 are two subspaces of BV we denote by $Q_1 * Q_2$ the minimal closed space which contain Q_1, Q_2 and $Q_1 \cdot Q_2$.

Theorem B: $pFL = pM \subsetneq bv'FL \subsetneq pM * bv'FL \subsetneq A * bv'FL \subsetneq$ ASYMP.

We would like to replace (in Theorem B) FL by M. This leads us to the following,

Open Problem: Is $bv'M \subset$ ASYMP, or even is $bv'M_a \subset$ ASYMP or $bv'M_b \subset$ ASYMP, or even is $f_q \circ \mu \in$ ASYMP for every $\mu \in M_b^1$, or for every $\mu \in M_a^1$.

This problem has proved to be very stubborn. The last part turns out to be equivalent to the open problem raised by Shapley (|20|) as to whether or not every weighted majority game with countable many players is regular. Observe, that our method implies, that if $\mu \in M^1$ and $f_q \circ \mu \in$ ASYMP for every $0 < q < 1$ than $f \circ \mu \in$ ASYMP for every $f \in bv'$, and thus these problems reduces to those of the jump functions. A positive solution to this question would imply in particular the existence of a partition value (in particular a value) on $bv'M$. Even this is unknown. Indeed we state

<u>Open Problem</u>: Does there exist a partition value on bv'M, or, even, does there exist a value on bv'M.

Along these lines, we can state the following:

<u>Theorem C</u>: (a) There exists a partition value on $bv'M_b$.

(b) If $f \in bv'$ and $\lambda \in M^1$, then for every subset B of the set of atoms of the measure λ, and every B-admissible sequence of partitions $(\Pi_k)_{k=1}^{\infty}$, the limit $\lim_{k\to\infty} \phi(f \circ \lambda)_{\Pi_k}(B)$ exists and is independent of the particular sequence $(\Pi_k)_{k=1}^{\infty}$.

We turn now to results concerning asymptotic pre-values; for each Borel measure λ on $[0,1]$ the λ-<u>pre value</u> ϕ_λ of the finite game (N,v) is the measure on N given by,

$$\phi_\lambda v(i) = \int_0^1 E\,(\Delta(i,\omega)\,|\,X_i = t) \cdot d\lambda(t) \quad .$$

The main conceptual interest is in those that are semi values, i.e., where λ is a probability measure (see |6|, |21|). However the pre values are mathematically convenient.

As with values, we may investigate the limiting λ-pre values, and we define (in the natural way) the space $ASYMP_\lambda$ to be the space of all games having an asymptotic λ-pre value. The proof of Lemma 5, reveals that the λ-pre values for the major player in the sequence of games v_k (we use notations from previous sections), converges to the limit, $\int f(t) \cdot d\lambda(t)$, whenever λ does not have an atom in the set B_q. (When λ has an atom in the set B_q the λ-pre value of the major player does not necessarily converge.) Thus if λ is non-atomic, the λ-pre values of the major players converge to a limit for every $0 < q < 1$. However, in order that the λ-pre values of the sequence of games v_k will converge (in the same manner that the values do) further assumptions (on λ) are needed.

<u>Theorem D</u>: (a) $bv'FL \subset ASYMP_\lambda$ iff λ is absolutely continuous with respect to Lebesque measure ℓ, and such that $d\lambda/d\ell$ is continuous.

(b) $pM \subset ASYMP_\lambda$ iff λ is absolutely continuous with respect to the Lebesque measure ℓ, and $d\lambda/d\ell \in L_\infty$.

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