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CONTINUOUS VALUES ARE DIAGONAL*†

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It is proved that every continuous value is diagonal, which in particular implies that every value on a closed reproducing space is diagonal. We deduce also that there are noncontinuous values.

1. Introduction. The axiomatic definition of the value of a nonatomic game was introduced by R. J. Aumann and L. S. Shapley in their book, *Values of Non-Atomic Games* [1]. [1, Chapter 7] discusses the diagonal property, which appears to be basic in the study of values. All the values examined there enjoy the diagonal property, and the authors raised the problem as to whether or not this property is a consequence of the axioms defining the value. This question was answered by A. Neyman and Y. Tauman [4] in the negative, by means of a counterexample. As a result of the counterexample, new problems arise: Can one formulate an additional "natural" axiom that will guarantee the diagonality of the value, or alternatively can one point to a class of spaces on which any value is diagonal. Regarding the second question, Y. Tauman [5] has shown that even on reproducing spaces there exist nondiagonal values.

In this paper we shall show that any continuous value is diagonal, which in particular implies that any value on a closed reproducing space is diagonal, thus solving these open problems. It is worth mentioning that the proofs do not make use of the efficiency axiom. Thus, the results remain valid for semivalues, i.e., linear symmetric positive operators from set functions to additive set functions.

Besides the wish to understand the diagonality of all the values occurring in [1], there is another strong reason to consider diagonal values. As nonatomic games are models for games with large masses of players, in which no individual player can affect the overall outcome, it is desirable to consider values that are limits of values of such finite games. Using the known formula for the unique value of a finite game together with some probability arguments, one can show that whenever a value of a nonatomic game is a limit of a sequence of values of finite games that approximate the nonatomic one, then this value is diagonal.

2. Definitions and statement of results. All the definitions and notations are as in [1].

Let $(I, \mathcal{C})$ be the measurable space $([0, 1], \mathfrak{B})$ where $\mathfrak{B}$ is the σ-field of Borel sets in $[0, 1]$. A *set function* is a real-valued function v on $\mathcal{C}$ such that $v(\emptyset) = 0$. The members of I are called *players*, the members of $\mathcal{C}$ *coalitions*, and the set functions *games*. A game v is *monotonic* if for each $S, T \in \mathcal{C}$, $S \subset T \Rightarrow v(S) \leqslant v(T)$. If Q is a set of games, Q^+ denotes the subset of monotonic games in Q. A game v is of *bounded variation* if it is the difference between two monotonic games. The *variation norm* of v is defined by

$$\|v\| = \inf(u(I) + w(I)),$$

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where the inf ranges over all monotonic set functions u and w such that $v = u - w$. The space of all games of bounded variation is called BV. The subspace of BV consisting of all bounded, finitely additive set functions is denoted FA.

Let Q be any subspace of BV. A mapping of Q into BV is *positive* if it maps Q^+ into BV^+. Q is reproducing if $Q = Q^+ - Q^+$.

Let $\mathcal{G}$ be the group of automorphisms of $(I, \mathcal{C})$ (i.e., one to one mappings of I onto itself that are measurable in both directions). Each $\theta \in \mathcal{G}$ induces a linear mapping θ^* of BV onto itself, defined by $(\theta^* v)(S) = v(\theta S)$ for all $s \in \mathcal{C}$.

Let Q be a symmetric subspace of BV. A *value* on Q is a positive linear mapping φ from Q into FA that satisfies:

$$\varphi \text{ is } \textit{symmetric}, \text{ i.e., } \varphi\theta^* = \theta^*\varphi \text{ for all } \theta \in \mathcal{G}. \tag{2.1}$$

$$\varphi \text{ is } \textit{efficient}, \text{ i.e., } \varphi v(I) = v(I) \text{ for all } v \in Q. \tag{2.2}$$

Define DIAG to be the set of all $v \in BV$ satisfying:

There exists a positive integer k, a k-dimensional vector ξ, of nonatomic probability measures, and a neighborhood U in E^K of the diagonal $[0, \xi(I)]$ *such that if $\xi(S) \in U$ then $v(S) = 0$.*

Note that DIAG is a symmetric subspace of BV. Let Q be a symmetric subspace of BV, γ a value on Q. We shall say that the pair (Q, γ) enjoys the *diagonal property*, and that γ is a *diagonal value*, if we have $\gamma v = 0$ for all $v \in Q \cap$ DIAG. We might paraphrase this by saying that if v vanishes in a neighborhood of some "diagonal", then its value also vanishes.

