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COOPERATION IN REPEATED GAMES WHEN THE NUMBER OF STAGES IS NOT COMMONLY KNOWN[1]

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It is shown that an exponentially small departure from the common knowledge assumption on the number T of repetitions of the prisoners' dilemma already enables cooperation. More generally, with such a departure, any feasible individually rational outcome of any one-shot game can be approximated by a subgame perfect equilibrium of a finitely repeated version of that game.

The sense in which the departure from common knowledge is small is as follows: (i) With probability one, the players know T with precision $\pm K$. (ii) With probability $1-\varepsilon$, the players know T precisely; moreover, this knowledge is mutual of order εT. (iii) The deviation of T from its finite expectation is exponentially small.

KEYWORDS: Game theory, finitely repeated games, common knowledge, prisoners' dilemma, cooperation.

1. INTRODUCTION[3]

IT HAS OFTEN BEEN OBSERVED that cooperative behavior may emerge in noncooperative situations when the nature of the interaction is long term. A fundamental message of the theory of repeated games is that the cooperative outcomes of a multi-person game are consistent with the usual "selfish" utility maximizing behavior assumed in economic theory, provided those games are repeated over and over. This fundamental message is delivered through the folk theorem and several related results (Abreu, Dutta, and Smith (1994), Aumann (1959, 1960, 1981), Aumann and Shapley (1994), Fudenberg and Maskin (1986), Rubinstein (1994), Smale (1980), Sorin (1986, 1990, 1992), and Wen (1994)), which assert that cooperative outcomes of the one-shot game are Nash equilibria (and also perfect equilibria) of the infinite repetition of that game. In some cases, the Nash equilibria of finitely repeated games also rationalize cooperation; this happens when the convex hull of the (vector) equilibrium payoffs contains a point that strictly dominates the smallest individually rational payoffs (Benoir and Krishna (1985)). However, the prisoners' dilemma is not in this class of games; indeed, in any finite repetition of the prisoners' dilemma, all equilibria and all correlated or even communication equilibria lead to the noncooperative outcome at each stage. This contrasts with a common observation in the experiments involving finite repetitions of the prisoners' dilemma, that players

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[3]Parts of the Introduction are taken from Neyman (1985).

do not always choose the single-period dominant actions, but instead achieve some mode of cooperation.

In the literature, there are several approaches that rationalize some measure of cooperation in the finitely repeated prisoners' dilemma. Radner (1980, 1986) explores a departure from the strict "rationality" of Nash equilibria. If each player is satisfied to get "close" (in utility) to the best response to the other players' strategies (epsilon-equilibria), then as the number of repetitions increases, the corresponding sets of epsilon-equilibria allow longer and longer periods of cooperation. Kreps, Milgrom, Roberts, and Wilson (1982), and Fudenberg and Maskin (1986), show how incomplete information about players' options or motivation or behavior can explain the observed cooperation. Neyman (1985, 1995) shows that cooperation in the finitely repeated prisoners' dilemma is justified under the assumption that there are bounds (possibly very large) to the complexity of the strategies that a player may use.

Each of the several approaches represents a kind of perturbation of the game. The definition of Nash equilibria in n-person games is based on a strict preference relation of each player on n-tuples of strategies. An n-tuple of strategies f is a Nash equilibrium if no player can unilaterally change his strategy in such a way that he will strictly prefer the new n-tuple of strategies. The epsilon-equilibrium can be interpreted as a perturbation that coarsens the strict preference relation of each player on n-tuples of strategies. The Kreps, Milgrom, Roberts, and Wilson (1982) and Fudenberg and Maskin (1986) approach can be considered as a perturbation of the strategy sets of the players in which at least one of the two players is restricted to mixed strategies that with probability epsilon choose a pre-specified strategy. The Neyman (1985, 1995) bounded complexity approach perturbs the repeated game by assuming that the players' strategies are restricted to those below some level of complexity. At least one of the two players is restricted to strategies implementable by automata of size that is subexponential in the number of repetitions.

In the epsilon-equilibria approach and the bounded complexity approach, the same perturbation rationalizes all individually rational and feasible payoffs as equilibrium payoffs. In Kreps, Milgrom, Roberts, and Wilson (1982) and Fudenberg and Maskin (1986), different individually rational and feasible payoffs are rationalized by different perturbations. Thus the equilibrium payoff set depends on the precise form of irrationality specified. The Fudenberg and Maskin "incomplete information" Folk Theorem shows that "by varying the kind of irrationality specified, one can trace out the entire set of individually rational and feasible payoffs" (Fudenberg and Maskin (1986, p. 535)). The Fudenberg and Maskin approach can also be interpreted as an incomplete information model in which the incomplete information is about the stage payoff, by assuming that with probability epsilon, the stage payoff is zero. On this interpretation, this single perturbation rationalizes all the individually rational and feasible payoffs.

The present paper explores the possibility of cooperation in the finitely repeated prisoners' dilemma, as well as in other finitely repeated games, in a different class of perturbations. Instead of altering the strict preference relations on outcomes or the strategies of the players in the finitely repeated prisoners' dilemma, or placing an uncertainty about the payoffs of the stage game, we analyze the equilibria of games in which at any state of nature the players "face" a finitely repeated prisoners' dilemma. The incomplete information is about the finite duration of the repeated game. The present approach seems to capture a smaller departure from rationality.

It is not surprising that cooperation can emerge in games when the players are not informed of the number of repetitions. The arguments that show that all equilibria of the finitely repeated prisoners' dilemma lead to the noncooperative outcome are usually based on the fact that both players know (are informed of) the number of repetitions. Moreover, each has to know that the other knows and so on. These assumptions are difficult to justify in most applications in which finitely repeated games are used to model the interactive decision problem. Even when both players are informed of the number of repetitions they might not be informed of the knowledge of the other player, or at least the number of repetitions might not be common knowledge. This paper shows that, in a finite repetition of the prisoners' dilemma, cooperation can result when there is an exponentially small departure from the common knowledge assumption. More generally, for every strategic game G, there is (for any value of n) a model of uncertain duration such that the expectation of the number of repetitions is exponentially (as a function of n) close to n, and the number of repetitions is exponentially close to common knowledge (in a sense to be defined precisely later), and the set of equilibrium payoffs approximate the set of individually rational and feasible payoffs.

An alternative approximation of common knowledge of the number of iterations is by means of common belief (Monderer and Samet (1989)). However, imposing common $(1-\varepsilon)$-belief of the event $T=n$ does not put any restriction on the number of repetitions at a set of states with small probability. Thus with a small probability the difference between the payoffs in the T-stage game and the payoffs in the n-stage game can be arbitrarily large. In particular, the expected payoffs can differ dramatically from the payoffs in the n-stage game. Therefore, for any $\varepsilon > 0$, a model of the repeated game with uncertain duration T with common $(1-\varepsilon)$-belief of the event $T=n$ is not necessarily an "approximation" of the n-stage repeated game. Thus, it is not surprising that, for any $\varepsilon > 0$, common $(1-\varepsilon)$-belief of the event $T=n$ does not disable cooperation. Indeed it allows an uncertainty structure that enables cooperation for essentially the same reasons as in the λ discounted game. Assume that the distribution of T is such that $T=n$ with probability $1-\varepsilon$, and with probability $\varepsilon(1-\lambda)\lambda^k$, $k=0,1,\ldots,$ $T=n+k+l$. With the trivial uncertainty structure (i.e., no information about the value of T), the event $T=n$ is everywhere common $(1-\varepsilon)$-belief, and for sufficiently large values of l and $\lambda < 1$ sufficiently close to

1 the trigger strategies are an equilibrium of G^T where G is the prisoners' dilemma.

