DIAGONALITY OF COST ALLOCATION PRICES*†

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The problem of allocating the production cost of a finite bundle of divisible consumption goods (or services) by means of per unit costs or prices is a basic problem in economics. Recently an axiomatic approach has been proposed (Billera and Heath (1981) and Mirman and Tauman (1981)) in which one considers a class of cost problems and studies the mappings from that class of cost problems to prices by means of the properties these prices satisfy. We look for a list of properties on the class of cost problems and the price mechanism that imply the "diagonality" of the price mechanism.

1. Introduction. The problem of allocating the production cost of a finite bundle of divisible consumption goods (or services) by means of per unit costs or prices is a basic problem in economics. Recently an axiomatic approach has been proposed (Billera–Heath (1981) and Mirman–Tauman (1981)) in which one considers a class of cost problems and studies the mappings from that class of cost problems to prices by means of the properties these prices satisfy. In that approach a cost function is a function \( F: \mathbb{R}_+^n \to \mathbb{R} \), with \( F(0) = 0 \), where for \( x \) in \( \mathbb{R}_+^n \), \( F(x) \) is interpreted as the cost of producing the bundle \( x = (x_1, \ldots, x_n) \) of commodities. A cost problem is a pair \((F, \alpha)\) where \( F \) is a cost function and \( \alpha \) is in \( \mathbb{R}_+^n \), the strictly positive orthant of \( \mathbb{R}^n \) (i.e., all components \( a_i \) of \( \alpha \) are strictly positive). The vector \( \alpha \) is interpreted as the vector of quantities actually produced. A mapping \( P(\cdot, \cdot) \) that associates with each cost problem, \((F, \alpha)\), \((F: \mathbb{R}_+^n \to \mathbb{R} \text{ and } \alpha \in \mathbb{R}_+^n)\) from a given class of cost problems, a vector of prices,

\[
P(F, \alpha) = (P_1(F, \alpha), \ldots, P_n(F, \alpha)),
\]

is called a price mechanism.

The previous papers dealing with this axiomatic approach considered the class of cost problems \((F, \alpha)\) in which \( F \) is continuously differentiable on \( \mathbb{R}_+^n \). For that class of cost problems one could define the marginal cost price mechanism given by

\[
P_i(F, \alpha) = \frac{\partial F}{\partial x_i}(\alpha).
\]

The marginal cost price mechanism satisfies several natural properties, among which are rescaling invariance, i.e., independence of the units of measurement; additivity, i.e., \( P(F + G, \alpha) = P(F, \alpha) + P(G, \alpha) \) for all \( \alpha \) in \( \mathbb{R}_+^n \) and all \( F, G: \mathbb{R}_+^n \to \mathbb{R} \), in the class of cost functions; positivity, i.e., for \( \alpha \) in \( \mathbb{R}_+^n \), if \( F \) is a nondecreasing cost function at each \( x \leq \alpha \), then \( P(F, \alpha) \geq 0 \), and consistency which (roughly speaking)
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means that the prices of two commodities having the same effect on costs are the same. A property that the marginal cost price mechanism does not in general satisfy is cost sharing. A price mechanism \( P \) is cost sharing if for all \((F, \alpha)\) in the domain of \( P \),

\[
\sum P_i(F, \alpha) \alpha_i = F(\alpha).
\]

Both Mirman-Tauman (1981) and Billera-Heath (1981) proved that there is a unique price mechanism for the class of continuously differentiable cost problems which satisfy the above-mentioned five properties, i.e., additivity, rescaling invariance, consistency, positivity and cost sharing. The prices associated by this price mechanism to the cost problem \((F, \alpha)\) are given by

\[
P_i(F, \alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t \alpha) \, dt,
\]

and are called Aumann-Shapley prices.

This formula has a starting aspect: it shows that these prices are completely determined by the behavior of \( F \) near the diagonal \( \{ t\alpha : 0 < t < 1 \} \). The behavior of \( F \) away from the diagonal is totally irrelevant. This remarkable phenomenon is the subject of the present paper. We look for lists of properties on the class of cost problems and the price mechanism that imply the "diagonality" of the price mechanism.

Our present contribution to the axiomatic approach to cost allocation prices is inspired by the theory of values of nonatomic games. Aumann and Shapley (1974) called attention to the diagonal property of values and raised the question of whether or not all values are diagonal. This has been answered in the negative by Neyman and Tauman (1976) and Tauman (1977). However, as proved by Neyman (1977), continuous values are diagonal.