The main result of the present paper is the following:

MAIN THEOREM. *Any continuous value is diagonal.*

From this we obtain the following:

MAIN COROLLARY. *Any value on a closed reproducing space is diagonal.*

PROOF. This is an immediate consequence of the main theorem and [1, proposition 4.15], which asserts that any positive linear operator from a closed reproducing subspace of BV into BV is continuous.

It was proven in [1] that the unique value on $bv'NA$, the asymptotic value on ASYMP and the mixing value on MIX are all diagonal [1, propositions 43.1, 43.2 and 43.11]. Since all those values are continuous it now follows immediately from the main theorem that they are all diagonal. The main theorem enables us also to sharpen proposition 43.13 of [1] to the following one.

PROPOSITION. *There is a unique continuous value on $pNAD$.*

Another direct implication of the main theorem (in view of either [4] or [5]) is the existence of noncontinuous values.

3. Proof of the main theorem. The idea of the proof is as follows: We assume that φ is a continuous value on Q and $v \in Q \cap$ DIAG. Then we shall show that there exist an automorphism τ of the measurable space and a constant K such that for any n and a_i with $|a_i| \leqslant 1$ we have

$$\left\| \sum_{i=1}^{n} a_i (\tau^i)^* v \right\| \leqslant K. \tag{*}$$

On the other hand, as $\varphi v \in FA$ we shall see that for any automorphism τ there exists a constant K_2 such that for each n there exist constants b_i with $|b_i| \leqslant 1$, such that

$$\left\| \sum_{i=1}^{n} b_i(\tau^i)^* \varphi v \right\| \geqslant K_2 \cdot n^{\frac{1}{2}} \|\varphi v\|. \tag{**}$$

Then, as φ is linear and symmetric, by looking at games of the form $\sum_{i=1}^{n} b_i(\tau^i)^* v$, (*) and (**) contradict the continuity of φ, unless $\varphi v = 0$.

Let φ be a continuous value on Q and let $v \in \mathrm{DIAG}$. $v \in \mathrm{DIAG}$ implies that there exist an integer k, nonatomic probability measures $\mu_1, \ldots, \mu_k$ and a positive constant $\epsilon > 0$ such that if $|\mu_i(S) - \mu_j(S)| < \epsilon$ for each $i, j = 1, \ldots, k$ then $v(S) = 0$. Denote by μ the scalar measure $\sum_{i=1}^{k} \mu_i$ and by f_i the Radon Nikodym derivative of μ_i with respect to μ. (Note that $f_i \in L_1(\mu)$.)

Let τ be an automorphism of the measurable space $(I, \mathcal{C})$ that is μ-measure preserving and μ-mixing, i.e.,

$$\mu(\tau S) = \mu(S) \quad \text{for each} \quad S \in \mathcal{C}. \tag{3.1}$$

$$\mu(B \cap \tau^n A) \underset{n \to \infty}{\to} \frac{\mu(A) \cdot \mu(B)}{\mu(I)} \quad \text{for all measurable sets} \quad A, B. \tag{3.2}$$

The automorphism τ induces a positive isometry P of L_1 defined by $Pf = f \cdot \tau$. Condition (3.2) implies that for each $f \in L_1^0$ (i.e., $f \in L_1$ and $\int f = 0$) $p^m f$ converge weakly to 0. The Blum-Hanson theorem [3] (for further references see A. Bellow [2]) asserts that in such a case

for each $f \in L_1^0$, $\frac{1}{n} \sum_{1}^{n} P^{k_i} f$ converges strongly (to 0) in L_1, for every sequence of integers k_1, $0 \leqslant k_1 < k_2 < \cdots$. (3.3)

Let $f \in L_1^0$, then (3.3) implies the existence of constants $K_f(\delta)$ $(\delta > 0)$ such that

$$\text{for each } g \in L_\infty(\mu) \text{ with } \|g\| \leqslant 1 \#\{m : |\langle p^m f, g\rangle| \geqslant \delta\} \leqslant K_f(\delta) \tag{3.4}$$

(where for $f \in L_1$ and $g \in L_\infty$ we denote by $\langle f, g\rangle$ the integral $\int fg \, d\mu$, and $\#$ means the cardinality of).