The results in the present paper do quantify the size of the perturbation that is needed to enable cooperation in the n-stage repeated game. In Radner's epsilon-equilibria approach and the Fudenberg and Maskin incomplete information Folk Theorem, the size of the perturbations that enable rational cooperation is of the order of $1/n$, where n is the number of repetitions. In the present contribution a significantly smaller perturbation is needed. An exponentially small (in the expected number of repetitions) perturbation is needed to enable rational cooperation.

2. THE MAIN RESULTS

2.1. *The Model*

The uncertainty of the players, including their uncertainty on the number of repetitions, is described by means of the standard model of games with incomplete information. The model of a finitely repeated game with incomplete information on the number of repetitions consists of a strategic game G and an uncertainty structure described by the following:

(i) a probability space $(\Omega, \mathscr{B}, P)$;
(ii) for each player i, a σ-field $\mathscr{B}^i \subset \mathscr{B}$; and
(iii) a measurable function T: $\Omega \to \mathbb{N}$.

The uncertainty structure is thus represented by $((\Omega, \mathscr{B}, P), (\mathscr{B}^i), T)$ and for simplicity of notation we will call it T for short. The induced game, called G^T for short, starts by nature choosing a point $\omega \in \Omega$ according to the probability P. Each player is informed about the choice of nature via $\mathscr{B}^i$.

A *countable uncertainty structure* is an uncertainty structure for which Ω is a countable set. In a countable uncertainty structure, $\mathscr{B}^i$ is identified with a partition of Ω, and then each player i is informed of the element $\mathscr{B}^i(\omega)$ of the partition $\mathscr{B}^i$ that contains ω. Thereafter, the game G is played repeatedly for $T(\omega)$ stages, and after each stage players are informed of the action taken by the other players.

All uncertainty structures in the present paper are countable uncertainty structures such that for any $\omega \in \Omega$, $\bigcap_{i \in N} B^i(\omega) = \{\omega\}$ where N is the set of players. Therefore, setting $X^i = \{B^i(\omega)\colon\ \omega \in \Omega\}$, we identify Ω with a subset of $\times_{i \in N} X^i$. Viewing X^i as the set of types of player i, Ω is the set of vector of types. Thus the model of uncertain duration fits the classical theory of games with incomplete information.

An event is a ($\mathscr{B}$-measurable) subset of Ω. Given a random variable f defined on the probability space $(\Omega, \mathscr{B}, P)$, we denote by $E(f)$ its expectations and by $E(f|\mathscr{B}^i)$ its conditional expectation given $\mathscr{B}^i$. The payoff to each player in G^T is the sum of his payoffs in all stages of the game, divided by $E(T)$. Note that G^T is strategically equivalent to the game in which the payoff to each player is the sum of his payoffs in all stages.[4] The division by the expected number of periods serves as a normalization that allows us to compare payoffs in G^T to payoffs in the one-shot game.

Following Aumann (1992), we say that an event is *mutual* knowledge at a point ω if all players know it at ω; *second-order* mutual knowledge if it is mutual knowledge that it is mutual knowledge; and $(k+1)$th *order* mutual knowledge if it is mutual knowledge that it is kth order mutual knowledge.

2.2. *The Results*

Our construction shows that when G is the prisoners' dilemma and n is sufficiently large, then there exists a countable uncertainty structure T such that:

(1) $T(\omega) \geq n$ for all $\omega \in \Omega$,

(2) $E(T)$ is very close to n,

(3) the probability that the event[5] $T = n$ is mutual knowledge of high order is close to 1, and

(4) G^T has an equilibrium[6] in which the players cooperate at most stages.

Such an uncertainty structure exists for any game and all strictly individual rational and feasible payoffs. In order to state the general result we introduce additional notation. A game in strategic form G is described by a finite set of players, N, a set of finitely many pure strategies A^i for each player i, and a vector function h: $A \to \mathbb{R}^N$ where A denotes the cartesian product $\times_{i \in N} A^i$, i.e., A is the set of N-tuples of pure strategies. For each player i we denote as usual by A^{-i} the set $\times_{j \neq i} A^j$ of $N \setminus \{i\}$-tuples of pure strategies of the other players.

The ith coordinate of h is the payoff function of player i and is denoted h^i. For any finite set B we denote by $\Delta(B)$ all probability distributions on B. Thus, $\Delta(A^i)$ is the set of mixed strategies of player i in G, and $\Delta(A^{-i})$ is the set of correlated mixed strategies of the other players. The unique linear extension of the payoff function h to $\Delta(A)$ is denoted also by h. For any game G in strategic form, let $w^i(G)$ denote *player i's correlated individual rational payoff* in G, i.e.,

$$w^i(G) = \min_{x^{-i} \in \Delta(A^{-i})} \max_{x^i \in A^i} h^i(x^i, x^{-i}).$$

In other words $w^i(G)$ is the smallest payoff to which the other players can hold player i when they are allowed to correlate their strategies. In view of the

[4]One may define a different model by defining the payoff in G^T as the expected average payoff per stage. I.e., at each $\omega \in \Omega$ one computes the average payoff per stage and then takes expectation. The main result of the present paper remains intact in this alternative model; the proofs need minor modifications.

[5]The event $T = n$ refers to (the more explicit) $\{\omega \in \Omega\colon T(\omega) = n\}$.

[6]As the constructed game G^T does not have any subgame, the equilibrium is obviously a subgame perfect equilibrium.

minimax theorem, $w^i(G) = \max_{x^i \in \Delta(A^i)} \min_{x^{-i} \in A^{-i}} h^i(x^i, x^{-i})$. The set of all correlated equilibrium payoffs of a game G is denoted by $CE(G)$. The convex hull of a set X is denoted $\text{co}(X)$, and $h(A) = \{h(a) | a \in A\}$. Let

$$F(G) = \{x \in \text{co}(h(A)) | \forall i \in N, x^i > w^i(G)\}.$$

The following proposition follows from the proof of the main result of Benoit and Krishna (1985). It illustrates the natural role of the correlated minimax level in the setup of an uncertainty structure.

PROPOSITION 1: *Let G be a game in strategic form such that for every $i \in N$ there is $y \in CE(G)$ with $y^i > w^i(G)$. For every $\varepsilon > 0$ and every sufficiently large n, there is an uncertainty structure $((\Omega, \mathscr{B}, P), (\mathscr{B}^i), T)$ with $T(\omega) = n$ for every $\omega \in \Omega$, such that for every $x \in F(G)$, there is an equilibrium of G^T with payoffs that are ε close to x.*

The above proposition handles those games for which there is a correlated equilibrium payoff which strictly dominates the correlated individual rational payoff. In that case we obtain a "folk type" theorem with an uncertainty structure for which the number of repetitions is common knowledge at each state $\omega \in \Omega$. The main theorem handles all finitely repeated games. Here we obtain a folk theorem in which the event $T = n$ is approximately common knowledge. Moreover, it quantifies the approximation of common knowledge, obtaining a "folk type" theorem in which the event $T = n$ is "exponentially" close (as a function of n) to common knowledge, in a sense that we define precisely below. The first requirement is that given $\varepsilon > 0$, the probability of the event $T = n$ is at least $1 - \exp(-\varepsilon n)$. Notice that outside the event $T = n$, the sum of payoffs in the T stages of the repeated game differs from the sum of payoffs in the first n stages of the repeated game by at most a constant times $T - n$, and may differ by as much. Thus the expected sum of the payoffs in the T stages of the repeated game differs from the sum of the payoffs in the first n stages by at most a constant times $E(T - n)$. Therefore, we require that the expectation of $T - n$ be exponentially small, i.e., that $E(T - n) \leq \exp(-\varepsilon n)$. Note that as $T \geq n$, the inequality $E(T - n) \leq \exp(-\varepsilon n)$ implies in particular that the probability of the event $T = n$ is at least $1 - \exp(-\varepsilon n)$. The following proposition (which is a special case of Theorem 1) shows that rational cooperation is feasible with an uncertainty structure in which the expectation of $T - n$ is exponentially small.

