1 Course Outline

Exchange economies, efficient allocations, competitive equilibria, efficiency of competitive equilibria, existence of competitive equilibrium, generic finiteness of competitive equilibria, core and competitive equilibria.

2 References


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Exchange Economies

An exchange economy consists of:

1) A set of traders $T$.

2) A vector space $V$ - the space of goods.

3) For every $t \in T$ a subset $X(t) \subset V$ - the consumption set of trader $t$.

4) For every $t \in T$, an element $e(t) \in X(t)$ - the initial endowment of trader $t$. 

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5) For every \( t \in T \) a binary relation \( \succ_t \) on \( X(t) \) - the *preference relation* of trader \( t \).

For the mean time we assume that the trader set \( T \) consists of finitely many traders, i.e., \( T \) is a finite set.

If there are \( \ell \) goods, \( V = \mathbb{R}^\ell \), and an element \( x \in V \), \( x = (x_1, \ldots, x_\ell) \) is interpreted as a bundle of \( x_i \) units of commodity \( i \).

The set \( X(t) \) represents the consumption set of trader \( t \), i.e. the set of all feasible consumption bundles of trader \( t \). For instance, if one of the commodities is leisure in a given day, it is bounded by 24 hours. Restrictions of the consumption set may also result by interrelations of several commodities.

The initial endowment \( e(t) \) represents the initial bundle of goods of trader \( t \) before trading takes place.

The preference relation \( \succ_t \) represents trader’s \( t \) preference over his feasible consumption. Given \( x, y \in X(t) \), \( x \succ_t y \) means that trader \( t \) prefers (strictly) bundle \( x \) over bundle \( y \).

**Remarks** The binary (preference) relation \( \succ_t \) is called *transitive* if \( x \succ_t y \) and \( y \succ_t z \) imply \( x \succ_t z \). It is *continuous* if \( \{ (x,y) \in X(t) \times X(t) : x \succ_t y \} \) is open in \( X(t) \times X(t) \).
A utility function on $X(t)$ is a real valued function $u : X(t) \rightarrow \mathbb{R}$. The preference relation $\succ$ induced by a utility function $u$ is the binary relation $x \succ y \iff u(x) > u(y)$. A binary relation induced by a utility function is transitive, and a binary relation induced by a continuous utility function is continuous.

An important implicit assumption in our present modeling of an exchange economy is that there are no externalities. Each individual cares about his consumption “alone”.

**Example** of an exchange economy with 2 traders and 2 commodities:

$T = \{1, 2\}$

$V = \mathbb{R}^2$

$\mathbb{R}_+^2 = X(1) = X(2)$

$e(1) = (4, 0)$ and $e(2) = (0, 2)$

$\succ_1$ is the preference relation induced by the utility function $u(x_1, x_2) = x_1x_2$, and $\succ_2$ is the preference relation induced by the utility function $u(x_1, x_2) = \min(x_1, x_2)$.

**Definition 1** An allocation in an exchange economy $(T; V; (X(t))_{t\in T}; (e(t))_{t\in T}; \succ_{t\in T})$ is a function $z : T \rightarrow V$ such that

1. $\forall t \in T$, $z(t) \in X(t)$
2. $\sum_{t \in T} z(t) = \sum_{t \in T} e(t)$.

**Question 1** What is the set of all allocations in the above example?

An allocation is a pair of 2-dimensional vectors, $z(1)$ and $z(2)$, in $\mathbb{R}_+^2$ such that $z(1) + z(2) = e(1) + e(2) = (4, 2)$. Therefore $z(1)$ obeys $(0, 0) \leq z(1) \leq (4, 2)$ (where for two finite dimensional vectors, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, $x \leq y \iff x_i \leq y_i$ for every $i = 1, \ldots, n$) and $z(2) = (4, 2) - z(1)$. Therefore specifying $z(1)$ determines the allocation $z$, thus the set of all allocation in this example is represented by the points in the rectangle $\{z \in \mathbb{R}_+^2 : (0, 0) \leq z \leq (4, 2)\}$. Note that whenever there are two traders and two commodities, and the consumption sets are the nonnegative orthant, there is a 1 – 1 correspondence between the rectangle $\{z \in \mathbb{R}_+^2 : (0, 0) \leq z \leq e(1) + e(2)\}$ and the set of allocations. This rectangle is called the **Edgeworth box**.
Question 2 How can we describe the preferences in the above example?