Otherwise we can construct by induction a sequence of finite sets K_n of positive integers satisfying: if $n < m$, $i \in K_n$ and $j \in K_m$ then $i < j$ and $\# K_n > n(\#\tilde{K}_{n-1})$ where $\tilde{K}_m = \bigcup_{i=1}^{m} K_i$ and:

$$\frac{1}{\# K_n} \left(\sum_{i \in K_n} \langle p^i f, g\rangle \right) > \delta \quad \text{for some} \quad g \in L_\infty \quad \text{with} \quad \|g\| \leqslant 1. \tag{3.5}$$

Therefore

$$\begin{aligned} \left\| \frac{1}{\#\tilde{K}_n} \left(\sum_{i \in \tilde{K}_n} p^i f \right) \right\| &\geqslant \left\| \frac{1}{\#\tilde{K}_n} \sum_{i \in K_n} p^i f \right\| - \left\| \frac{1}{\#\tilde{K}_n} \sum_{i \in \tilde{K}_{n-1}} p^i f \right\| \\ &\geqslant \frac{n}{n+1} \delta - \frac{1}{1+n} \cdot \|f\| \end{aligned} \tag{3.6}$$

and by looking on the sequence of increasing positive integers determined by $\bigcup K_n$, (3.6) contradicts (3.3) and thus the proof of (3.4) is complete.

Now we shall show that there exists a constant K such that for any sequence a_i with $|a_i| \leqslant 1$ and any positive integer n, (*) holds.

Let ξ be the k-dimensional vector of measures $\mu_1, \ldots, \mu_k$. For $\delta > 0$ denote by $U(\xi, \delta)$ the set of all $S \in \mathcal{C}$ such that $|\mu_i(S) - \mu_j(S)| < \delta$ for each $i, j = 1, \ldots, k$. For any $i, j = 1, \ldots, k$, let $K_{i,j}(\delta)$ be the constant $K_f(\delta)$ given by (3.4) where $f = f_i - f_j$.

For any positive integer m and any $S \in \mathcal{C}$ we have

$$\mu_i(\tau^m S) - \mu_j(\tau^m S) = \langle p^m(f_i - f_j), \chi_S \rangle \tag{3.7}$$

where χ_S is the characteristic function of S.

Denote $K(\delta) = \sum_{i,j=1}^{k} K_{i,j}(\delta)$. Then, as $\chi_S \in L_\infty$ with $\|\chi_S\| \leqslant 1$ for any $S \in \mathcal{C}$, (3.4) and (3.7) imply that

$$\#\{m : \tau^m S \notin U(\xi, \delta)\} \leqslant K(\delta). \tag{3.8}$$

Observe that if $\mu(S \Delta T) < \eta$ (Δ denotes the symmetric difference) then $\tau^m S \in U(\xi, \delta)$ implies that $\tau^m T \in U(\xi, \delta + \eta)$. Thus for any $S \in C$

$$\begin{aligned} \#\{m\colon & \text{there exist } T \in \mathcal{C} \text{ with } \mu(T \Delta S) < \eta \text{ and} \\ & \tau^m T \notin U(\xi, \delta + \eta)\} \leqslant K(\delta). \end{aligned} \tag{3.9}$$

Let Ω be a chain, $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_L = I$. Then the variation of $u \in BV$ over Ω is

$$\|u\|_\Omega = \sum_{i=1}^{L} |u(S_i) - u(S_{i-1})|, \tag{3.10}$$

and it is known that $\|u\| = \sup \|u\|_\Omega$, where the sup is taken over all chains Ω [1, proposition 4.1]. Thus to prove (*) it is enough to show that

$$\left\| \sum_{i=1}^{n} a_i (\tau^i)^* v \right\|_\Omega \leqslant K. \tag{3.11}$$

For given $\eta > 0$, the positivity of μ and the fact that $\mu(I) = k$ yield the existence of indices $i_1, \ldots, i_N$ with $N \leqslant k/\eta + 1$ and $0 \leqslant i_j \leqslant L$ such that for each l, $l = 1, \ldots, L$ there exists j, $1 \leqslant j \leqslant N$ such that $\mu(S_l \Delta S_{i_j}) < \eta$.