PROPOSITION 2: *Let G be a game in strategic form. There exists a positive constant $K > 0$, such that for every $\varepsilon > 0$, and every sufficiently large n, there is an uncertainty structure $((\Omega, \mathscr{B}, P), (\mathscr{B}^i), T)$, with $T \geq n$ and*

$$E(T - n) \leq \exp(-\varepsilon n),$$

such that for every $x \in F(G)$, there is an equilibrium of G^T with payoffs that are $K\varepsilon$ close to x.

Notice that the constant K depends only on the stage game G. It should be pointed out that finding an uncertainty structure T with $T \geq n$ and $E(T-n) \leq \exp(-\varepsilon n)$, which enables cooperation in G^T where G is the prisoners' dilemma, could seem at first counterintuitive. After all, if $T \geq n$ and $E(T) \leq n + \theta$, then at most states ω (with probability $\geq 1 - 1/m$) the conditional expectation of T, given the information of player i, is at most $n + m\theta$, and when this is the case for a sufficiently small θ, no possible gain from potential future stages offsets the loss resulting from not playing the dominant action at stage n. Therefore, given an equilibrium of G^T, in most states $\omega \in \Omega$ the players will play the one-shot equilibrium actions of stage n. Let A_0 denote all states $\omega \in \Omega$ for which the conditional expectation at each player on the duration T is very close to n and both players play the dominant action at stage n. Then the probability of A_0 is close to 1. It seems that one could start with the usual backward induction. For every nonnegative integer l let A_l be the set of all states for which the equilibrium play repeats the dominant actions in stages $t = n - l, \dots, n$, and the conditional expectation (of each player) of T is close to n. It follows that if the probability of A_l is very close to 1, so is the probability of A_{l+1}. However, this argument fails; the probabilities of the sets A_l can increase exponentially in l and this enables the above proposition.

In addition to the requirement that $E(T-n)$ be small, we would like the probability of the event $\{\omega\colon T = n \text{ is } [\varepsilon n]\text{th order mutual knowledge at } \omega\}$ to be at least $1 - \exp(-\varepsilon n)$. We will impose even stricter requirements. Fix a number $\infty \geq \alpha \geq 1$. Define T_α as the smallest positive integer m such that for every positive integer $k \leq \alpha$ and every list of players $i_1, \dots, i_k$, player i_1 knows that player i_2 knows that...player i_k knows that the number of repetitions is at most m. Set $T_0 = T$.

An alternative equivalent definition of T_α is by means of the accessibility metric on Ω. The accessibility metric on Ω is the largest metric d on Ω such that $d(\omega, \omega') = 1$ whenever $\omega \neq \omega'$ and there is a player i that does not distinguish between ω and ω', i.e., there is an atom of $(\Omega, \mathscr{B}^i)$ that contains both ω and ω'. Let $B(\omega, \alpha)$ be the closed ball in Ω that is centered at ω and with radius α. Then $T_\alpha(\omega)$ is the maximal value of T on $B(\omega, \alpha)$, i.e.,

$$T_\alpha(\omega) = \sup\{T(\omega')\colon d(\omega, \omega') \leq \alpha\}.$$

Note that $T_\alpha \geq T$ and that T_α is monotonic nondecreasing, i.e., $T_0 = T \leq T_1 \leq \cdots \leq T_k$. Note that whenever $T \geq n$ everywhere, the event $T = n$ is kth order mutual knowledge at ω if and only if $T_k(\omega) = n$, and it is common knowledge at ω if and only if $T_\infty(\omega) = n$.

DEFINITION 1: The event $T = n$ is (k, ε)*-exponentially close to common knowledge* if $T \geq n$ and $E(T_k - n) \leq \exp(-\varepsilon n)$.

Notice that $T_1 \geq T$, and thus $E(T_1) \geq E(T)$. It follows that if the event $T = n$ is $(1, \varepsilon)$-exponentially close to common knowledge, then $E(T) \leq n + \exp(-\varepsilon n)$.

Note also that as T_k is monotonic nondecreasing, if the event $T=n$ is $(k+1,\varepsilon)$-exponentially close to common knowledge, then it is (k,ε)-exponentially close to common knowledge.

If the event $T=n$ is (k,ε)-exponentially close to common knowledge, then $E(T_k-n)\leq \exp(-\varepsilon n)$. As T_k is integer valued, we deduce that $\text{prob}(T_k=n)\geq 1-\exp(-\varepsilon n)$. Recall that on $T_k=n$, the event $T=n$ is kth order mutual knowledge.

We conclude that, if the event $T=n$ is $([\varepsilon n],\varepsilon)$-exponentially close to common knowledge, then

(a) $T\geq n$ and $E(T)\leq E(T_{[\varepsilon n]})\leq n+\exp(-\varepsilon n)$, and

(b) $\text{Prob}(T_{[\varepsilon n]}=n)=\text{Prob}(\{\omega\in\Omega|\text{the event } T=n \text{ is } [\varepsilon n]\text{th order mutual knowledge at } \omega\})\geq 1-\exp(-\varepsilon n)$.

Notice also that $|T-E(T|\mathscr{B}^i)|\leq K$ wherever $T_1\leq T+K$. Therefore the condition $T_1\leq T+K$ implies that $|T-E(T|\mathscr{B}^i)|\leq K$ everywhere.

THEOREM 1: *Let G be a game in strategic form. There exists a positive constant $K>0$, such that for every $\varepsilon>0$, and every sufficiently large n, there is an uncertainty structure $((\Omega,\mathscr{B},P),(\mathscr{B}^i),T)$, with $T\geq n$, and such that:*

(i) *the event $T=n$ is $([\varepsilon n],\varepsilon)$-exponentially close to common knowledge,*

(ii) $\forall\omega\in\Omega$, $T_1(\omega)\leq T(\omega)+K$, *and* $T_{m+1}(\omega)\leq T_m(\omega)+K$

and for every $x\in F(G)$, there is an equilibrium of G^T with payoffs that are $K\varepsilon$ close to x.

In the next section we provide a detailed construction of an uncertainty structure T and an explicit description of equilibrium strategies for G^T where G is a specific prisoners' dilemma game. The equilibrium strategies constructed for the prisoners' dilemma are both extensive form perfect and sequential. Such an explicit construction of (extensive form perfect and sequential) equilibrium is possible whenever G has a correlated equilibrium payoff y with $y^i=w^i(G)$ for every player i.

The proof of the main theorem is presented in Section 4. The proofs are by means of an explicit uncertainty structure. However, the existence of an equilibrium is not by means of an explicit construction. Section 5 discusses the tightness of the assumptions of Theorem 1.

3. THE CONSTRUCTION FOR A FINITELY REPEATED PRISONERS' DILEMMA

Let G be the prisoners' dilemma depicted in the following figure.

	D_2	F_2
D_1	1,1	5,0
F_1	0,5	4,4

Each player i ($i = 1,2$) has two actions available, labeled F_i (Friendly) and D_i (Unfriendly). The payoff to player i is given by the function h^i, where $h^1(D_1, D_2) = 1 = h^2(D_1, D_2)$, $h^1(D_1, F_2) = 5 = h^2(F_1, D_2)$, $h^1(F_1, D_2) = h^2(D_1, F_2) = 0$, and $h^1(F_1, F_2) = h^2(F_1, F_2) = 4$. The set of actions of player i, $\{D_i, F_i\}$ is denoted A^i, and the cartesian product $A^1 \times A^2$ is denoted A.