The consumption set of trader 1 is the positive orthant of $\mathbb{R}^2$ and his preference relation, $\succ_1$ can be described by means of the indifference curves. E.g., consider all points $x = (x_1, x_2) \in \mathbb{R}^2_+$ with $x_1 x_2 = 1$. The geometric location of all these points is called an indifference curve. Similarly, for any given $\alpha \geq 0$, the geometric location of all points $x = (x_1, x_2) \in \mathbb{R}^2_+$ with $x_1 x_2 = \alpha$ is denoted $I^1_\alpha$ and is an indifference curve. Trader 1 is indifferent between any two bundles in $I^1_\alpha$, and if $x \in I^1_\alpha$ and $y \in I^1_\beta$ then $x \succ_1 y \iff \alpha > \beta$.

Similarly, the consumption set of trader 2 is the positive orthant of $\mathbb{R}^2$ and his preference relation, $\succ_2$ can be described by means of the indifference curves. E.g., consider all points $y = (y_1, y_2) \in \mathbb{R}^2_+$ with $\min(y_1, y_2) = 1$. The geometric location of all these points is an indifference curve of the
preference relation $\succsim_2$. Similarly, for any given $\alpha \geq 0$, the geometric location of all points $x = (x_1, x_2) \in \mathbb{R}_+^2$ with $\min(x_1, x_2) = \alpha$ is denoted $I^2_{\alpha}$ and is an indifference curve of $\succsim_2$. Trader 2 is indifferent between any two bundles in $I^2_{\alpha}$, and if $x \in I^2_{\alpha}$ and $y \in I^2_{\beta}$ then $x \succsim_2 y \iff \alpha > \beta$.

**Definition 2** An allocation $z$ is (weakly) efficient if there is no allocation $y$ with $y(t) \succsim_t z(t)$ for every $t \in T$.

**Question 3** What are the (weakly) efficient allocations in the above example?

**Question 4** Are there efficient allocation in each exchange economy?

**Definition 3** A competitive equilibrium of an exchange economy $E$ is an ordered pair $(p, z)$ such that:

- $p$ is a nontrivial linear functional on $V$.
- $z$ is an allocation with $p(z(t)) = p(e(t))$ for every $t \in T$, and $\forall t \in T, \{x \in X(t) : p(x) \leq p(e(t)) \text{ and } x \succsim_t z(t)\} = \emptyset$.

**Remarks** If $V = \mathbb{R}^\ell$ then any linear functional over $V$ is represented by a vector $p = (p_1, \ldots, p_\ell)$ where

$$p(x) = \sum_{i=1}^{\ell} p_i x_i$$
The vector $p$ is called the vector of prices and $p_i$ is interpreted as the price of commodity $i$.

**Question 5** What are the competitive equilibria in the above example?

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4.1 Efficiency of Competitive Equilibrium

**Theorem 1** Let $\mathcal{E} = \langle T; V; (X(t))_{t \in T}; (e(t))_{t \in T}; (\succ_t)_{t \in T} \rangle$ be an exchange economy, and assume that $(p, x)$ is a competitive equilibrium. Then the allocation $x$ is weakly efficient.

**proof** Assume that $z$ is an allocation with $z(t) \succ_t x(t)$ for every trader $t \in T$. As $(p, x)$ is a competitive equilibrium, $p(z(t)) > p(e(t))$. As $p$ is a linear functional, $p(\sum_{t \in T} z(t)) = \sum_{t \in T} p(z(t))$ and $p(\sum_{t \in T} e(t)) = \sum_{t \in T} p(e(t))$ and therefore $p(\sum_{t \in T} z(t)) > p(\sum_{t \in T} e(t))$ which contradicts the equality $\sum_{t \in T} z(t) = \sum_{t \in T} e(t)$.

4.2 The Core

A *coalition* is a subset of $T$. Let $S \subset T$ be a coalition. An $S$ - *allocation* is a function $x : S \to V$ such that for every $t \in S$ $x(t) \in X(t)$ and $\sum_{t \in S} x(t) =$
Definition 4 \textit{The core of an exchange economy} \( E \) \textit{is the set of all allocations} \( y \) \textit{for which there is no coalition} \( S \) \textit{and an} \( S \)-\textit{allocation} \( x \) \textit{with} \( x(t) \succ_t y(t) \) \textit{for every} \( t \in S \).

Definition 5 \textit{A competitive allocation is an allocation} \( z \) \textit{for which there is a competitive equilibrium} \( (p, z) \).

Consider the following exchange economy \( E \). The set of traders is \( T = \{1, 2, 3, 4\} \). There are two goods in the market; the space of goods is \( \mathbb{R}^2 \). The consumption sets of each trader \( t \in T \) is \( X(t) = \mathbb{R}^2_+ \). The initial endowments are: \( e(1) = e(2) = (1, 0) \) and \( e(2) = e(4) = (0, 1) \). The preference relation of each trader \( t \in T \) is induced by the utility function \( u_t(x_1, x_2) = \sqrt{x_1x_2} \).