The formal definitions and statements of the results are presented in §2. The proofs of Propositions 2.12 and 2.14 are similar to those of Tauman (1977) and Proposition 4.7 of Aumann and Shapley (1974) and are therefore omitted. The proof of the main theorem differs significantly from the one in Neyman (1977) and is presented in §3.

2. Definitions and statements of results. The class of all functions \( F: \mathbb{R}^n \to \mathbb{R} \) with \( F(0) = 0 \) is denoted by \( \mathcal{F}_n \). A cost problem is an ordered pair \((F, \alpha)\) where \( F \in \mathcal{F}_n \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is in \( \mathbb{R}^n_+ \) (i.e., \( \alpha_i > 0 \) for all \( i = 1, \ldots, n \)). The set, \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \times \mathbb{R}^n_+ \), of all cost problems is denoted by \( \mathcal{F} \). If \( \mathcal{I} \) is a subset of \( \mathcal{F} \) we denote by \( \mathcal{I}_n \) the set \( \mathcal{I} \cap (\mathcal{F}_n \times \mathbb{R}^n_+) \). Let \( \mathcal{I} \) be a class of cost problems, i.e., \( \mathcal{I} \subset \mathcal{F} \). A price mechanism for \( \mathcal{I} \) is a mapping \( P: \mathcal{I} \to \bigcup_{n=1}^{\infty} \mathbb{R}^n \) such that if \((F, \alpha) \in \mathcal{I}_n\) then \( P(F, \alpha) \in \mathbb{R}^n \). We write \( P_i(F, \alpha) \) for the \( i \)th component of the vector \( P(F, \alpha) \).

Each vector \( \lambda = (\lambda_1, \ldots, \lambda_n) \) in \( \mathbb{R}^n_+ \) induces a mapping \( \lambda: \mathbb{R}^n \to \mathbb{R}^n \) that is given by

\[
\lambda x \equiv \lambda(x_1, \ldots, x_n) = (x_1/\lambda_1, \ldots, x_n/\lambda_n),
\]

and also induces a mapping \( \lambda: \mathcal{F}_n \to \mathcal{F}_n \) that is given by

\[
(\lambda F)(x) = F(\lambda x).
\]

For \( \lambda = (\lambda_1, \ldots, \lambda_n) \) in \( \mathbb{R}^n_+ \), \( \lambda^{-1} \) denotes the vector \( (\lambda_1^{-1}, \ldots, \lambda_n^{-1}) \). A subset \( \mathcal{I} \) of

3 Although the main theorems in Mirman-Tauman (1981) and Billera-Heath (1981) are slightly different they both essentially boil down to this statement.

4 As well as M.C. prices and the entire class of prices studied in Tauman and Samet (1981).
**DEFINITION 2.1.** Let \( \mathcal{I} \) be a resealing invariant set of cost problems. A price mechanism \( P \) for \( \mathcal{I} \) is resealing invariant, if for every positive integer \( n \), every \( \lambda \) in \( \mathbb{R}_+^n \), and every \( (F, \alpha) \) in \( \mathcal{I}_n \),

\[
P(\lambda F, \lambda^{-1} \alpha) = \lambda P(F, \alpha).
\]

**DEFINITION 2.2.** Let \( \mathcal{I} \) be an additive set of cost problems; A price mechanism \( P \) for \( \mathcal{I} \) is additive, if for every positive integer \( n \), and every \( (F, a), (G, a) \) in \( \mathcal{I}_n \),

\[
P(F + G, a) = P(F, a) + P(G, a).
\]

By an ordered partition \( \mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_m) \) of \( \{1, \ldots, m\} \) we mean an ordered tuple of nonempty disjoint subsets of \( \{1, \ldots, m\} \), such that \( \bigcup_{i=1}^m \mathcal{T}_i = \{1, \ldots, m\} \). Each ordered partition \( \mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_m) \) of \( \{1, \ldots, m\} \) induces a mapping \( \mathcal{T} : \mathcal{T}_n \rightarrow \mathcal{T}_m \) that is given by

\[
\mathcal{T} x = (x(\mathcal{T}_1), x(\mathcal{T}_2), \ldots, x(\mathcal{T}_m))
\]

where for \( \mathcal{T}_i \subset \{1, \ldots, m\} \) and \( x \) in \( \mathcal{T}_n \), \( x(\mathcal{T}_i) = \sum_{i \in \mathcal{T}_i} x_i \). It also induces a mapping \( \mathcal{T} : \mathcal{T}_m \rightarrow \mathcal{T}_n \) that is given by

\[
\mathcal{T} F(x) = F(\mathcal{T} x), \quad \text{for all } F \text{ in } \mathcal{T}_n \text{ and } x \text{ in } \mathcal{T}_m.
\]

A subset \( \mathcal{J} \) of \( \mathcal{F} \) is said to be consistent if for every positive integer \( n \) and every ordered partition \( \mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_m) \) of \( \{1, \ldots, m\} \) and every \( (F, \alpha) \) in \( \mathcal{I}_n \), the cost problem \( (\mathcal{T} F, \beta) \) is in \( \mathcal{I}_m \) whenever \( \beta \in \mathbb{R}_+^m \) is such that \( \mathcal{T} \beta = \alpha \).