Let $(\Omega, \mathscr{B}, P)$ be a probability space, $\mathscr{B}^i$, $i = 1,2$ are σ-fields with $\mathscr{B}^i \subset \mathscr{B}$. We interpret Ω as the states of nature and P as the common prior of all players on Ω. The σ-field $\mathscr{B}^i$ describes the information available to player i.

For a fixed measurable function T: $\Omega \to \mathbb{N}$ with finite expectation, we consider the "finitely repeated" prisoners' dilemma G^T. Recall that in G^T, a state of nature, $\omega \in \Omega$, is being chosen by nature according to the probability P, and the "atom" of $\mathscr{B}^i$ is told to player i. The prisoners' dilemma G is repeated $T(\omega)$ times, and recall that the payoff to player i is $\sum_{t=1}^{T(\omega)} h^i(a_t)/E(T)$, where a_t is the 2-tuple of actions taken at stage t. We denote this game by $\Gamma = (G; (\Omega, \mathscr{B}, P); (\mathscr{B}^1, \mathscr{B}^2); T)$. Recall that the event $T(\omega) = m$ is mutual knowledge of order k at ω, if at ω each player knows that $T(\omega) = m$, each knows (that the (any) other player knows)$^{k-1}$ that $T(\omega) = m$. To make that formal, let C be a given event. We identify the characteristic function of C with C. Therefore, player i knows C at ω if the conditional expectation of C with respect to the σ-field $\mathscr{B}^i$ equals 1 at ω, i.e., if $E(C|\mathscr{B}^i)(\omega) = 1$. Player 1 knows that player 2 knows C at ω if $E(E(C|\mathscr{B}^2)|\mathscr{B}^1)(\omega) = 1$. Thus, an operation K^i from the integrable functions on $(\Omega, \mathscr{B}, P)$ into the integrable functions on $(\Omega, \mathscr{B}^i, P)$ is defined by $K^i(f) = E(f|\mathscr{B}^i)$. We say that C is mutual knowledge of order k, $k = 0,1,2,\ldots$ at ω if $(K^{i_1}K^{i_2}\ldots K^{i_k})(C)(\omega) = 1$, where $i_1, i_2, \ldots, i_k$ is any ordered list of players. A pure strategy for player i is a sequence of functions $(f_t^i)_{t=1}^{\infty}$ where f_t^i: $\Omega \times A^{t-1} \to \{D_i, F_i\}$ and f_t^i is measurable with respect to $\mathscr{B}^i$. Given two strategies f^1, f^2, we denote by $a_t(\langle f^1, f^2 \rangle, \omega)$ the induced action in stage t, at ω. Note that the payoff to player i as a function of the strategy pair $\langle f^1, f^2 \rangle$ in the game G^T is $E_P(H^i(\langle f^1, f^2 \rangle, \omega))$ where $H^i(\langle f^1, f^2 \rangle, \omega) = \sum_{t=1}^{T(\omega)} h^i(a_t(\langle f^1, f^2 \rangle, \omega))/E(T)$.

It is possible to show that if at some ω with $P(\omega) > 0$ the value of $T(\omega)$ is mutual knowledge of order k, then any equilibrium of the game will lead to playing the Unfriendly actions in the last k stages of the game. The following example will specify $(\Omega, \mathscr{B}, P)$, $\mathscr{B}^1$, $\mathscr{B}^2$, and T and will demonstrate an equilibrium that will repeat the Friendly outcome up to (but not including) stage $T(\omega) - k(\omega)$ where $k(\omega)$ is at the maximum k for which the value of $T(\omega)$ is mutual knowledge of order k at ω.

We start by describing the idea of the construction. The uncertainty structure has the following property. At each point ω, player i ($i = 1,2$ and note that $3 - i$ refers thus to the other player) either knows that the duration is exactly n or is uncertain among two possible successive numbers of periods, k and $k + 1$ where $k \geq n$. When player i is uncertain about the number of periods (either k or $k + 1$), his conditional probability of the event $T = k$ is approximately $1/3$ and thus his conditional probability of the event $T = k + 1$ is approximately $2/3$.

The equilibrium strategy of player i selects the Friendly action up to a stage $l^i(\omega)$ (which depends on the players initial information) as long as the other player has played the Friendly action, and thereafter plays the Unfriendly action. The functions l^i, $i = 1, 2$, will satisfy the following conditions. Assume that his uncertainty is $k \leq T \leq k + 1$. Conditional on the event $T = k$ he is certain that the other player $(3 - i)$ starts with the Unfriendly action in stage k, (i.e., that the value of l^{3-i} equals $k - 1$). Conditional on the event $T = k + 1$ he is certain that $k \leq l^{3-i} \leq k + 1$, and the conditional probabilities that he assigns for the other player $(3 - i)$ starting with the Unfriendly action in stages $k + 1$ (i.e., that the value of l^{3-i} equals k) is approximately $1/2$. Therefore, simple arithmetic involving the payoffs of the game matrix and the above approximate probabilities illustrates that his conditional best reply is to start with the Unfriendly action in stage $k + 1$, i.e., the value of l^i equals k. When player i is certain that the duration $T(\omega)$ equals n, the possible values of the function $l^i(\omega)$ are $0, \ldots, n - 1$. Given $l^i(\omega)$, the possible values of $l^{3-i}(\omega)$ are $l^i(\omega) - 1$, $l^i(\omega)$, and $l^i(\omega) + 1$, each with conditional probability (given player i's information) of approximately $1/3$, and therefore his conditional best reply is to start with the Unfriendly action in stage $l^i(\omega) + 1$. It remains to construct such an uncertainty structure and corresponding functions l^i (with the above-mentioned properties) such that the value of l^i depends on the information of player i, and the probability of $l^i(\omega) > n - \varepsilon n$ is sufficiently close to 1. Note that, at each point ω with $l^i(\omega) = n - k$, there is mutual knowledge of order k of the event $T = n$.

Set $\Omega = \{(i, j) \in Z^2 | -1 + i \leq j \leq i + 1\}, \mathscr{B} = 2^{\Omega}$. Fix $0 < p < 1$, and $0 < q < 1$ such that $3p(1 + q)/(1 - q) = 1$ with p sufficiently small (e.g., $p < 1/17$). Note that $q \to 1$ as $p \to 0$. Let r be a fixed nonnegative integer. Define a probability measure P or $(\Omega, \mathscr{B})$ by $P((i, j)) = pq^{|\min\{i, j\} + r|}$. Note that most of the probability is concentrated on $\{(i, j) | -2r < i < -r/2\}$.

For $k = 1, 2$ let $\mathscr{B}^k$ be the σ-field generated by the kth coordinate of the points in Ω, i.e., $\mathscr{B}^1$ is the smallest σ-field for which $(i, j) \to i$ is measurable and $\mathscr{B}^2$ is the smallest σ-field for which $(i, j) \to j$ is measurable. In other words, the information of player 1 is represented by the partition of Ω according to the different possible values i of the first coordinate of a point $\omega = (i, j) \in \Omega$. Let n be a fixed positive integer and define T: $\Omega \to \mathbb{N}$ by

$$T((i, j)) = \max\{n, [(i + j)/2] + n\}$$

where for any number x, $[x]$ denotes the largest integer which is less than or equal to x.

Part of the uncertainty structure is depicted in Figure 1. Each square in the figure represents a point $\omega = (i, j) \in \Omega$. The value of i indicates the row, and the value of j indicates the column that includes ω. Thus, player 1 is informed of the row, and player 2 is informed about the column. The number of repetitions $T(\omega)$ is marked in the upper left part of each square.

