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Large Symmetric Games Are Characterized by Completeness of the Desirability Relation

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The paper presents a characterization of continuous cooperative games (set functions) which are monotonic functions of countably additive non-atomic measures. The characterization is done through a natural desirablity relation defined on the set of coalitions of players. A coalition S is at least as desirable as a coalition T (with respect to a given game v (in colational form)), if for each coalition U that is disjoint from $S \cup T$, $v(S \cup U) \geqslant v(T \cup U)$. The characterization asserts, that a game v is of the form $v = f \circ \mu$, where μ is a non-atomic signed measure and f is a monotonic and continuous function on the range of μ, if, and only if, it is in pNA' (i.e., it is a uniform limit of polynomials in non-atomic measures or equivalently it is uniformly continuous function in the NA-topology) and has a complete desirability relation. *Journal of Economic Literature* Classification Number: 026. © 1989 Academic Press, Inc.

1. INTRODUCTION

A coalitional game is a function v defined on a σ-algebra C of subsets of a set I, and satisfying $v(\emptyset) = 0$. The elements of C are called coalitions and I represents the set of players. For a coalition S and a coalitional game v, $v(S)$ is called the worth of the colation S. Coalitional games are the cornerstone in cooperative game theory and they also arise in various other social science models. In a production economic model with one output, $v(S)$ stands for the maximal production that the group S of agents can produce, or alternatively, I represents the set of raw materials and $v(S)$ is the maximum output that can be produced from the set S of raw materials. A dual interpretation views the coalitions as sets of projects or outputs and $v(S)$ represents the cost of performing (or producing) all elements in S.

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Coalitional games arise also in non-additive expected utility theory (see [7] and the motivation section and the reference there).

Every coalitional game $v: C \to \mathbb{R}$ induces a desirability relation on C: a coalition S is *at least as desirable as* a coalition T, if for each coalition U that is disjoint from $S \cup T$ (i.e., $U \cap (S \cup T) = \emptyset$) we have $v(S \cup U) \geqslant v(T \cup U)$, or equivalently, $v(S \cup U) - v(U) \geqslant v(T \cup U) - v(U)$. Thus, the desirability relation is a basic binary relation on coalitions reflecting their marginal contributions. The desirability relation for coalitions in a coalitional game with finitely many players was first introduced in [9]. It generalizes the relation of desirability for players that was defined in [11]. The relation was used in [15, 16] to develop a theory of coalition formation in finite simple games.

The desirability relation is complete if for every pair of coalitions S and T, either S is at least as desirable as T or T is at least as desirable as S. For many games the induced desirability relation is not complete. One class of games for which the desirability relation is complete are the scalar measure games of the form $f \circ \mu$, where μ is a measure on C and f is a monotonic non-decreasing function. In that case the inequality $\mu(S) \geqslant \mu(T)$ implies that S is at least as desirable as T. One spacial subclass of these games are the monotonic weighted majority games; here μ is a probability measure and f is a monotonic function from the closed interval $[0, 1]$ to the set $\{0, 1\}$.

The desirability relation (induced by a game v) will be denoted by $\succcurlyeq$. The desirability relation $\succcurlyeq$ induces a strict desirability relation $\succ$ by $S \succ T$ if and only if $S \succcurlyeq T$ but $T \not\succcurlyeq S$. Another property of the scalar measure game $f \circ \mu$ with f monotonic, is that their desirability relation is acyclic, i.e., there does not exist a finite sequence $S_1, ..., S_k$ of coalitions with $S_1 \succ S_2 \succ \cdots \succ S_k \succ S_1$.

Thus, the games of the form $f \circ \mu$, where μ is a measure on the players space and f is a monotonic function, have a complete and acyclic desirability relation. A natural inquiry that arises is whether it is possible to characterize these scalar measure games (where f is monotonic) by means of properties of the desirability relation. In particular, a natural question that arises is whether or not any game that has a complete and acyclic desirability relation has the form $f \circ \mu$, where μ is a measure and f is a monotonic function, or for what classes of games one could deduce that if their desirability relation is complete and acyclic then they have such a form. Peleg conjectured that all monotonic simple games, with finitely many players that have a complete and acyclic desirability relation are weighted majority games, i.e., games of the form $f \circ \mu$, where μ is a probability measure and f is a monotonic function $f: [0, 1] \to \{0, 1\}$. A counterexample to Peleg's conjecture was given by Einy [5]. The present paper shows that an even stronger version of this conjecture in games with

2. SRT, if and only if $(S \cup U)\, R(T \cup U)$ whenever $(S \cup T) \cap U = \varnothing$;
3. $SR\varnothing$ and $\varnothing \not R I$.

A probability measure P on (I, C) is said to agree with R if for all events A and B, $P(A) \geqslant P(B)$ if and only if ARB. Savage [18] studied conditions on the qualitative probability probability which guarantees the existence of a probability measure P that agrees with R. We will draw now the analogy between Savage's program and our development.

Given a game v we can define the relation $R(v)$ on C by $SR(v)T$ if, and only if $v(S) \geqslant v(T)$. Then $R(v)$ is obviously a simple ordering and if we further assume that for all coalitions S, $0 \leqslant v(S)$ and $v(I) > 0$ we also have $SR(v)\varnothing$ and $\varnothing \not R(v) I$. If we assume that for all coalitions S, T, and U, $SR(v)T$ if and only if $(S \cup U)R(v)(T \cup U)$ whenever $(S \cup T) \cap U = \varnothing$, then the desirability relation of v is complete but the converse is not necessarily true. In Savage's development one assumes additional monotonicity and continuity assumptions on the qualitative probability R that assert the existence of a non-atomic probability measure P that agrees with R. In our paper the continuity assumption is embodied in the fact that v is in pNA' and we do not have a monotonicity assumption. Our conclusion is that if the desirability relation of v in pNA' is complete then v is of the form $f \circ \mu$, where μ is a non-atomic (countably additive) measure and f is monotonic on the range of μ. Assuming monotonicity of v (or equivalently, of $R(v)$) we would conclude that v is of the form $f \circ \mu$, where μ is a non-atomic probability measure that is uniquely determined by v.