**DEFINITION 2.3.** Let \( \mathcal{J} \) be a consistent class of cost problems. A price mechanism \( P \) for \( \mathcal{J} \) is consistent, if for every ordered partition \( \mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_m) \) of \( \{1, \ldots, m\} \) where \( m > n \), and every \( \beta \) in \( \mathbb{R}_+^m \) such that \( (F, \mathcal{T} \beta) \in \mathcal{I}_n \),

\[
P_j(\mathcal{T} F, \beta) = P_i(F, \mathcal{T} \beta) \quad \text{where } j \in \mathcal{T}_i.
\]

**DEFINITION 2.4.** A class of cost problems \( \mathcal{J} \) that is resealing invariant, additive and consistent is called admissible. A cost allocation price function, a C.A.P.F., for short, for an admissible class of cost problems \( \mathcal{J} \) is a price mechanism \( \psi \) for \( \mathcal{J} \) that is additive, consistent and resealing invariant.

For each \( \alpha \) in \( \mathbb{R}_+^n \), we define a semimetric \( d_\alpha \) on \( \mathcal{F}_n \) by:

\[
d_\alpha(F, G) = \sup \sum_{i=1}^k |(F(x^i) - F(x^{i-1})) - (G(x^i) - G(x^{i-1}))| \tag{2.5}
\]

where the supremum in (2.5) is taken over all chains \( 0 = x^0 < x^1 < \cdots < x^k = \alpha \).

**REMARKS.** (i) The range of \( d_\alpha \) is \([0, \infty)\).

(ii) An equivalent definition for this semimetric is the following:

\[
d_\alpha(F, G) = \inf \epsilon \in \mathbb{R}_+ : \text{there are increasing functions } F_i, G_i \text{ in } \mathcal{F}_n \text{ with } F_i(\alpha) + G_i(\alpha) = \epsilon \text{ and } F + F_i - G, G + G_i - F \text{ are both (monotonic) nondecreasing on } \{x \in \mathbb{R}_+^n : 0 < x < \alpha\}, \text{where } \inf \epsilon = \infty.
\]

**DEFINITION 2.6.** A price mechanism \( \psi \) for a set of cost problems \( \mathcal{J} \) is said to be continuous if there exists a constant \( C \) such that for all \( n \), and all \( (F, \alpha), (G, \alpha) \) in \( \mathcal{I}_n \),

\[
(\psi(F, \alpha) - \psi(G, \alpha)) \cdot \beta < Cd_\alpha(F, G), \quad \text{for all } 0 < \beta < \alpha.
\]

\(^5\)This statement could be proved using arguments similar to those found in Aumann–Shapley (1974).
REMARK. The reason for considering this bounded variation semimetric (in the definition of continuity of a price mechanism) is its close relation to positivity⁶ of a price mechanism.

DEFINITION 2.7. A price mechanism $\psi$ for a set of cost problems $\mathcal{I}$ is diagonal if for every $n$, and every $(F, \alpha), (G, \alpha)$ in $\mathcal{I}_n$ with $d_\alpha(F, G) < \infty$, if there is an $\epsilon > 0$ such that for all $x \in \mathbb{R}_+^n$ with $0 < x < \alpha$ and $|x_i/\alpha - x_j/\alpha| < \epsilon$, $\forall 1 < i, j < n$, $F(x) = G(x)$

then $\psi(F, \alpha) = \psi(G, \alpha)$.

MAIN THEOREM. Every continuous C.A.P.F. for an admissible class of cost problems is diagonal.

The assumptions of the main theorem include, in particular, the consistency of both the class of cost problems and the price mechanism. These properties (of consistency) have not been defined or used in previous papers. A property similar to our consistency, introduced in previous papers, is (what we call in the present paper) weak consistency.⁷

DEFINITION 2.8. A price mechanism $\psi_i$ for a class of cost problems $\mathcal{I}$ is weakly consistent if for all $n$ and all $(F, \alpha)$ in $\mathcal{I}_n$ if $F(x_1, \ldots, x_n) = f(x_1 + \cdots + x_n)$ on $I^a = \{x \in \mathbb{R}_+: 0 < x < \alpha\}$ for some $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ and $(f, \sum \alpha_i)$ is in $\mathcal{I}_1$, then $\psi_i(F, \alpha) = \psi_i(f, \sum \alpha_i)$.