Denote by M the set of all positive integers m for which there exists l, $1 \leqslant l \leqslant L$, with $\tau^m S_l \notin U(\xi, \delta + \eta)$. The last argument and (3.9) yield

$$\#M \leqslant N \cdot K(\delta) \leqslant (k/\eta + 1) \cdot K(\delta). \tag{3.12}$$

If δ and η are chosen so that $\delta + \eta < \epsilon$, then $i \notin M$ implies that $v(\tau^i S_l) = 0$ for each $l = 1, \ldots, L$ and thus $\|(\tau^i)^* v\|_\Omega = 0$. Therefore

$$\begin{aligned} \left\| \sum_{i=1}^{n} a_i (\tau^i)^* v \right\|_\Omega &\leqslant \sum_{i=1}^{n} \|a_i (\tau^i)^* v\|_\Omega \leqslant \sum_{i=1}^{n} \|(\tau^i)^* v\|_\Omega \\ &\leqslant \left(\frac{k}{\eta} + 1 \right) \cdot K(\delta) \cdot \|v\|. \end{aligned}$$

As this holds for any chain Ω, this completes the proof of (*).

Assume now that $\varphi v \neq 0$. We shall show that there exists a constant K_2 such that for each n there exists a selection of signs $\epsilon_i = \pm 1$ with

$$\left\| \sum_{i=1}^{n} \epsilon_i (\tau^i)^* \varphi v \right\| \geqslant K_2 \cdot n^{\frac{1}{2}} \|\varphi v\|. \tag{3.13}$$

Let n and $\delta > 0$ be given. As $(\tau^i)^*\varphi v \in FA$ there exists a chain $\Omega : \emptyset = S_0 \subset S_1 \subset \cdots \subset S_m = I$ such that

$$\|(\tau^i)^*\varphi v\|_\Omega \geqslant \|(\tau^i)^*\varphi v\|(1-\delta) = \|\varphi v\|(1-\delta) \tag{3.14}$$

for each $i = 1, \ldots, n$.

Note that if $T_i = S_i \backslash S_{i-1}$, $i = 1, \ldots, m$, then for each $u \in FA$ $\|u\|_\Omega = \sum_{i=1}^m |u(T_i)|$. Thus, for a given selection of signs $(\epsilon_1, \ldots, \epsilon_n)$,

$$\left\| \sum_{i=1}^n \epsilon_i (\tau^i)^* \varphi v \right\|_\Omega = \sum_{j=1}^m \left| \sum_{i=1}^n \epsilon_i ((\tau^i)^*\varphi v)(T_j) \right| = \sum_{j=1}^m \left| \sum_{i=1}^n \epsilon_i a_{ij} \right| \tag{3.15}$$

where $a_{ij} = (\tau^i)^*\varphi v(T_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Therefore, using first Khinchin's inequality [6, p. 213 (8.5)] and then Schwarz's inequality, we find that if we let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ range over all selections of signs, then

$$\begin{aligned}
\frac{1}{2^n} \sum_\epsilon \sum_{j=1}^m \left| \sum_{i=1}^n \epsilon_i a_{ij} \right| &= \sum_{j=1}^m \frac{1}{2^n} \sum_\epsilon \left| \sum_{i=1}^n \epsilon_i a_{ij} \right| \\
&\geqslant K_2 \cdot \sum_{j=1}^m \left(\sum_{i=1}^n a_{ij}^2 \right)^{\frac{1}{2}} = K_2 \cdot \sum_{j=1}^m \left(\sum_{i=1}^n a_{ij}^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n \left(\frac{1}{\sqrt{n}} \right)^2 \right)^{\frac{1}{2}} \\
&\geqslant K_2 \cdot n^{-\frac{1}{2}} \cdot \sum_{j=1}^m \sum_{i=1}^n |a_{ij}| = K_2 \cdot n^{-\frac{1}{2}} \cdot \sum_{i=1}^n \sum_{j=1}^m |a_{ij}| \\
&= K_2 \cdot n^{-\frac{1}{2}} \cdot \sum_{i=1}^n \|(\tau^i)^*\varphi v\|_\Omega \geqslant K_2 \cdot n^{-\frac{1}{2}} \cdot n \cdot \|\varphi v\|(1-\delta).
\end{aligned} \tag{3.16}$$

Therefore ((3.15) and (3.16)) there exists a selection of signs $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ $\epsilon_i = \pm 1$ with

$$\left\| \sum_{i=1}^n \epsilon_1 (\tau^i)^*\varphi v \right\| \geqslant \left\| \sum_{i=1}^n \epsilon_i (\tau^i)^*\varphi v \right\|_\Omega \geqslant K_2 \cdot n^{\frac{1}{2}} \cdot \|\varphi v\|(1-\delta) \tag{3.17}$$

and as this holds for any $\delta > 0$, (**) is proved. This completes the proof of the theorem.

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