In view of the above analogy between Savage's program and our contribution, it is of interest to establish conditions on a binary relation R on C for which there is a real valued function v on C that belongs to pNA' and realizes the relation, i.e., $R = R(v)$.

In Section 2 we present the basic definitions and notations that are relevant to our paper. In Section 3 we state and prove our main theorem. In Section 4 we discuss some examples, and present some corollaries of the main theorem.

2. Preliminaries

Most of the definitions and notations in this section are according to [1]. Let (I, C) be a measurable space. The members of I are called *players*, the members of C *coalitions*. A *game* is a function $v\colon C \to \mathbb{R}$ such that $v(\varnothing) = 0$. A game v is *monotonic* if $v(S) \geqslant v(T)$ for each S, T such that $S \supset T$. The space of all games on (I, C) that are the difference of two monotonic games is denoted by BV. A nondecreasing sequence of sets in C of the form Ω: $S_0 \subset S_1 \subset \cdots \subset S_n$ is called a *chain*. Let $v \in BV$, the *variation of v over a chain Ω* is defined by $\|v\|_\Omega = \sum_{i=1}^{n} |v(S_i) - v(S_{i-1})|$. The *varia-*

tion norm of a game $v \in BV$ is defined by $\|v\|_{BV} = \sup\{\|v\|_\Omega \mid \Omega \text{ is a chain}\}$. It is well known that $(BV, \| \ \|_{BV})$ is a Banach space (see Proposition 4.3 in [1]). Denote by NA the set of all non-atomic measures on (I, C), and by NA^+ the subset of NA consisting of non-negative measures. By pNA we denote the closed linear subspace of BV spanned by all powers of NA^+ measures. Let BS be the Banach space of all bounded games with the supremum norm. By pNA' we denote the closed linear subspace of BS spanned by all power of NA^+ measures. It is clear that $pNA \subset pNA'$. Let $B(I, C)$ be the Banach space of all bounded and measurable real valued functions on (I, C) (measurable with respect to the σ-field C and the σ-field of Borel subsets of $\mathbb{R}$) with the supremum norm. Denote by $B_1(I, C)$ the subset of $B(I, C)$ consisting of all functions from I to $[0, 1]$. Each member μ of NA induces a function $\bar{\mu}$ on $B_1(I, C)$ defined by $\bar{\mu}(f) = \int_I f \, d\mu$, for each $f \in B_1(I, C)$. The NA-*topology* on $B_1(I, C)$ is the smallest topology for which all these functions are continuous. It is shown by Aumann and Shapley (see Proposition 22.16 in [1]) that there is unique mapping that associates with each $v \in pNA'$ a function $v^*\colon B_1(I, C) \to \mathbb{R}$ such that

$$v^*(\chi_S) = v(S) \quad \text{for each } S \quad (\text{i.e., } v^* \text{ is an extension of } v \text{ to } B_1(I, C)). \tag{2.1}$$

$$v^* \text{ is uniformly continuous on } B_1(I, C) \text{ in the NA-topology.} \tag{2.2}$$

$$(\alpha v + \beta w)^* = \alpha v^* + \beta w^*, \qquad \text{for each } \alpha, \beta \in R \quad \text{and} \quad v, w \in pNA'. \tag{2.3}$$

$$(vw)^* = v^* w^* \qquad \text{for each } v, w \in pNA'. \tag{2.4}$$

$$\mu^*(f) = \int_I f \, d\mu \qquad \text{for each } \mu \in NA. \tag{2.5}$$

$\|v^*\|' = \|v\|'$ for each $v \in pNA'$ (where here and in the sequel, if $w \in pNA'$ then $\|w\|' = \sup_{S \in C} |w(S)|$, and $\|w^*\|' = \sup_{f \in B_1(I,C)} |w^*(f|$. (2.6)

We note that by Proposition 1 in [12] each game v on (I, C) that has an extension v^* on $B_1(I, C)$ which is continuous in the NA-topology (such an extension is unique by Proposition 22.4 of [1]) is in pNA'.

For given $f, g \in B_1(I, C)$, $f \vee g$ $(f \wedge g)$ denoted the maximum (minimum) of the two functions f and g. The constant functions in $B(I, C)$ will be denoted by their value.

3. Characterization of Symmetric (Scalar Measure) Games in pNA'

Let v be a game on (I, C). A coalition S is *at least as desirable as* a coalition T (with respect to v), written $S \succcurlyeq T$, if for each $U \in C$ such that $U \cap (S \cup T) = \emptyset$ we have $v(S \cup U) \geqslant v(T \cup U)$.

If $S \succcurlyeq T$, but $T \not\succcurlyeq S$, then we write $S \succ T$. The relation $\succcurlyeq$ was introduced in [9]. It generalizes the relation of desirability for players (see Definition 9.1 in [11]).

If μ is a finite dimensional vector of measures, then the range of μ is denoted by $R(\mu)$.

We are now ready to state our main results.

THEOREM A. *Let $v \in pNA'$. Then v is of the form $v = f \circ \mu$, where $\mu \in NA$ and f is a monotonic and continuous function on $R(\mu)$, if and only if it has a complete desirability relation.*

We start with an outline of the proof.

Outline of the Proof. Let v be a game in pNA' that has a complete desirability relation. We consider the extension v^* of v (v^* is the NA-continuous function on the class $B_1(I, C)$ of "ideal coalitions"). It is shown that v^* is quasi-affine (Lemma 3.4). Using the quasi-affinity it is shown that there exist a non-zero, continuous linear functional x^* on $B(I, C)$ such that for every f, g in $B_1(I, C)$, $x^*(f) = x^*(g)$ implie $v^*(f) = v^*(g)$ (Lemma 3.7). Therefore, v^* is a composition of a monotonic function (using quasi-affinity) with x^*. The linear functionals on $B(I, C)$ can be represented by finitely additive measures; thus $v = f \circ \nu$, where ν is a finitely additive measure and f is a monotonic function on the range of ν. Using the continuity properties of v^* we first deduce that f is continuous and ν is actually countably additive. Using again the NA-continuity of v we conclude that there exists a countably additive non-atomic measure μ with $f \circ \mu = f \circ \nu = v$, which completes the proof.

For the proof of Theorem A we need a number of lemmata, but first we need some definitions.