The rationale that has been given for weak consistency⁸ applies as well for consistency. We will show that consistency is essential for our theorem by showing that the conclusion (of diagonality) is false once consistency is replaced by weak consistency, even if we assume that the price mechanism obeys two additional properties—cost sharing and positivity.

DEFINITION 2.9. A price mechanism $\psi$ for a class of cost problems $\mathcal{I}$ is cost sharing if for all integers $n > 1$, and all $(F, \alpha)$ in $\mathcal{I}_n$,

$$\sum_{i=1}^n \psi_i(F, \alpha) \alpha_i = F(\alpha).$$

DEFINITION 2.10. A price mechanism $\psi$ for a class of cost problems $\mathcal{I}$ is positive if for all integers $n > 1$, and all $(F, \alpha)$ in $\mathcal{I}_n$, if $F - G$ is monotonic (nondecreasing) on $I^a$ then $\psi_i(F, \alpha) > \psi_i(G, \alpha)$ for all $1 < i < n$.

We are now ready for

PROPOSITION 2.11. There is an admissible class of cost problems $\mathcal{I}$ and a continuous, additive, rescaling invariant, positive and weakly consistent price mechanism that is not diagonal.

The next proposition shows that continuity is essential in our theorem.

PROPOSITION 2.12. There is an admissible class of cost problems $\mathcal{I}$ and a C.A.P.F. $\psi$ for $\mathcal{I}$ that is cost sharing and positive but not diagonal.

⁶See, e.g., Proposition 2.14.
⁷This property (of weak consistency) has been called consistency by Mirman-Tauman (1981) and Samet-Tauman (1981) and aggregation invariant by Billera-Heath (1981).
⁸See Samet-Tauman (1981) for example.
DEFINITION 2.13. A class of cost problems $\mathcal{I}$ is internal if for all integers $n > 1$, and all $(F, \alpha), (G, \alpha)$ in $\mathcal{I}_n$, with $d_\alpha(F, G) < \infty$, and every $\epsilon > 0$, there are functions $F_\epsilon, G_\epsilon$ in $\mathcal{F}_n$ with $(F_\epsilon, \alpha), (G_\epsilon, \alpha)$ in $\mathcal{I}_n$, $F_\epsilon(\alpha) + G_\epsilon(\alpha) = \epsilon + d_\alpha(F, G)$, and such that $F + F_\epsilon - G, G + G_\epsilon - F$ are both nondecreasing on $I^\alpha$.

PROPOSITION 2.14. Every positive, cost-sharing C.A.P.F. for an admissible and internal class of cost problems is continuous and thus also diagonal.

3. Proof of the main theorem. In what follows $e$ denotes the vector in $R^n_+$ with $e_i = 1/n$ for all $1 \leq i \leq n$. We use the same symbol $e$ with different values of $n$; no confusion should result. For $x$ in $R^n_+$ we denote by $\|x\|_1$ the norm $\sum|x_i|$ and by $\|x\|_2$ the norm $(\sum x_i^2)^{1/2}$.

Let $\mathcal{I}$ be an admissible class of cost problems and let $\psi$ be a continuous C.A.P.F. for $\mathcal{I}$. Assume that $(F, e), (G, e) \in \mathcal{I}_n$ are such that $F(x) = G(x)$ for every $x$ in $R^n_+$ with $0 < x < e$ and $|x_i - x_j| < \epsilon$, where $0 < \epsilon < 1$ is a given positive constant, and that $d_\epsilon(F, G) < \infty$. To prove the theorem it is sufficient (using the rescaling invariance) to prove that these assumptions imply that $\psi(F, e) = \psi(G, e)$.

Fix a positive integer $K$ and let $\mathcal{M}$ be the set $\{1, 2, \ldots, n^K\}$. Let $\mathcal{T}^k = (\mathcal{T}_1^k, \ldots, \mathcal{T}_n^k)$, $1 \leq k \leq K$, be $k$ ordered partitions of $\mathcal{M}$ that satisfy the following properties:

(i) $|\mathcal{T}_i^k| = n^{k-1}$ for all $1 \leq k \leq K$ and all $1 \leq i \leq n$; where for a set $A$, $|A|$ denotes the number of elements of $A$.