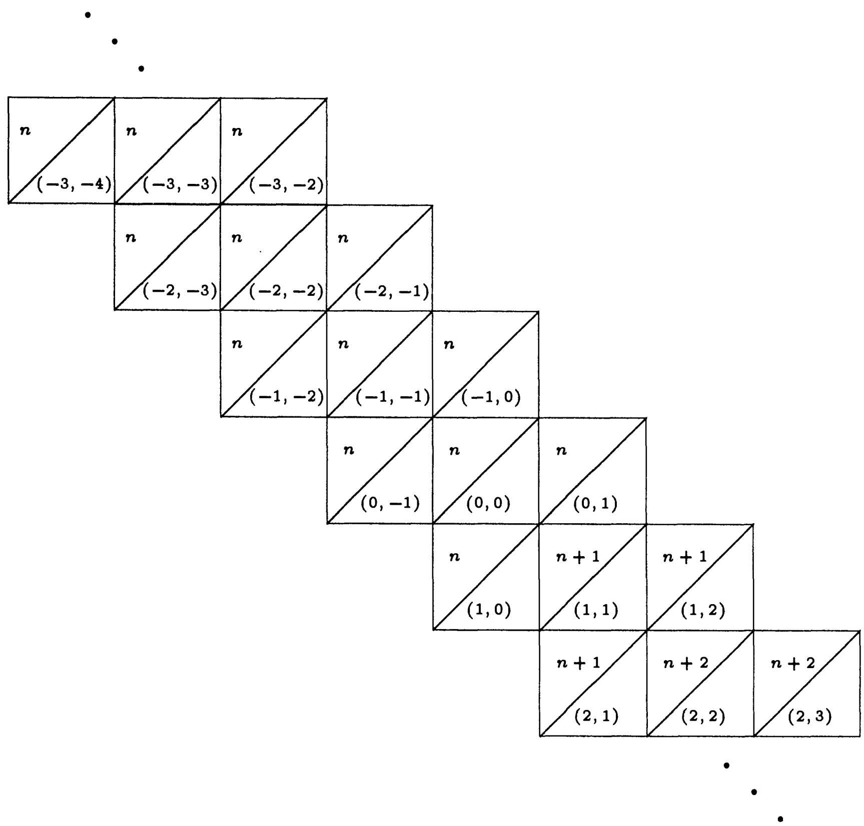

FIGURE 1.—The uncertainty structure.

For every point $\omega=(i,j)\in\Omega$, with $\max\{i,j\}\leq 0$, consider the smallest number $k(\omega)$ of vertical and horizontal line segments that connect (i,j) with $(0,0)$ within the pictured Ω. Then the event $T=n$ is mutual knowledge of order $k(\omega)$ at ω. For example, at $\omega=(-1,0)$, both players know that the number of repetitions is n. Moreover, player 1 knows that player 2 knows that $T=n$. Because at $\omega=(1,0)$ player 1 does not know whether the number of repetitions is n or $n+1$, at $\omega=(-1,0)$ player 2 does not know that player 1 knows that $T=n$. Therefore, at $\omega=(-1,0)$, the event $T=n$ is mutual knowledge of order 1 but is not mutual knowledge of order 2, and (using the symmetry of the uncertainty structure) at $\omega=(-1,-1)$, the event $T=n$ is mutual knowledge of order 2, but not of order 3. Similarly, at $(-2,-2)$, the event $T=n$ is mutual knowledge of order 3. More generally, for every nonnegative integer k, at each of the points $(-k,-k)$, $(-k-1,-k)$, and $(-k,-k-1)$, the event $T=n$ is mutual knowledge of order $k+1$.

We define now a pure strategy equilibrium of $\Gamma=(G;(\Omega,\mathscr{B},P);(\mathscr{B}^1,\mathscr{B}^2);T)$ that will have the following property: at $\omega=(i,j)$ with $\max\{i,j\}\leq 0$, the first stage at which player 1 (player 2) plays the D_1 (D_2) action is $n+i$ ($n+j$). Thus in particular, if the value of T at $\omega=(i,j)$ is not of mutual knowledge of order k, both players will play the friendly outcome in the first $T(i,j)-k+1$ stages.

$$f_t^1((i,j),(a_1,\dots,a_{t-1}))=\begin{cases} D_1 & \text{if } t\ge i+n,\\ D_1 & \text{if } \exists 1\le s<t \text{ s.t. } a_s(2)=D_2,\\ F_1 & \text{otherwise;}\end{cases}$$

$$f_t^2((i,j),(a_1,\dots,a_{t-1}))=\begin{cases} D_2 & \text{if } t\ge j+n,\\ D_2 & \text{if } \exists 1\le s<t \text{ s.t. } a_s(1)=D_1,\\ F_2 & \text{otherwise.}\end{cases}$$

Note that the above strategies are those associated (in our description of the idea of the proof) with the functions l^1 and l^2 on Ω defined by: $l^1(i,j)=\max(0,n+i-1)$, and $l^2(i,j)=\max(0,n+j-1)$. We first show that (f^1,f^2) is an equilibrium. By the symmetry of the game and the strategies between the two players, it is enough to show that for every pure strategy g^1 of player 1, $E(H^1(g^1,f^2,\omega)|\mathscr{B}^1)\le E(H^1(f^1,f^2,\omega)|\mathscr{B}^1)$.

Assume that B is the atom $\{(i,i-1),(i,i),(i,i+1)\}$ of $\mathscr{B}^1$. Then the conditional probabilities of (i,j) given B are given by

$$P((i,i-1)|i)=\frac{1}{1+2q},\quad P((i,i)|i)=P((i,i+1)|i)=\frac{q}{1+2q}$$

if $i\ge 1-r$, and

$$P((i,i-1)|i)=\frac{q}{q+2},\quad P((i,i)|i)=P((i,i+1)|i)=\frac{1}{2+q}$$

if $i\le -r$. In both cases $q\to 1$ as $p\to 0$ and thus $P((i,j)|i)\to 1/3$. If $p<1/17$ then $q>7/10$, and

$$3P((i,i+1)|i)-P((i,i)|i)-P((i,i-1)|i)>0,$$

and

$$5P((i,i+1)|i)+P((i,i)|i)-4P((i,i-1)|i)>0.$$

We first compute $E(H^1(f^1,f^2,\omega)|\mathscr{B}^1)$ on B. Let $(i,j)\in B$.

If $i+n<1$, then $j+n\le 1$ and therefore $f_t^2((i,j),(\cdot))=D_2$ and therefore $H^1(g^1,f^2,(i,j))\le T(\omega)$. As $H^1(f^1,f^2,(i,j))=T(\omega)$, it follows that on $i+n<1$, for every strategy g^1 of player 1, $E(H^1(g^1,f^2,\omega)|i)\le E(H^1(f^1,f^2,\omega)|i)$.

If $i\ge 1$, then $T(i,j)=[(i+j)/2]+n$ and then

$$\begin{aligned}E(H^1(f^1,f^2,\omega)|i)&=P((i,i-1)|i)(i-2+n)4\\&\quad+P((i,i)|i)(4(i+n-1)+1)\\&\quad+P((i,i+1)|i)(4(i+n-1)+5)\\&=4(i+n-1)+5P((i,i+1)|i)\\&\quad+P((i,i)|i)-4P((i,i-1)|i).\end{aligned}$$

Let $F(t) = ((F_1, F_2), \ldots, (F_1, F_2))$ $(t-1$ times); i.e., $F(t)$ is a history of repeated play of the Friendly outcome for $t-1$ periods. Let $k(g^1) = \inf\{t\colon g_t^1((i,\cdot), F(t)) = D_1\}$. We distinguish a few cases:

(a) $k(g^1) > i + n + 1$. Then

$$\begin{aligned} E(H^1(g^1, f^2, \omega)|i) &= P((i, i-1)|i)4(i+n-2) \\ &\quad + P((i,i)|i)4(i+n-1) + P((i,i+1)|i)4(i+n) \\ &= 4(i+n-1) + 4P((i,i+1)|i) - 4P((i,i-1)|i) \\ &< E(H^1(f^1, f^2, \omega)|i). \end{aligned}$$

This inequality is independent of the value of p.