Let L be a linear space over $\mathbb{R}$, and let K be a convex subset of L. A function $f: K \to \mathbb{R}$ is *quasi convex* if for each $x, y \in K$ and each $t \in [0, 1]$, we have $f(tx + (1-t)\, y) \geqslant \max(f(x), f(y))$. f is *quasi concave* if $-f$ is quasi convex, or, equivalently, if for each $x, y \in K$ and $t \in [0, 1]$ we have $f(tx + (1-t)\, y) \geqslant \min(f(x), f(y))$. A function $f: K \to \mathbb{R}$ is *quasi-affine* if it is quasi convex and quasi concave.

Remark 3.1. Let $J \subset \mathbb{R}$ be an interval. Then a function $f: J \to \mathbb{R}$ is quasi-affine if and only if it is monotonic.

For a given game v in pNA' we define the desirability relation $\succcurlyeq^*$ on $B_1(I, C)$ by: for $f, g \in B_1(I, C)$, $f \succcurlyeq^* g$ (f is at least as desirable as g) if for every h in $B_1(I, C)$ with $0 \leqslant h \leqslant 1 - (f \vee g)$, $v^*(h + f) \geqslant v^*(h + g)$.

LEMMA 3.2. *Let $v \in pNA'$ be a game that has a complete desirability relation on C. Then v^* has a complete desirability relation on $B_1(I, C)$.*

For the proof of Lemma 3.2 we need the following result from [4, Theorem 4].

LEMMA 3.3. *Let μ be a finite dimensional vector of measures in NA, and let $f_1, ..., f_m$ be m functions in $B_1(I, C)$ such that $\sum_{i=1}^m f_i \in B_1(I, C)$. Then there are m disjoint sets $S_1, ..., S_m$ in C such that $\mu(S_i) = \int_I f_i \, d\mu$ for each $1 \leqslant i \leqslant m$ (where here, and in the sequel, if $\mu = (\mu_1, ..., \mu_n)$ is a vector of measures in NA and $f \in B_1(I, C)$, then $\int_I f \, d\mu = (\int_I f \, d\mu_1, ..., \int_I f \, d\mu_n)$).*

Proof of Lemma 3.2. Otherwise, there exist $g_1, g_2 \in B_1(I, C)$ and $h_1, h_2 \in B_1(I, C)$ with $0 \leqslant h_i \leqslant 1 - (g_1 \vee g_2)$ and

$$v^*(g_1 + h_1) > v^*(g_2 + h_1) \qquad \text{and} \qquad v^*(g_1 + h_2) < v^*(g_2 + h_2). \tag{3.1}$$

Define $f_1 = g_1 - (g_1 \wedge g_2)$, $f_2 = g_2 - (g_1 \wedge g_2)$, $f_3 = (g_1 \wedge g_2)$, $f_4 = (h_1 \wedge h_2)$, $f_5 = h_1 - (h_1 \wedge h_2)$, and $f_6 = h_2 - (h_1 \wedge h_2)$. Note that $0 \leqslant f_i$ and that $\sum_{i=1}^6 f_i \leqslant 1$. Let $\varepsilon > 0$ with $2\varepsilon < v^*(g_1 + h_1) - v^*(g_2 + h_1)$ and $2\varepsilon < v^*(g_2 + h_2) - v^*(g_1 + h_2)$.

Since v^* is continuous on $B_1(I, C)$ in the **NA**-topology, there exist a vector $\mu = (\mu_1, ..., \mu_k)$ of measures in NA and $\delta > 0$ such that for every h in $B_1(I, C)$ and every f that is a partial sum of $\sum_{i=1}^6 f_i$, i.e., $f = \sum_{i=1}^6 \alpha_i f_i$, where $\alpha_i \in \{0, 1\}$, we have

$$\left\| \int_I (h - f) \, d\mu \right\|_\infty < \delta \Rightarrow |v^*(h) - v^*(f)| < \varepsilon, \tag{3.2}$$

where $\| \ \|_\infty$ denotes here the maximum norm in the Euclidean space E^k. By Lemma 3.3, there exist disjoint coalitions S_i, $1 \leqslant i \leqslant 6$, with $\mu(S_i) = \int_I f_i \, d\mu$. Therefore, as $g_1 + h_1 = f_1 + f_3 + f_4 + f_5$ and $g_2 + h_1 = f_2 + f_3 + f_4 + f_5$, we deduce from (3.1) and (3.2) that

$$\begin{aligned} v(S_1 \cup (S_3 \cup S_4 \cup S_5)) > v^*(g_1 + h_1) - \varepsilon > v^*(g_2 + h_1) \\ + \varepsilon > v(S_2 \cup (S_3 \cup S_4 \cup S_5)) \end{aligned} \tag{3.3}$$

and similarly, as $g_2 + h_2 = f_2 + f_3 + f_4 + f_6$ and $g_1 + h_2 = f_1 + f_3 + f_4 + f_6$, we deduce from (3.1) and (3.2) that

$$v(S_1 \cup (S_3 \cup S_4 \cup S_6)) < v(S_2 \cup (S_3 \cup S_4 \cup S_6)). \tag{3.4}$$

The two inequalities (3.3) and (3.4) contradict the completeness of the desirability relation that is induced on C by v.

LEMMA 3.4. *Let $v \in pNA'$ be a game that has a complete desirability relation. Then v^* is quasi-affine on $B_1(I, C)$.*

Proof. First we will show that v^* is quasi convex. Let $f, g \in B_1(I, C)$. We have to prove that for each $0 \leqslant t \leqslant 1$, $v^*(tf + (1-t)g) \leqslant \max(v^*(f), v^*(g))$. Since v^* is continuous on $B_1(I, C)$ in the NA-topology, the deneseness of the dyatic rationals in $[0, 1]$ implies that it is sufficient to show that $v^*(f/2 + g/2) \leqslant \max(v^*(f), v^*(g))$. Assume, on the contrary, that $v^*((f+g)/2) > \max(v^*(f), v^*(g))$.