(ii) If for every pair $i, j$ with $1 \leq i < j \leq n$ and every $1 \leq k \leq K$ we denote by $f_{i,j}$ the function on $\mathcal{M}$ that is given by

$$f_{i,j}^k(m) = \begin{cases} 1 & \text{if } m \in \mathcal{T}_i^k, \\ -1 & \text{if } m \in \mathcal{T}_j^k, \\ 0 & \text{otherwise,} \end{cases}$$

then for all $1 \leq k, k' < K$ with $k \neq k'$,

$$\sum_{m \in \mathcal{M}} f_{i,j}^k(m)f_{i,j}^{k'}(m) = 0. \quad (3.1)$$

To show that such a sequence of ordered partitions does indeed exist, identify $\mathcal{M}$ with all functions $h : \{1, \ldots, K\} \rightarrow \{1, 2, \ldots, n\}$ (there are exactly $n^K$ such functions), and let $\mathcal{T}_i^k = \{h : h(k) = i\}$.

Let $p = (p_1, \ldots, p_n) = \psi(F, e) - \psi(G, e)$, and for each $1 < k < K$, let $p^k = \psi(\mathcal{T}_i^k F, e) - \psi(\mathcal{T}_i^k G, e)$. (Observe that $p$ is in $R^n_+$, that the $p^k$'s are in $R^{n^K}_+$, and that the same symbol $e$ is used for both a vector in $R^n$ and a vector in $R^{n^K}$.) Observe that if $m \in \mathcal{T}_i^k$ then $p^k_m = p_i$ (using the consistency) and thus the sequence $p^k$, $1 < k < K$, satisfies the following property:

For every $1 < k < K$,

$$\sum_{m=1}^{n^K} |p^k_m| = \sum_{i=1}^n \sum_{m \in \mathcal{T}_i^k} |p^k_m| = \sum_{i=1}^n n^{K-1} |p_i| = n^{K-1} \|p\|_1. \quad (3.2)$$

LEMMA 3.3. There is a universal constant $C > 0$ (i.e., independent of $n$ and $K$), such that for every $n$, every $K$, every such sequence of ordered partitions $\mathcal{T}_1^k, \ldots, \mathcal{T}_K^k$ and every $p$ in $R^n_+$ there is a subset $\overline{K}$ of $\{1, 2, \ldots, K\}$ such that

$$\|\sum_{k \in \overline{K}} p^k\|_1 \geq C|\overline{K}| \|p\|_1 n^{K-1}.$$
PROOF. By Khinchin's inequality (Zygmund [8, p. 213 (8.5)]) there is a positive constant \( C > 0 \) such that for every \( K \) and every \( (\alpha_1, \ldots, \alpha_K) \) in \( \mathbb{R}^K \),

\[
(1/2^K) \sum_{\epsilon} \left| \sum_{k=1}^{K} \epsilon_k \alpha_k \right| > 2C \left( \sum_{k=1}^{K} \alpha_k^2 \right)^{1/2}
\]

where \( \epsilon_1, \ldots, \epsilon_K \) range over all selection of signs. Therefore if we let \( \epsilon_1, \ldots, \epsilon_K \) range over all selection of signs, then

\[
(1/2^K) \sum_{\epsilon} \left\| \sum_{k=1}^{K} \epsilon_k p_k \right\| = \sum_{m=1}^{n^K} (1/2^K) \sum_{k=1}^{K} \epsilon_k p_m^k > 2C \sum_{m=1}^{n^K} \left( \sum_{k=1}^{K} (p_m^k)^2 \right)^{1/2}
\]

(3.4)

Using Schwarz's inequality, we find that

\[
(\sum_{k=1}^{K} (p_m^k)^2)^{1/2} = \left( \sum_{k=1}^{K} (1/\sqrt{K}) \right)^{1/2} > \sum_{k=1}^{K} |p_m^k| = (1/\sqrt{K}) \sum_{k=1}^{K} |p_m^k|.
\]

(3.5)

Combining (3.4), (3.5) and (3.2) we conclude that

\[
(1/2^K) \sum_{\epsilon} \left\| \sum_{k=1}^{K} \epsilon_k p_k \right\| > 2C \sum_{m=1}^{n^K} (1/\sqrt{K}) \sum_{k=1}^{K} |p_m^k|
\]

\[
= 2C \sum_{k=1}^{K} (1/\sqrt{K}) \sum_{m=1}^{n^K} |p_m^k| = 2C \sqrt{K} \| p \|_1 n^{K-1}.
\]