(b) If $k(g^1) = i + n$, then

$$E(H^1(g^1, f^2, \omega)|i) = E(H^1(f^1, f^2, \omega)|i).$$

(c) If $k(g^1) = i + n - 1$, then

$$\begin{aligned} E(H^1(g^1, f^2, \omega)|i) &= P((i, i-1)|i)(4(i+n-2)+1) \\ &\quad + P((i,i)|i)(4(i+n-2)+5+1) \\ &\quad + P((i,i+1)|i)(4(i+n-2)+5+1) \\ &= 4(i+n-1) - 3P((i,i-1)|i) + 2P((i,i)|i) \\ &\quad + 2P((i,i+1)|i) \\ &< E(H^1(f^1, f^2, \omega)|i). \end{aligned}$$

(d) If $k(g^1) < i + n - 1$, then

$$\begin{aligned} E(H^1(g^1, f^2, \omega)|i) &\le 4(i+n-2) + 1 + P((i,i-1)|i) + 2(P((i,i)|i) \\ &\quad + P((i,i+1)|i)) \\ &\le 4(i+n-1) - 1 < E(H^1(f^1, f^2, \omega)|i). \end{aligned}$$

Therefore, if $i \ge 1$,

$$E(H^1(g^1, f^2, \omega)|i) \le E(H^1(f^1, f^2, \omega)|i)$$

for every strategy g^1 of player 1. A similar computation shows that if $1 - n < i < 1$ or $i = 1 - n$ (in both cases $T(i,j) = n$), then for any strategy g^1 of player 1, $E(H^1(g^1, f^2, \omega)|i) \le E(H^1(f^1, f^2, \omega)|i)$. This completes the proof that f^1 is a best reply to f^2 and thus that (f^1, f^2) is an equilibrium.

The game $\Gamma = (G; (\Omega, \mathscr{B}, P); (\mathscr{B}^1, \mathscr{B}^2); T)$ that we have constructed depends on the two positive integers n and r, and therefore we denote it by $\Gamma(n,r)$. The parameters n and r determine the probability measure $P = P(r)$ on Ω and the number of repetitions $T = T(n,r)$. The equilibrium strategy f depends on n.

The game $\Gamma(n,r)$ and the equilibrium strategy $f = f(n)$ have the following properties:

(a) $T(n,r) \ge n$, and $E(T(n,r)) - n = 3pq^{r+1}/(1-q) \le q^{r+1} \to 0$ as $r \to \infty$.

(b) $E(H^1(f, \omega)) = E(H^2(f, \omega)) \ge 4 - [6p/(1-q)^2 - 6rp/(1-q)]/n$.

(c) $P(r)(\{\omega$: the event $T = n$ is mutual knowledge of order k at $\omega\}) \geq 1 - 3pq^{r-k}/(1-q) \geq q^{r-k} \to 1$ as $r \to \infty$; and for every positive integer m,

(d) $P(r)(\{\omega$: the event $T \leq n + m$ is mutual knowledge of order k at $\omega\}) \geq 1 - 3pq^{r+m-k}/(1-q) \geq q^{r+m-k} \to 1$ as $r \to \infty$.

Several instances of this example, obtained by choosing specific values for p, k, r, and n, could be used to demonstrate the implications of a slight weakening of the common knowledge assumption on the number of repetitions. One special class of examples could be generated by setting $r = r(n) = [n^{1/3}]$. We obtain a sequence of games $\Gamma(n, r(n))$ and corresponding equilibria $f = f(n)$ such that for sufficiently large n we have:

(a′) $T(n, r(n)) \geq n$ and $E(T(n, r(n)) \leq n + 1/n$.

(b′) $E(H^1(f(n), \omega)) = E(H^2(f(n), \omega)) \geq 4(1 - 1\sqrt{n}\,)$.

(c′) $P(r(n))(\{\omega$ the event $T = n$ is common knowledge of order $n^{1/4}\}) \geq 1 - 1/n$.

Another instance is obtained by choosing $r = [-2\varepsilon n/\ln q]$. We obtain a sequence of games $\Gamma(n)$ and corresponding equilibria $f = f(n)$ such that for sufficiently large n we have:

(a″) $T(n) \geq n$ and $E(T(n)) \leq n + \exp(-\varepsilon n)$.

(b″) $E(H^1(f(n), \omega)) = E(H^2(f(n), \omega)) \geq 4 - K(q)\varepsilon$.

(c″) $P(n)(\{\omega$: the event $T = n$ is $[\varepsilon n]$th order mutual knowledge at $\omega\}) \geq 1 - \exp(-\varepsilon n)$.

The examples show that cooperation in the finitely repeated prisoners' dilemma could be justified by a small departure from the common knowledge assumption on the number of repetitions.

REMARKS: (a) A related remark appears in Kreps, Milgrom, Roberts, and Wilson (1982), where it is pointed out that the classical backward induction argument relies on the assumption that the rationality of the players is common knowledge. Aumann (1992) shows that a small departure from the common knowledge assumption of rationality enables cooperation in the finitely repeated prisoners' dilemma. In his model of an information system—a simultaneous representation of all the players' uncertainties—a state of the world includes the choice of pure strategies by all players. It is shown there that cooperation in the finitely repeated prisoners' dilemma is possible with extremely small overall probabilities of irrationality, and with, in most states of the world, high order of mutual knowledge of rationality.

(b) In our example the number of repetitions T is unbounded. Indeed, in any repeated prisoners' dilemma in which the number of repetitions T is a random variable about which the players receive partial information, if $T \leq m$ everywhere, then any equilibrium of this game results in repeated play of the Unfriendly actions. However, even though we allow for unbounded numbers of repetition, the expected number of repetitions is very close to the minimum value of T. Moreover, at each state of the world and for any order k of mutual knowledge there is a kth order mutually known bound to the length of the game. And this bound is linear in k.

4. PROOF OF THE MAIN THEOREM

PROOF OF THEOREM 1: Let N be the finite set of players in the game G, and let $\varepsilon > 0$. Fix $\varepsilon_1 > 0$ sufficiently small. There exists a positive integer d, such that for each $x \in F(G)$ there are elements $b_1, \ldots, b_d \in A$ such that for each $i \in N$,

$$\left| x^i - \sum_{j=1}^{d} h^i(b_j)/d \right| < \varepsilon,$$

and

$$\sum_{j=1}^{d} h^i(b_j)/d \geq w^i(G) + \varepsilon_1.$$

We partition the set of players N into two subsets. The first one, $I \subset N$, includes all players i in N for which there is a correlated equilibrium payoff y with $y^i > w^i(G)$, and the second one is $J = N \setminus I$, i.e., J includes all players i for which $y^i = w^i(G)$ for any correlated equilibrium payoff y of G. Recall that the set of correlated equilibrium payoffs, $CE(G)$, is convex and that for every player $i \in N$ and every vector payoff $y \in CE(G)$, $y^i \geq w^i(G)$. Therefore, there is $y \in CE(G)$ with $y^i > w^i(G)$ for every $i \in I$. Each of the uncertainty structures to be constructed is a product of two uncertainty structures, $((\Omega_1, \mathscr{B}_1, P_1), (\mathscr{B}_1^i), T)$ and $((\Omega_2, \mathscr{B}_2, P_2), (\mathscr{B}_2^i))$. The first one serves to generate the uncertainty on the number of repetitions, and the second one enables both the effective punishment of each deviating player and the independently repeated implementation of a fixed correlated equilibrium μ of G with corresponding payoff y that obeys $y^i > w^i(G)$ for every $i \in I$. It will become evident from the sequel (as well as from the proofs of the corresponding results for finitely repeated games) that if $J = \emptyset$, there is no need for the first uncertainty structure.