Let $f_1 = ((f \wedge g) + f)/2$, $f_2 = ((f \wedge g) + g)/2$, $g_1 = f - f_1$, $g_2 = g - f_2$. Then $f_1 + g_1 = f$, $(f+g)/2 = f_1 + g_2 = f_2 + g_1$, and $f_2 + g_2 = g$. Therefore, $v^*(f_1 + g_1) = v^*(f) < v^*((f+g)/2) = v^*(f_1 + g_2)$ and $v^*(f_2 + g_2) = v^*(f) < v^*((f+g)/2) = v^*(f_2 + g_1)$, which contradicts, together with Lemma 3.2, the completeness of the desirability relation that is induced by v. In order to show that v^* is quasi concave, we consider the game $w = -v$. Then w has a complete desirability relation. Therefore by what we have just shown, w^* is quasi convex. Since $w^* = -v^*$ (see (2.3)), $v^* = -w^*$ is quasi concave.

COROLLARY 3.5. *Let $\mu \in NA$ and $f: R(\mu) \to \mathbb{R}$ be a continuous function. Then, the game $v = f \circ \mu$ has a complete desirability relation if and only if f is monotonic on $R(\mu)$.*

Proof. It is clear that if f is monotonic, then the game $v = f \circ \mu$ has a complete desirability relation. So assume that v has complete desirability relation. Since f is continuous, $v \in pNA'$. Therefore by Lemma 3.4, v^* is quasi-affine. Now for each $S \in C$, $f(\mu(S)) = v^*(\chi_S)$. Therefore f is quasi-affine on $R(\mu)$. By Remark 3.1, f is monotonic.

Remark 3.6. Let $v \in pNA'$. Then v^* is continuous on $B_1(I, C)$ in the supremum norm.

Remark 3.6 follows immediately from the definition of the NA-topology and the fact that v^* is continuous on $B_1(I, C)$ in the NA-topology.

LEMMA 3.7. *Let $v \in pNA'$ be a game with a complete desirability relation. Then there exists a non-zero continuous linear functional $x^*: B(I, C) \to \mathbb{R}$ that for each $f_1, f_2 \in B_1(I, C)$ we have*

$$x^*(f_1) = x^*(f_2) \Rightarrow v^*(f_1) = v^*(f_2).$$

Proof. If v^* is identically zero the lemma is obvious. So assume that v^* is not identically zero. Without loss of generality, there exists $c > 0$ with $c < \sup_{f \in B_1(I, C)} v^*(f)$ (for otherwise we consider $-v^*$). Consider now the sets

$$K_1 = \{f \in B_1(I, C) \mid v^*(f) \geqslant c\}, \qquad K_2 = \{f \in B_1(I, C) \mid v^*(f) < c\}.$$

It is clear that K_1 and K_2 are non-empty disjoint subsets of $B(I, C)$.

Moreover, by Lemma 3.4, K_1 and K_2 are convex, and by Remark 3.6 they have a non-empty interior in $B(I, C)$. Therefore by a standard separation theorem there exist $\alpha \in \mathbb{R}$ and a non-zero continuous linear functional $x^*: B(I, C) \to \mathbb{R}$ such that $\|x^*\| \leqslant 1$ and

$$x^*(f) \geqslant \alpha \qquad \text{for each} \quad f \in K_1, \tag{3.5}$$

$$x^*(f) \leqslant \alpha \qquad \text{for each} \quad f \in K_2. \tag{3.6}$$

Since $c > 0$ and $v^*(0) = 0$, $0 \in K_2$. Therefore, by (3.6) and the continuity of v^*, $\alpha > 0$.

Let $f_0 \in \operatorname{int}(K_1)$. Fix $\varepsilon > 0$ sufficiently small so that, $2\varepsilon < \alpha$ and

$$f_0 \leqslant 1 - 3\varepsilon, \tag{3.7}$$

$$x^*(f_0) \geqslant \alpha + 3\varepsilon, \tag{3.8}$$

and

$$3\varepsilon x^*(f_0)/\alpha \leqslant f_0. \tag{3.9}$$

We first show that if f_1, f_2 are interior points of $B_1(I, C)$ such that $x^*(f_1) = x^*(f_2)$, then $v^*(f_1) = v^*(f_2)$. Let $f_1, f_2 \in \operatorname{int}(B_1(I, C))$ such that $x^*(f_1) = x^*(f_2)$. Assume, on the contrary, that $v^*(f_1) \neq v^*(f_2)$. Consider the real valued function F on the closed interval $[0, 1]$ that is given by $F(\alpha) = v^*(\alpha f_1 + (1-\alpha) f_2)$. The function F is continuous (using the continuity of v^*) and $F(0) = v^*(f_2) \neq v^*(f_1) = F(1)$. Therefore there exists α_1, α_2 in $[0, 1]$ with $|\alpha_1 - \alpha_2| < \varepsilon$ and $F(\alpha_1) > F(\alpha_2)$. Setting $f_3 = \alpha_1 f_1 + (1-\alpha_1) f_2$ and $f_4 = \alpha_2 f_1 + (1-\alpha_2) f_2$ we have

$$v^*(f_3) > v^*(f_4), \tag{3.10}$$

$$x^*(f_3) = x^*(f_4) \qquad (= x^*(f_1) = x^*(f_2)), \tag{3.11}$$

$$\|f_3 - f_4\|_\infty < \varepsilon. \tag{3.12}$$

As $x^* \neq 0$ and $f_3 \in \operatorname{int} B_1(I, C)$, we deduce by the continuity of v^*, (3.10) and (3.11), that there exists f_5 in $B_1(I, C)$ with

$$x^*(f_5) < x^*(f_4), \tag{3.13}$$

$$v^*(f_5) > v^*(f_4), \tag{3.14}$$

and

$$\|f_5 - f_4\|_\infty < 2\varepsilon. \tag{3.15}$$

Consider the function $h_1 = f_5 \wedge f_4$, $g_1 = f_5 - h_1$, $g_2 = f_4 - h_1$. Note that

$0 \leqslant g_2 \leqslant 2\varepsilon$ and therefore $|x^*(g_2)| \leqslant 2\varepsilon \, \|x^*\| \leqslant 2\varepsilon$. Let $h_2 = ((\alpha - x^*(g_2))/x^*(f_0)) f_0$.