Observe that the left-hand side of the inequality is an average over all selections of signs \( \epsilon_1, \ldots, \epsilon_K \), of \( \| \sum_{k=1}^{K} \epsilon_k p_k \|_1 \). Since this average is at least \( 2C \sqrt{K} \| p \|_1 n^{K-1} \) there exists at least one selection of signs \( \epsilon_1, \ldots, \epsilon_K \) such that

\[
\left\| \sum_{k=1}^{K} \epsilon_k p_k \right\|_1 > 2C \sqrt{K} \| p \|_1 n^{K-1}.
\]

(3.6)

The left-hand side of (3.6) is at most

\[
\left\| \sum_{k : \epsilon_k = 1} p_k \right\|_1 + \left\| \sum_{k : \epsilon_k = -1} p_k \right\|_1,
\]

and thus at least one of these sums is at least \( C \sqrt{K} \| p \|_1 n^{K-1} \), which proves Lemma 3.3.

**Lemma 3.7.** For every subset \( \bar{K} \) of \( \{1, 2, \ldots, K\} \)

\[
d_{\epsilon} \left( \sum_{k \in \bar{K}} \mathcal{F}^{-k} F, \sum_{k \in \bar{K}} \mathcal{F}^{-k} G \right) \lesssim (12n/\epsilon^3) d_{\epsilon}(F, G).
\]

**Proof.** In what follows we identify each function \( f_{ij}^k : \mathcal{M} \to \mathcal{R} \), \( 1 < i < j < n \), \( 1 < k < K \) with the vector \( (f_{ij}^k(1), f_{ij}^k(2), \ldots, f_{ij}^k(n^K)) \) in the Euclidean space \( \mathbb{R}^{n^K} \). The inner product of two vectors \( x, y \) in \( \mathbb{R}^{n^K} \) will be denoted by \( \langle x, y \rangle \). For every \( \eta > 0 \), \( D(\eta) \) will denote the set of all vectors \( y \) in \( \mathbb{R}^{n^K} \) with \( |y_i - y_j| < \eta \) for all \( 1 < i < j < n \), and \( D(\eta) \) will denote the set of all vectors \( y \) in \( D(\eta) \) with \( 0 \leq y < \epsilon \).

Observe that for given \( 1 < k < K \) and \( x \) in \( \mathbb{R}^{n^K} \), \( \mathcal{F}^{-k} x \in D(\eta) \) if and only if for all \( 1 < i < j < n \), \( |\langle f_{ij}^k, x \rangle| < \eta \).
For given \(1 < i < j < n\) the sequence \(f_k^{ij}\), \(1 \leq k \leq K\) is orthogonal and therefore from Parserval's inequality,

\[
\sum_{k=1}^{K} \langle f_k^{ij}, x \rangle^2 \leq \langle x, x \rangle.
\]

Since for all \(1 < i < j < n\), \(\langle f_k^{ij}, f_k^{ij} \rangle = 2n^{K-1}\) we conclude that

\[
\sum_{k=1}^{K} \langle f_k^{ij}, x \rangle^2 \leq \langle x, x \rangle 2n^{K-1}.
\]

Therefore for given \(1 < i < j < n\),

\[
\left| \left\{ k : 1 < k \leq K, \langle f_k^{ij}, x \rangle > \eta \right\} \right| \leq \langle x, x \rangle \frac{2n^{K-1}}{\eta^2}.
\]  \hspace{1cm} (3.8)

Since there are \(n(n - 1)/2\) such pairs \(i, j\) we deduce that

\[
\left| \left\{ k : 1 < k \leq K, \exists 1 < i < j < n \text{ with } \langle f_k^{ij}, x \rangle > \eta \right\} \right| \leq \langle x, x \rangle \frac{n^{K}(n - 1)}{\eta^2}. \]  \hspace{1cm} (3.9)

Since \(T^{k}x \notin D(\eta)\) iff \(\exists 1 < i < j < n \text{ with } \langle f_k^{ij}, x \rangle > \eta\) we conclude that for every \(x\) in \(\mathbb{R}^{n^k}\),

\[
\left| \left\{ k : 1 < k \leq K, T^{k}x \notin D(\eta) \right\} \right| \leq \langle x, x \rangle \frac{n^{K}(n - 1)}{\eta^2}. \]  \hspace{1cm} (3.10)

For \(x\) in \(\mathbb{R}^{n^k}\) with \(0 < x < e\), \(\langle x, x \rangle = e^{-K}\) and for all \(1 < k \leq K\), \(0 < T^{k}x < e\). Therefore (3.10) implies that for every \(x\) in \(\mathbb{R}^{n^k}\) with \(0 < x < e\),

\[
\left| \left\{ k : 1 < k \leq K, T^{k}x \notin D(\eta) \right\} \right| \leq \langle x, x \rangle \frac{(n - 1)}{\eta^2}. \]  \hspace{1cm} (3.11)