We define now the first uncertainty structure. Let $|J|$ denote the number of elements of the set J. Set

$$Z(J) = \left\{ \langle j_i \rangle_{i \in J} \in \mathbb{Z}^J \middle| \forall i, i' \in J \text{ with } i \neq i',\, j_i \neq j_{i'} \right\},$$

$$\Omega_1 = \{ \langle j_i \rangle_{i \in J} \in Z(J) | \max\{ j_i | i \in J\} - \min\{ j_i | i \in J\} = |J| - 1 \}.$$

For every $i \in J$, $\mathscr{B}_1^i$ is the partition of Ω_1 induced by the different values of j_i. For $i \in I$, $\mathscr{B}_1^i = \mathscr{B}_1$, where $\mathscr{B}_1$ is the set of all subsets of the countable set Ω_1. Let m be sufficiently large. The number of repetitions $T(\omega)$ equals the constant m if $J = \emptyset$, and if $J \neq \emptyset$, $T(\omega)$ is given by

$$T(\langle j_i \rangle_{i \in J}) = \max\{ m, m + dL \min\{ j_i | i \in J\}\}.$$

The probability P_1 is specified by three parameters, $r, p > 0$, and $0 < q < 1$, where r is a sufficiently large positive integer, the value of q is sufficiently close to 1, and the value of p is such that the total probability of P_1, $P_1(\Omega_1)$, equals 1, where

$$P_1(\langle j_i \rangle_{i \in J}) = pq^{|\min\{j_i | i \in J\} + r|}.$$

Fix a sufficiently large positive integer L. Let $k(n)$ be the smallest positive integer for which $n - k(n)$ is divisible by Ld. We start by defining an auxiliary N-tuple of strategies, $\tau = \langle \tau_i \rangle_{i \in N}$. These strategies call for playing a specified equilibrium of the game G in the first $k(n)$ stages. Thereafter, they enter into playing a cycle of length Ld. The Ld-cycle consists of L_1 mini-cycles $b_1, \dots, b_d$ followed by repeating $L_2 d$ plays of the correlated equilibrium μ, where $L_1 + L_2 = L$, and L_2/L is sufficiently small so that the average payoff of the cycle is 2ε close to x. On the other hand, L_2 is sufficiently large to deter deviation by a player $i \in I$. As soon as a deviation by a player $i \in I$ from the cycle is observed, the strategies $\langle \tau_j \rangle_{j \neq i}$ punish the deviator i in all remaining stages. If a deviation by a player $i \in J$ is observed, the strategy τ reverts to playing the correlated equilibrium μ in all remaining stages.

Let $t^i = m + Ld(j_i - 1)$. Consider the N person game G^T with the following modification: each player $i \in N$ is restricted to a subset $\Sigma^i(\tau)$ of the set Σ^i of all strategies in G^T. For each player $i \in I$, $\Sigma^i(\tau) = \{\tau^i\}$. The restricted set of strategies $\Sigma^i(\tau)$ of a player $i \in J$ consists of all strategies that obey the following properties: (a) they follow in the first $k(n)$ stages the specified equilibrium of the one-shot game G, (b) they coincide with the strategy τ_i up to stage t^i, (c) they play in the last L_2 stages of each cycle (i.e., in all stages t with $t = k(n) + lLd + s$ with $L_1 d < s \leq Ld$) the correlated equilibrium μ, and (d) if the first deviator from the cycle is a player $i \in I$, he is punished forever thereafter with a specified correlated strategy punishment, and if the first deviator is a player $i \in J$, they revert to playing repeatedly the correlated equilibrium μ. Let $\sigma = \langle \sigma_i \rangle_{i \in N}$ be an equilibrium of this restricted game. It turns out that σ is also an equilibrium of G^T and obeys all the required inequalities. Indeed, if a player $i \in I$ is the first one to deviate from the proposed play, his payoff will decrease by at least $L_2(y^i - w^i(G))$ minus a constant times d. Therefore, for a sufficiently large value of L_2, for every player $i \in I$, τ^i is a best reply to σ. Assume that $i \in J$, and $t^i > k(n)$. We claim that player i is unable to increase his conditional payoff, given his information, by deviating prior to stage t^i. Player i's conditional probability that $t^i < t^j$ for every $j \in J \setminus \{i\}$ is approximately $1/|J|$. Therefore, by deviating prior to stage t, the expected loss from the possibility of earlier punishments exceeds the expected gain from higher payoffs in those cases where another player had deviated prior to stage t^i, and therefore any best reply of player $i \in J$ to σ is in $\Sigma^i(\tau)$. This completes the proof of the theorem. *Q.E.D.*

5. TIGHTNESS OF THE ASSUMPTIONS

Together, conditions (i) and (ii) of Theorem 1 quantify the small departure from the assumption that the number of repetitions is commonly known. Condition (ii) asserts that in all states ω, both players know the number of repetitions with accuracy $\pm K$, and condition (i) asserts in particular that with high probability there is mutual knowledge of high order that the number of periods equals n.

It is impossible to require a larger order of mutual knowledge (e.g., $[n/4]$th instead of $[\varepsilon n]$th) of the event $T=n$ from the uncertainty structure T. Indeed, if G is the prisoners' dilemma and the event $T=n$ is kth order mutual knowledge at ω, then at ω all equilibrium strategies of G^T will result in the Unfriendly actions in the last k rounds.

Condition (i) asserts also that the deviation of (the expectation of) T from n is exponentially (as a function of n) small. Note that it implies in particular that the aggregate (expected) payoffs in all stages $t>n$ is exponentially smaller (as a function of n) than any one-stage potential gain.

A natural question that arises is whether one can require the distribution of T to be even closer asymptotically to the constant value n, and still enable cooperation. In particular, what are the bounds on $E(T)-n$ (as a function of n) that disable cooperation in G^T. The next result provides such a bound for the prisoners' dilemma.

A prisoners' dilemma is a two-player game G in which each player i $(i=1,2)$ has two actions labeled F_i (Friendly) and D_i (Unfriendly). The payoff to player i is given by the function h^i which satisfies $h^1(D_1,*)>h^1(F_1,*)$, $h^2(*,D_2)>h^2(*,F_2)$, and $h^i(F_1,F_2)>h^i(D_1,D_2)$.

THEOREM 2: *Let G be a prisoners' dilemma game. Then there exists* $K>0$, *such that for any* $\varepsilon>0$ *there is N such that for any* $n\geq N$ *and any uncertainty structure* $((\Omega,\mathscr{B},P),(\mathscr{B}^i),T)$, *with* $T\geq n$ *and* $E(T)\leq n+\exp(-Kn)$, *all equilibrium payoffs of* G^T *are* ε *close to the equilibrium payoff of G.*

The proof of Theorem 2 is given in the Appendix. The following corollary follows from Theorem 2.