Then by (3.15), $0 \leqslant g_i \leqslant 2\varepsilon$ and, using (3.7) and (3.8),

$$0 \leqslant h_2 \leqslant 1 - (g_1 \vee g_2) \tag{3.16}$$

(and obviously also $0 \leqslant h_1 \leqslant 1 - (g_1 \vee g_2)$).

Note that

$$v^*(h_1 + g_1) = v^*(f_5) > v^*(f_4) = v^*(h_1 + g_2), \tag{3.17}$$

and as $x^*(h_2 + g_1) = x^*(h_2) + x^*(g_1) = \alpha - x^*(g_2) + x^*(g_1) = \alpha - x^*(f_4) + x^*(f_5) < \alpha$ and $x^*(h_2 + g_2) = x^*(h_2) + x^*(g_2) = \alpha$, we deduce from (3.5), (3.6), and the continuity of v^* that

$$v^*(h_2 + g_2) \geqslant c > v^*(h_2 + g_1). \tag{3.18}$$

The three inequalities (3.16), (3.17), and (3.18) contradict the completeness of the desirability relation $\succcurlyeq^*$ on $B_1(I, C)$, and thus it follows that $v^*(f_1) = v^*(f_2)$ whenever f_1, f_2 are interior points of $B_1(I, C)$ and $v \in pNA'$ has a complete desirability relation.

We will now show that $x^*(f_1) = x^*(f_2)$ implies $v^*(f_1) = v^*(f_2)$ for each $f_1, f_2 \in B_1(I, C)$. Indeed let $f_1, f_2 \in B_1(I, C)$. For each natural number n let $g_n = (1 - 1/n) f_1 + (1/2n) \chi_I$, $h_n = (1 - 1/n) f_2 + (1/2n) \chi_I$. Then, $g_n, h_n \in \operatorname{int}(B_1(I, C))$ for each n. Since $x^*(g_n) = x^*(h_n)$, $v^*(g_n) = v^*(h_n)$ for each n. By Remark 3.5, we have $v^*(g_n) \to v^*(f_1)$ and $v^*(h_n) \to v^*(f_2)$. Therefore $v^*(f_1) = v^*(f_2)$.

LEMMA 3.8. *Let $v \in pNA'$, and let $\{S_n\}_{n=1}^{\infty}$ be a non-increasing sequence of coalitions in C such that $\bigcap_{n=1}^{\infty} S_n = \varnothing$. Then for each $g \in B_1(I, C)$ and each $0 \leqslant t \leqslant 1$, we have $v^*(tg + (1-t)\chi_{S_n}) \to v^*(tg)$.*

Proof. Let $g \in B_1(I, C)$ and $0 \leqslant t \leqslant 1$. If $\mu \in NA^+$, then by the countable additivity of μ we have $\mu^*(tg + (1-t)\chi_S) \to \mu^*(tg)$. Therefore $tg + (1-t)\chi_{S_n}$ converges to tg in $B_1(I, C)$ in the NA-topology. As v^* is continuous in the NA-topology, $v^*(tg + (1-t)\chi_{S_n}) \to v^*(tg)$.

LEMMA 3.9. *Let $J \subset R$ be an interval which contains 0, and let $f\colon J \to R$ be a continuous function. Assume that there exists $a \in \operatorname{cl}(J)$, $a \neq 0$, such that*

$$f(ta) = 0 \qquad \textit{for each } 0 \leqslant t < 1, \tag{3.19}$$

$$\textit{If} \quad x \in J \quad \textit{then} \quad f(tx) = f(a + t(x - a)) \qquad \textit{for each } 0 < t \leqslant 1. \tag{3.20}$$

Then $f(x) = 0$ for each $x \in J$.

Proof. Let $x_0 \in J$. We will show that $f(x_0)=0$. If $0<x_0<a$ or $a<x_0 \leqslant 0$, then by (3.19), $f(x_0)=0$. It remains to distinguish the following possibilities:

(a) $0<a<x_0$. We show that $f(x)=0$ for each $a<x<x_0$, and then by the continuty of f at x_0 we will obtain that $f(x_0)=0$. Let $0<x \leqslant x_0$. Then by (3.20) we have

$$f(x)=f(a+(x_0-a)x/x_0)=f(a+(1-a/x_0)x).$$

Let $\alpha=1-a/x_0$. Then $\alpha>0$. For each $x \in \mathbb{R}$, let $A(x)=(x-a)/\alpha$. Then for each $a<x \leqslant x_0$ we have $0<A(x) \leqslant x_0$. Therefore,

$$f(x)=f(A(x)) \qquad \text{for each } a<x \leqslant x_0. \tag{3.21}$$

Let $a<x<x_0$. Then $A(x)<x$. Since A is an increasing function, the sequence $\{A^n(x)\}_{n=0}^{\infty}$ is decreasing. Now, there exists a natural number n such that $A^n(x) \leqslant a$. For otherwise, the sequence $\{A^n(x)\}_{n=0}^{\infty}$ converges to a point $\hat{x}$. Since A is continuous, $\hat{x}$ is a fixed point of A. But this is impossible because $x<x_0$, and A has only x_0 as a fixed point. Let n_0 be the minimal natural number such that $A^{n_0}(x) \leqslant a$. Then $A^{n_0}(x)=(A^{n_0-1}(x)-a)/\alpha>0$. Thus $0<A^{n_0}(x) \leqslant a$. Therefore, by (3.19), $f(A^{n_0}(x))=0$. Since $a<A^n(x)<x_0$ for each $n<n_0$, by (3.21), $f(x)=f(A^{n_0}(x))=0$.

(b) $a<0<x_0$. In this case we will show that $f(x)=0$ for each $0<x<x_0$. Let $\alpha=1-a/x_0$. Then $\alpha>0$. For each $x \in \mathbb{R}$, we define

$$A(x)=\alpha x+a.$$

Then, by (3.20), we have $f(x)=f(A(x))$, for each $0<x \leqslant x_0$. Let $0<x<x_0$. As $a<0$, $A(x)<x$. Since A is increasing, the sequence $\{A^n(x)\}_{n=0}^{\infty}$ is decreasing. Now, by a similar argument to that which was used in (a), there exists a natural n such that $A^n(x) \leqslant 0$. Let n_0 be the minimal natural number such that $A^{n_0}(x) \leqslant 0$. Then $A^{n_0}(x)=a+\alpha A^{n_0-1}(x)>a$. Thus, $a<A^{n_0}(x) \leqslant 0$. Therefore, by (3.19), $f(A^{n_0}(x))=0$. Hence, $f(x)=f(A^{n_0}(x))=0$.