In particular, by setting \(\eta = \epsilon/2\),

\[
\left| \left\{ k : 1 < k \leq K, T^{k}x \notin D(\epsilon/2) \right\} \right| \leq 4(n - 1)/\epsilon^2. \]  \hspace{1cm} (3.12)

In order to prove the lemma we have to prove that for every subset \(K\) of \(\{1, 2, \ldots, K\}\) and every increasing sequence \(0 = x^0 < x^1 < \ldots < x^I = e\) in \(\mathbb{R}^{n^k}\),

\[
\sum_{s=1}^{I} \left| \left( \sum_{k \in K} T^{k}F(x^s) - \sum_{k \in K} T^{k}F(x^{s-1}) \right) - \left( \sum_{k \in K} T^{k}G(x^s) - \sum_{k \in K} T^{k}G(x^{s-1}) \right) \right|
\equiv \sum_{s=1}^{I} \left| \sum_{k \in K} \left( F(T^{k}x^s) - F(T^{k}x^{s-1}) \right) - \left( G(T^{k}x^s) - G(T^{k}x^{s-1}) \right) \right|
\leq (12n/\epsilon^2) \delta_*(F, G).
\]

As the left-hand side of the last equality is bounded from above by

\[
\sum_{k \in K} \sum_{s=1}^{I} \left| \left( F(T^{k}x^s) - F(T^{k}x^{s-1}) \right) - \left( G(T^{k}x^s) - G(T^{k}x^{s-1}) \right) \right|
\]

it is enough to prove that

\[
\sum_{k \in K} \sum_{s=1}^{I} \left| \left( F(T^{k}x^s) - F(T^{k}x^{s-1}) \right) - \left( G(T^{k}x^s) - G(T^{k}x^{s-1}) \right) \right|
\leq (12n/\epsilon^3) \delta_*(F, G). \hspace{1cm} (3.13)
\]

Since \(\|x^0\| = 0 < \|x^1\| < \ldots < \|x^I\| = 1\), there is a subset \(L\) of \(\{1, \ldots, I\}\) with \(|L| < \lfloor I/\epsilon \rfloor\) (where by \(\lfloor a \rfloor\) we mean the smallest integer \(> a\)) such that for each \(1 < s < l\) there is an \(r\) in \(L\) with \(s < r\) and \(\|x^r - x^s\| < \epsilon/2\).
Observe that if \( x, y \in \mathbb{R}^n \) with \( 0 \leq x \leq y \leq e \) and \( \| x - y \|_1 \leq \epsilon / 2 \) then, for every \( 1 \leq k \leq K \),
\[
\mathcal{T}^k y \in D(\epsilon / 2) \Rightarrow \mathcal{T}^k x \in D(\epsilon). \tag{3.14}
\]
Therefore if \( 1 \leq k \leq K \) is such that \( \mathcal{T}^k x^r \in D(\epsilon / 2) \) for every \( r \) in \( L \) then \( \mathcal{T}^k x^r \in D(\epsilon) \) for every \( 1 \leq s \leq l \), and therefore for such \( k \),
\[
\sum_{s=1}^l \left| \left( F(\mathcal{T}^k x^s) - F(\mathcal{T}^k x^{s-1}) \right) - \left( G(\mathcal{T}^k x^s) - G(\mathcal{T}^k x^{s-1}) \right) \right| = 0, \tag{3.15}
\]
and for every (other) \( k \),
\[
\sum_{s=1}^l \left| \left( F(\mathcal{T}^k x^s) - F(\mathcal{T}^k x^{s-1}) \right) - \left( G(\mathcal{T}^k x^s) - G(\mathcal{T}^k x^{s-1}) \right) \right| \leq d_e(F, G). \tag{3.16}
\]
Let
\[
A = \{ k : 1 \leq k \leq K, \exists r \in L, \mathcal{T}^k x^r \notin D(\epsilon / 2) \}.
\]
Then (3.15) holds for \( k \) in \( \overline{K} \setminus A \) and (3.16) holds for every \( k \) in \( \overline{K} \). Therefore the left-hand side of (3.13) is bounded by \( |A| d_e(F, G) \).
Using (3.12) we have
\[
|A| \leq \sum_{r \in L} \left| \left\{ k : 1 \leq k \leq K, \mathcal{T}^k x^r \notin D(\epsilon / 2) \right\} \right| \leq \left[ 2 / \epsilon \right] 4(n - 1) / \epsilon^2,
\]
and by recalling the assumption \( \epsilon < 1 \) which implies \( [2 / \epsilon] < 3 / \epsilon \) we conclude that \( |A| \leq 12n / \epsilon^2 \) which implies (3.13) and thus Lemma 3.7.
We now complete the proof that \( \psi(F, e) = \psi(G, e) \). By considering sufficiently large \( K \) and the subset \( \overline{K} \) of \( \{1, \ldots, K\} \) that is given by Lemma 3.3 and the cost problem \( (\sum_{k \in K} \mathcal{T}^k F, e) \) and \( (\sum_{k \in K} \mathcal{T}^k G, e) \) we observe that \( d_e(\sum_{k \in K} \mathcal{T}^k F, \sum_{k \in K} \mathcal{T}^k G) \) is bounded (by Lemma 3.7) while
\[
\sum_{m=1}^{n^K} \left| \left( \psi \left( \sum_{k \in K} \mathcal{T}^k F, e \right) \right)_m - \left( \psi \left( \sum_{k \in K} \mathcal{T}^k G, e \right) \right)_m \right| / n^K
\]
is bounded only if \( \psi(F, e) = \psi(G, e) \). Therefore the continuity assumption implies that \( \psi(F, e) = \psi(G, e) \).
This completes the proof of the main theorem.