COROLLARY 1: *Let G be a prisoners' dilemma. Then for every* $\varepsilon>0$, *there exists* $K=K(\varepsilon)>0$ *such that for any n and any uncertainty structure* $((\Omega,\mathscr{B},P),(\mathscr{B}^i),T)$, *with* $T\geq n$ *and* $|E(T)-n|\leq\exp(-Kn)$, *all equilibrium payoffs of* G^T *are* ε *close to the equilibrium payoff of G.*

In Theorem 1, the set of payoffs which is approximated by equilibrium payoffs of G^T is the set $F(G)$. The set $F(G)$ is defined as the set of all vector payoffs $x\in\operatorname{co}(h(A))$ with the strict inequality $x^i>w^i(G)$ for every player $i\in N$. In the case of two-person games, we can replace the strict inequality with the weak inequality. More generally, we can replace $F(G)$ in the statement of Theorem 1 with $F^*(G)$ where

$$F^*(G)=\cup_{j\in N}F^j(G),$$

and for every player $j\in N$,

$$F^j(G)=\{x\in\operatorname{co}(h(A))|\forall i\in N\setminus\{j\},x^i>w^i(G),\text{ and }x^j\geq w^j(G)\}.$$

However, we cannot dispense with the strict inequalities appearing in the definition of $F(G)$ altogether, as will turn out by examining the following[7] 3-person game G:

2,0,1	1,1,0
1,1,0	1,1,0

1,1,0	1,1,0
1,1,0	0,2,1

Player one chooses the row, player two the column, and player three the matrix. The correlated individual rational payoffs of players 1, 2, and 3 are 1, 1, and 0 respectively, and the vector payoff $x=(1,1,1)$ is in $\text{co}(h(A))$. However, for any uncertainty structure T, all equilibrium payoffs of G^T are $(1,1,0)$. Therefore, we cannot replace in the theorem the set $F(T)$ with $F'(T)=\{x \in \text{co}(h(A)) | x^i \geq w^i(G)\}$.

Recall that the constructed games G^T do not have subgames. Therefore, the results stay intact if we replace the words "an equilibrium" with "a subgame perfect equilibrium." There are however other refinements that attempt to capture the notion of "perfect" equilibrium, and the question that arises is whether the results hold for these refinements.

The equilibrium strategies constructed for the prisoners' dilemma are both extensive form perfect and sequential. However, the theory of refinements in games with infinitely many strategies is not yet mature, and therefore no attempt is made here to exhibit "perfect" equilibria. We would like to point out however, that (in the main theorem) it is possible to replace the word equilibrium with extensive form perfect and sequential equilibrium, whenever G is a game with a correlated equilibrium payoff y with $y^i = w^i(G)$.

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APPENDIX

PROOF OF THEOREM 2: Assume that σ is an equilibrium of G^T, and without loss of generality we assume that σ is a pure strategy equilibrium. Otherwise, we modify the uncertainty structure to enable a pure strategy equilibrium σ^* that induces the same distribution on plays as σ. For any $1 \leq k \leq n$, let

$$A^i_k(\sigma) = \{\omega \in \Omega | a^i_k(\sigma, \omega) = D_i\},$$

$P^i_k = 1 - P(A^i_k(\sigma))$, and set $P^i_{n+1} = E(T-n)$.

Without loss of generality we assume that $h^i(a|D_i) \geq h^i(a|F_i) + 1$, for every action pair a, and set $C = \max\{h^i(a) - h^i(D_1, D_2) | a \in A, i = 1,2\}$.

[7]This game is strategically equivalent to the example in Forges, Mertens, and Neyman (1986).

For every $0 \le k < n$ let

$$g_k^i(\omega) = \sum_{k < t \le T(\omega)} (h^i(a_t(\sigma, \omega)) - h^i(D_1, D_2)).$$

Note that $g_k^i(\omega)$ is the difference between the accumulated payoffs of player i in stages $t = k + 1, \ldots, T(\omega)$ given σ and ω, and the accumulated payoffs of player i in stages $t = k+1, \ldots, T(\omega)$ assuming repeated play of the unfriendly action pair (D_1, D_2). Note that $h^{3-i}(a_k(\sigma, \omega)) - h^{3-i}(D_1, D_2) \le C$, and that for every $\omega \in A_k^i(\sigma), h^{3-i}(a_k(\sigma, \omega)) - h^{3-i}(D_1, D_2) \le 0$. Therefore,

$$E(g_k^i) \le \sum_{k < t \le n+1} CP_t^{3-i}.$$

On the other hand, as σ is an equilibrium of G^T and $h^i(* | D_i) \ge h^i(D_1, D_2), E(g_0^i) \ge 0$, and therefore the theorem follows once we prove that $\Sigma_{0 < t \le n+1} CP_t^i$ is sufficiently small.

Let $S = \{D_1, F_1\} \times \{D_2, F_2\}$ denote the set of action pairs in the one-shot game, and S^t denotes the set of sequences of length t of action pairs. For every positive integer k, the pure strategy σ induces a measurable function φ_k: $\Omega \to S^{k-1}$ by $\varphi_k(\omega) = (a_1(\sigma, \omega), \ldots, a_{k-1}(\sigma, \omega))$. Let $\mathscr{B}_k^i$ be the σ-field generated by $\mathscr{B}^i$ and the play induced by σ up to stage k, i.e., $\mathscr{B}_k^i$ is the smallest σ-field which contains $\mathscr{B}^i$ and for which φ_k is measurable. Set

$$B_k^i = \{\omega \in \Omega | E(g_k^i | \mathscr{B}_k^i)(\omega) < 1\},$$

and note that B_k^i is measurable with respect to $\mathscr{B}_k^i$. We claim that for every $\omega \in B_k^i, a_k^i(\sigma, \omega) = D_i$ (i.e., $B_k^i \subset A_k^i(\sigma)$). Otherwise, define a strategy $\tau^{i,k}$ of player i as follows: On $(\Omega \setminus B_k^i) \cup \{a_k^i = D_i\}, \tau^{i,k} = \sigma^i$, i.e., for every t and all $\omega \in \Omega$ with either $\omega \notin B_k^i$, or $a_k^i(\sigma, \omega) = D_i, \tau_t^{i,k}(\ldots, \omega) = \sigma_t^i(\ldots, \omega)$. On $B_k^i \cap \{a_k^i \ne D_i\}, \tau_t^{i,k} = \sigma_t^i$ if $t \le k-1$ and $\tau_t^{i,k}(\ldots, \omega) = D_i$ if $t \ge k$. Then, if $\text{prob}_\sigma(\{\omega \in B_k^i | a_k^i(\sigma, \omega) = F_i\}) > 0$,

$$H^i(\sigma | \tau^{i,k}) > H^i(\sigma).$$

Obviously $E(g_k^i) = E(E(g_k^i | \mathscr{B}_k^i)) \ge \text{prob}(E(g_k^i | \mathscr{B}_k^i) \ge 1) = 1 - \text{prob}(B_k^i)$, and therefore

$$\sum_{k < t \le n+1} CP_t^{3-i} \ge 1 - \text{prob}(B_k^i).$$

As $B_k^i \subset A_k^i(\sigma), P_k^i \le 1 - \text{prob}(B_k^i) \le \Sigma_{k < t \le n+1} CP_t^{3-i}$. By induction on $n - k$ it follows that for every $0 \le k \le n$,

$$P_k^i \le C(C+1)^{n-k} E(T-n),$$

and therefore in particular

$$E(g_0^i) \le \sum_{0 < t \le n+1} CP_t^{3-i} \le (C+1)^{n+1} E(T-n).$$

As $E(T-n) \le \exp(-Kn)$,

$$(C+1)^{n+1} E(T-n) \le \exp((n+1)\ln(C+1) - Kn).$$

Therefore, if $K > \ln(C+1)$,

$$(C+1)^{n+1} E(T-n) \to 0 \quad \text{as} \quad n \to \infty,$$

which completes the proof of the theorem. *Q.E.D.*

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