(c) $x_0<a<0$. In this case we will show that $f(x)=0$ for each $x_0<x<a$. Let $\alpha=1-a/x_0$. For each $x \in \mathbb{R}$, define

$$A(x)=(x-a)/\alpha.$$

Then, by (3.20), $f(x)=f(A(x))$, for each $x_0 \leqslant x<a$. Let $x_0<x<a$. Then $A(x)>x$. Since A is increasing, the sequence $\{A^n(x)\}_{n=0}^{\infty}$ is increasing. Therefore there exists n such that $A^n(x) \geqslant a$. For otherwise the sequence $\{A^n(x)\}_{n=0}^{\infty}$ converges to a point $\hat{x}$ which is a fixed point of A. But this is impossible because $\hat{x} > x_0$, and A has only x_0 as a fixed point. Let n_0 be the minimal natural number such that $A^{n_0}(x) \geqslant a$. Then, $A^{n_0}(x) = (A^{n_0-1}(x) - a)/\alpha < 0$. Thus, $a \leqslant A^{n_0}(x) < 0$. Therefore, by (3.19), $f(A^{n_0}(x)) = 0$. Hence, $f(x) = f(A^{n_0}(x)) = 0$.

(d) $x_0 < 0 < a$. In this case we will show that $f(x) = 0$, for each $x_0 < x < a$. Let $\alpha = 1 - a/x_0$. Then $\alpha > 0$. For each $x \in \mathbb{R}$ define

$$A(x) = \alpha x + a.$$

Then, by (3.20) we have $f(x) = f(A(x))$, for each $x_0 \leqslant x < 0$. Let $x_0 < x < 0$. As $a > 0$, $A(x) > x$. Since A is increasing, the sequence $\{A^n(x)\}_{n=0}^{\infty}$ is increasing. Therefore there exists n such that $A^n(x) \geqslant 0$. Let n_0 be the minimal natural number such that $A^{n_0}(x) \geqslant 0$. Then $A^{n_0}(x) = a + \alpha A^{n_0-1}(x) < a$. Thus, $0 \leqslant A^{n_0}(x) < a$. Therefore by (3.19), $f(A^{n_0}(x)) = 0$. Hence, $f(x) = f(A^{n_0}(x)) = 0$.

We denote by FA the set of all bounded and finitely additive measure on (I, C).

Proof of Theorem A. It is clear that if $\mu \in NA$ and f is a monotonic function on $R(\mu)$, then the game $v = f \circ \mu$ has a complete desirability relation. Assume that v has a complete desirability relation. If v is identically zero, the theorem is trivial. Assume that v is not identically zero. Now, by Lemma 3.7, there exists a non-zero continuous linear functional $x^*: B(I, C) \to \mathbb{R}$ such that for each $g_1, g_2 \in B_1(I, C)$,

$$x^*(g_1) = x^*(g_2) \Rightarrow v^*(g_1) = v^*(g_2). \tag{3.22}$$

We now use the fact that the dual of $B(I, C)$ is FA (see Theorem IV.5.1, p. 258 in [3]). This yields the existence of a measure $\nu \in FA$ such that for each $g \in B(I, C)$

$$x^*(g) = \int_I g \, d\nu.$$

Let $J = x^*(B_1, (I, C))$. Define a function $f: J \to R$ by $f(x^*(g)) = v^*(g)$, for each $g \in B_1(I, C)$. Then, by (3.22), f is well defined. It is clear that $v = f \circ \nu$. Now, by Lemma 3.4, v^* is quasi-affine on J. By Remark 3.1 it is monotonic on J. Since $B_1(I, C)$ is a connected subset of $B(I, C)$ and v^* is continuous on $B_1(I, C)$, the set $v^*(B_1(I, C))$ is an interval. Since f is monotonic on J and $f(J)$ is an interval, f must be continuous on J. We first show that ν is countably additive. Assume, on the contrary, that ν is not countably additive. Then there exists a monotonic non-increasing sequence $\{S_n\}_{n=1}^{\infty}$ of coalitions in C such that $\bigcap_{n=1}^{\infty} S_n = \emptyset$, and the sequence $\{\nu(S_n)\}_{n=1}^{\infty}$ does not converge to 0. Since $\nu \in FA$, the sequence $\{\nu(S_n)\}_{n=1}^{\infty}$ is bounded.

Therefore it has a convergent subsequence $\{v(S_{n_i})\}_{i=1}^{\infty}$ which converges to a number $a \neq 0$. Clearly, $a \in \text{cl}(J)$. By Lemma 3.8 we have

$$v^*(tg) = \lim_{i \to \infty} v^*(tg + (1-t)\, \chi_{S_{n_i}}), \tag{3.23}$$

for each $0 \leqslant t \leqslant 1$ and $g \in B_1(I, C)$. Let $0 \leqslant t < 1$. Then $ta \in J$. Since f is continuous on J, we obtain by (3.23) that

$$f(ta) = \lim_{i \to \infty} f(tv(S_{n_i})) = \lim_{i \to \infty} v^*(t\chi_{S_{n_i}}) = v^*(0) = 0.$$

Let $x \in J$. Then there is $g \in B_1(I, C)$ such that $x = x^*(g)$. Let $0 < t \leqslant 1$. Then

$$\begin{aligned} f(tx) = f(x^*(tg)) = v^*(tg) &= \lim_{i \to \infty} v^*(tg + (1-t)\, \chi_{S_{n_i}}) \\ &= \lim_{i \to \infty} f(tx + (1-t)\, v(S_{n_i})) \\ &= f(tx + (1-t)a) = f(a + t(x-a)). \end{aligned}$$