4. Proof of Proposition 2.11. We start by defining a price mechanism \( \psi \) for the class of all cost problems. For every vector \( \alpha \) in \( \mathbb{R}_{++}^n \) and every subset \( \mathcal{J} \) of \( \{1, \ldots, n\} \) we denote by \( \alpha(\mathcal{J}) \) the vector in \( \mathbb{R}_{++}^n \) that is given by
\[
(\alpha(\mathcal{J}))_i = \begin{cases} 
\alpha_i & \text{if } i \in \mathcal{J}, \\
0 & \text{if } i \notin \mathcal{J}.
\end{cases}
\]
For every integer \( n > 1 \) and every cost problem \( (F, \alpha) \) in \( \mathcal{F}_n \times \mathbb{R}_{++}^n \), associate the finite game \( v \equiv v(F, \alpha) \), in coalitional form on the set \( N = \{1, 2, \ldots, n\} \) of players that is given by \( v(\mathcal{J}) = F(\alpha(\mathcal{J})) \).
Let \( \phi \) denote the Shapley value for finite games and define the price\(^9\) mechanism

\(^9\)This price mechanism for the class of continuously differentiable cost problems was suggested by Andras Simonovitz in order to show that weak consistency is essential for the results of Mirman–Tauman (1981) and could not be replaced by symmetry, i.e., by permutational invariance.
\[ \psi_i(F, \alpha) = \frac{(\psi_0(F, \alpha))(\{i\})}{\alpha_i} \quad \text{for all } 1 \leq i \leq n. \]

It is easy to verify that this price mechanism is additive, cost sharing, positive, continuous and rescaling invariant on all of \( F \) and thus obeys these properties on every class of cost problems. Let \( \mathcal{I} \) be the class of all cost problems \((F, \alpha)\) that are linear in some neighborhood in \( I^a \) of the diagonal \( \{tx: 0 < t < 1\} \), i.e., there is \( \epsilon > 0 \) and constants \( a_1, \ldots, a_n \) such that

\[ F(x) = \sum_{i=1}^{n} a_i x_i \]

for all \( x \) in \( I^a \) with \( |x_i/\alpha_i - x_j/\alpha_j| < \epsilon. \)

It is easy to verify that \( \mathcal{I} \) is an admissible class of cost problems. We claim that \( \psi \) is weakly consistent on \( \mathcal{I} \). Indeed, let \((F, \alpha) \in \mathcal{I}_n\) be such that on \( I^a \), \( F \) is of the form,

\[ F(x_1, \ldots, x_n) = f(x_1 + \cdots + x_n). \]

As \( F \) is linear on a neighborhood of the diagonal \([0, \alpha]\), it follows that on \( I^a \),

\[ F(x_1, \ldots, x_n) = a(x_1 + \cdots + x_n). \]

Thus \( v(F, \alpha)(\mathcal{I}) = a(\alpha(\mathcal{I})) \) and therefore \( \psi_i(F, \alpha) = a \) which proves that \( \psi \) is weakly consistent on \( \mathcal{I} \).

Altogether, \( \psi \) is a continuous, additive, rescaling invariant, positive and weakly consistent price mechanism for the admissible class of cost problems \( \mathcal{I} \), which completes the proof of Proposition 2.11.

References


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