Thus we have shown that f satisfies the conditions of Lemma 3.9 on J. Therefore $f(x) = 0$ for each $x \in J$, which implies that v^* is identically zero. But this contradicts our assumption that v is not identically zero. Therefore, v is countably additive. Now if v is nonatomic[1] we will take $\mu = v$ and the proof is complete. So assume that v have atoms. Let A be an atom of v. We will show that $v(S \cup A) = v(S)$ for each $S \in C$ such that $S \cap A = \varnothing$. Assume, on the contrary, that there exists $S \in C$ such that $S \cap A = \varnothing$ and $v(S \cup A) \neq v(S)$. Let $\varepsilon = |v(S \cup A) - v(S)|$. Then $\varepsilon > 0$. Since $v \in pNA'$, there exists a vector ξ of measures in NA and a polynomial p on $R(\xi)$ such that $\|v - p \circ \xi\|' < \varepsilon/3$. Let n be a natural number such that $|p(\xi(S) + \xi(A)/n) - p(\xi(S))| < \varepsilon/3$. Now, because of Lyapunov's theorem it is possible to partition A into disjoint sets $T_1, \ldots, T_n$ such that $\xi(T_i) = \xi(A)/n$ for each $1 \leqslant i \leqslant n$. Since A is an atom of v, there exists $1 \leqslant i \leqslant n$ such that $v(A) = v(T_i)$. Therefore $v(S \cup A) = f(v(S \cup A)) = f(v(S \cup T_i)) = v(S \cup T_i)$. Now, $|p(\xi(S) + \xi(A)/n) - v(S \cup A)| = |p(\xi(S \cup T_i)) - v(S \cup T_i)| \leqslant \varepsilon/3$. Hence

$$\begin{aligned} |v(S \cup A) - v(S)| \leqslant |p(\xi(S)) - v(S)| &+ |p(\xi(S) + \xi(A)/n) - p(\xi(S))| \\ &+ |v(S \cup A) - p(\xi(S) + \xi(A)/n)| < \varepsilon. \end{aligned}$$

But this contradicts the choice of ε. Therefore $v(S \cup A) = v(S)$. Now what we have just shown implies that $v(S) = v(S \setminus A)$ for each $S \in C$ and each atom A of v. By induction we have $v(S) = v(S \setminus \bigcup_{i=1}^{n} A_i)$ for each $S \in C$

[1] Note that it can be shown by Lemma 3.9 that $v(\{s\}) = 0$ for each $s \in I$. Therefore if (I, C) is a standard measurable space (i.e., it is isomorphic to $[0, 1]$ with its Borel subsets), then v is nonatomic.

and each finite set $\{A_i\}_{i=1}^{n}$ of atoms of ν. Let $(A_i)_{i=1}^{\infty}$ be a sequence of ("all") atoms of ν for which the measure μ on (I, C) that is given by $\mu(S) = \nu(S \setminus \bigcup_{i=1}^{\infty} A_i)$ is in NA. Since μ is countably additive and f is continuous, for each $S \in C$ we have

$$f\left(\nu\left(S \setminus \bigcup_{i=1}^{\infty} A_i\right)\right) = \lim_{n \to \infty} f\left(\nu\left(S \setminus \bigcup_{i=1}^{n} A_i\right)\right) = f(\nu(S)).$$

Therefore, $v = f \circ \mu$. Since $R(\mu) \subset J$, f is continuous and monotonic on $R(\mu)$.

4. Some Corollaries and Examples

A game v on (I, C) is called a *scalar measure game* if it is of the form $v = f \circ \mu$, where $\mu \in NA^+$ and f is a real valued function on $R(\mu)$. The following corollary is an immediate consequence of Theorem A.

Corollary 4.1. *Let $v \in pNA'$ be a monotonic game. Then v is a scalar measure game if and only if it has complete desirability relation.*

Corollary 4.2. *Let ν be a finite dimensional vector of measures in NA, and let g: $R(\nu) \to \mathbb{R}$ be a continuous function. If the desirability relation of the game $v = g \circ \nu$ is complete then $v = f \circ \mu$, where $\mu \in NA$ and f is a continuous and monotonic function on $R(\mu)$.*

Proof. Since g is continuous on $R(\nu)$, $v \in pNA'$ (see Proposition 27 in [20]). Now Corollary 4.2 is a direct consequence of Theorem A.

The following example shows that there are games of the form $g \circ \nu$, where ν is a finite dimensional vector of measures in NA and g is a real valued function on $R(\nu)$, that have a complete desirability relation and are not of the form $f \circ \mu$, where $\mu \in NA$.

Example 4.3. Consider the measurable space $([0, 2], C)$, where C is the σ-field of Borel subsets of $[0, 2]$. Let λ be the Lebesgue measure on $[0, 2]$. We define two measures ν_1 and ν_2 on $([0, 2], C)$ by $\nu_1(S) = \lambda(S \cap [0, 1])$ and $\nu_2(S) = \lambda(S \cap [1, 2])$ for each $S \in C$. Let $\nu = (\nu_1, \nu_2)$. Then $R(\nu) = [0, 1]^2$. Define now a function g: $[0, 1]^2 \to \{0, 1\}$ by

$$g(x_1, x_2) = \begin{cases} 1, & x_1 + x_2 > 1, \\ 1, & x_1 + x_2 = 1, \quad x_1 \geqslant x_2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $v = g \circ \nu$, and let $\succcurlyeq$ be the desirability relation of v. We will show that

$\succcurlyeq$ is complete. Indeed, let $S, T \in C$. If $\nu_1(S)+\nu_2(S) > \nu_1(T)+\nu_2(T)$, then $S \succcurlyeq T$. If $\nu_1(S)+\nu_2(S)=\nu_1(T)+\nu_2(T)$, then $S \succcurlyeq T$, if $\nu_1(S) \geqslant \nu_1(T)$. We will show that there is no measure $\mu \in NA$ and no function $f\colon \mathbb{R}(\mu) \to \{0, 1\}$ such that $v = f \circ \mu$. Assume, on the contrary, that there exist such μ and f. Since the sets $\{x \in [0, 1]^2 \mid g(x) \geqslant 1\}$ and $\{x \in [0, 1]^2 \mid g(x) \leqslant 0\}$ are convex, g is quasi-affine on $[0, 1]^2$. Now, $f(\mu(S)) = g(\nu(S))$, for each S. Therefore f is quasi-affine on $R(\mu)$. By Remark 3.1, it is monotonic on $R(\mu)$. Without loss of generality, assume that f is non-decreasing on $R(\mu)$. Let $q = \inf\{t \in R(\mu) \mid f(t) = 1\}$. Then

$$f(t) = \begin{cases} 1, & t \geqslant q, \\ 0, & t < q; \end{cases} \tag{4.1}$$

or

$$f(t) = \begin{cases} 1, & t > q, \\ 0, & t \leqslant q. \end{cases} \tag{4.2}$$

We first assume that f is given by (4.1). For each natural number $n \geqslant 3$, let $S_n = [2/3, 5/3 + 1/n]$. Then $v(S_n) = 1$ for each $n \geqslant 3$. Therefore $\mu(S_n) \geqslant q$ for each $n \geqslant 3$. Since $S_{n+1} \subset S_n$, $\lim_{n\to\infty} \mu(S_n) = \mu(\bigcap_{n=3}^{\infty} S_n) = \mu([\frac{2}{3}, \frac{5}{3}])$. Hence, $\mu([\frac{2}{3}, \frac{5}{3}]) \geqslant q$, but this is impossible because $v([\frac{2}{3}, \frac{5}{3}]) = 0$. If v is given by (4.2), then for each natural number $n \geqslant 2$ we define $S_n = [1/2 + 1/n, 3/2]$. Then $v(S_n) = 0$ for each $n \geqslant 2$. Therefore $\mu(S_n) \leqslant q$ for each $n \geqslant 2$. Since $S_{n+1} \supset S_n$, $\lim_{n\to\infty} \mu(S_n) = \mu(\bigcup_{n=3}^{\infty} S_n) = \mu([\frac{1}{2}, \frac{3}{2}])$. Hence, $\mu([\frac{1}{2}, \frac{3}{2}]) \leqslant q$. But this is impossible because $v([\frac{1}{2}, \frac{3}{2}]) = 1$.

We now present an example of a game $v \in pNA$ which has a cyclic desirability relation (i.e., the strict relation which is derived from the desirability relation of v is cyclic).

EXAMPLE 4.4. Consider the measurable space $([0, 6], C)$, where C is the σ-field of Borel subsets of $[0, 6]$. For each $i = 1, \ldots, 6$, we define a measure μ_i on C by $\mu_i(S) = \lambda(S \cap [i-1, i])$, where λ is the Lebesgue measure on $[0, 6]$. Let $\mu = (\mu_1, \ldots, \mu_6)$. Then $R(\mu) = [0, 1]^6$. Let $N = \{1, \ldots, 6\}$, and let

$$\theta = \{\{1, 2, 5\}, \{1, 3, 4\}, \{4, 5, 6\}\}.$$

We define a monotonic finite simple game $c\colon 2^N \to \{0, 1\}$ by:

$$c(T) = 1. \quad \Leftrightarrow \quad \text{There exists } T_0 \in \theta \text{ such that } T_0 \subset T$$

(i.e., θ is the set of minimal winning coalitions in the game c). Let

$f: [0, 1]^6 \to \mathbb{R}$ be the multilinear extension of the game c (see [14, Chap. X]); i.e.,

$$f(x_1, ..., x_6) = \sum_{T \subset N} c(T) \prod_{i \in T} x_i \prod_{i \notin T} (1 - x_i).$$

Let $v = f \circ \mu$. Then by Lemma 7.2 of [1], $v \in pNA$. Let $\succcurlyeq$ be the desirability relation of v. Consider the sets

$$S_1 = (0, 2), \qquad S_2 = (2, 4), \qquad S_3 = (4, 6).$$

We will show that $S_1 \succ S_2 \succ S_3 \succ S_1$. Let $U \in C$ such that $U \cap (S_1 \cup S_2) = \varnothing$. Then $v(U \cup S_1) = f(1, 1, 0, 0, \mu_5(U), \mu_6(U)) = \mu_5(U)$, and $v(U \cup S_2) = f(0, 0, 1, 1, \mu_5(U), \mu_6(U)) = \mu_5(U)\, \mu_6(U)$. Hence, $v(U \cup S_1) \geqslant v(U \cup S_2)$. Therefore, $S_1 \succcurlyeq S_2$. Since $v(S_1 \cup (4, 5)) = 1$ and $v(S_2 \cup (4, 5)) = 0$, $S_1 \succ S_2$. Let now $U \in C$ with $U \cap (S_2 \cup S_3) = \varnothing$. Then $v(U \cup S_2) = \mu_1(U) \geqslant \mu_1(U)\, \mu_2(U) = v(U \cup S_3)$. Hence, $S_2 \succcurlyeq S_3$. Since $v(S_2 \cup (0, 1)) = 1$ and $v(S_3 \cup (0, 1)) = 0$, $S_2 \succ S_3$. Let now $U \in C$ such that $U \cap (S_1 \cup S_3) = \varnothing$. Then $v(U \cup S_3) = \mu_4(U) \geqslant \mu_3(U)\, \mu_4(U) = v(U \cup S_1)$. Since $v(S_3 \cup (3, 4)) = 1$ and $v(S_1 \cup (3, 4)) = 0$, $S_3 \succ S_1$. Thus we have shown that $S_1 \succ S_2 \succ S_3 \succ S_1$. Therefore $\succ$ is cyclic.

We note that in a finite game $\succ$ may be cyclic even if $\succcurlyeq$ is complete (see Remark 6.5 in [5]). The following immediate corollary of Theorem A shows that if $v \in pNA'$, this is impossible.

COROLLARY 4.5. *Let $v \in pNA'$. If the desirability relation of v is complete, then it is acyclic (i.e., the strict relation which is derived from the desirability relation of v is acyclic).*

Let $v \in pNA'$. Then v is *homogeneous of degree one* if $v^*(\alpha \chi_S) = \alpha v(S)$ for each $0 \leqslant \alpha \leqslant 1$ and each set $S \in C$. Games that are homogeneous of degree one arise in various economic models (e.g., the market game that is investigated in Section 30 of [1]). The following corollary of Theorem A characterizes those games in pNA' that are homogeneous of degree one and have a complete desirability relation.

COROLLARY 4.6. *Let $v \in pNA'$ be homogeneous of degree one. Then $v \in NA$ if and only if the desirability relation of v is complete.*

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