Consider the allocation \( x \) where \( x(1) = x(3) = (1/4, 1/4) \) and \( x(2) = x(4) = (3/4, 3/4) \). Prove that \( x \) is an efficient allocation.

Question 6 \textit{Find all efficient allocations} \( y \) \textit{with} \( y(1) = y(3) \) \textit{and} \( y(2) = y(4) \) \textit{of the above exchange economy}.

Consider the above allocation \( x \). Show that it is not a core allocation by exhibiting a \( \{1, 2, 3\} \)-allocation \( y \) where \( y(t) \succ x(t) \) for every \( 1 \leq t \leq 3 \).
Question 7 Find all core allocations \( y \) with \( y(1) = y(3) \) and \( y(2) = y(4) \) of the above exchange economy.

Show that every core allocation \( y \) of the above exchange economy satisfies \( y(1) = y(3) \) and \( y(2) = y(4) \). Therefore the solution of the previous question describes all allocations in the core of \( E \).

Theorem 2 Any competitive allocation is in the core.

Proof. Assume that \((p, z)\) is a competitive equilibrium, \( S \) a nonempty subset of \( T \), and \( x : S \to V \) with \( x(t) \succ_t z(t) \) for every \( t \in S \). As \((p, z)\) is a competitive equilibrium, \( px(t) > pz(t) \) for every \( t \in S \), and therefore \( \sum_{t \in S} px(t) > \sum_{t \in S} pz(t) = \sum_{t \in S} pe(t) \), i.e., \( p \sum_{t \in S} x(t) > p \sum_{t \in S} z(t) = p \sum_{t \in S} e(t) \), implying that \( \sum_{t \in S} x(t) \neq \sum_{t \in S} e(t) \). Thus \( x \) is not an \( S \)-allocation. We have thus proved that there is no coalition \( S \) and an \( S \)-allocation \( x \) with \( x(t) \succ z(t) \) for every \( t \in S \), i.e., that \( z \) is in the core of the exchange economy. \( \blacksquare \)

4.3 Preferences, Utility, and Demand

A binary relation \( \mathcal{R} \) on a set \( X \) is a subset \( \mathcal{R} \subseteq X \times X \). We write \( x \mathcal{R} y \) whenever \((x, y) \in \mathcal{R} \). The binary relation \( \mathcal{R} \) is reflexive if \( x \mathcal{R} x \) for every \( x \in X \). It is transitive if for every \( x, y, z \in X \), “\( x \mathcal{R} y \) and \( y \mathcal{R} z \)” implies...
A binary relation $\mathcal{R}$ on $X$ is complete if for every $x, y \in X$ either $x\mathcal{R}y$ or $y\mathcal{R}x$.

4.4 Brouwer’s Fixed Point Theorem

**Theorem 3** (Brouwer’s fixed point theorem): Let $C$ be a nonempty convex compact subset of $\mathbb{IR}^n$ and let $f : C \to C$ be a continuous function. Then $f$ has a fixed point, i.e., there is a point $x \in C$ such that $f(x) = x$.

**Remarks** The proof of Brouwer’s fixed point theorem in the 1-dimensional case is equivalent to the mean value theorem of continuous functions. Indeed, a convex compact subset of $\mathbb{IR}$ is a closed interval $[a, b]$, and a continuous function $f : [a, b] \to [a, b]$ satisfies $f(a) \geq a$ and $f(b) \leq b$. Therefore, the continuous function $g : [a, b] \to \mathbb{IR}$ defined by $g(x) = f(x) - x$ obeys $g(a) \geq 0$ and $g(b) \leq 0$ and therefore there is $x \in [a, b]$ such that $g(x) = 0$ i.e. $f(x) = x$.

Each one of the assumptions in Brouwer’s fixed point theorem necessary. Indeed, if $S^1 = \{(x, y) \in \mathbb{IR}^2 : x^2 + y^2 = 1\}$, the continuous function $f : S^1 \to S^1$ defined by $f(x, y) = -(x, y)$ does not have a fixed point. The continuous function $f : (0, 1] \to (0, 1]$ defined by $f(x) = x/2$ does not have a fixed point. The continuous function $f : \mathbb{IR} \to \mathbb{IR}$ defined by $f(x) = x + 1$ does not have a fixed point. The function $f : [0, 1] \to [0, 1]$ defined by $f(x) = x/2$ if $x > 0$
and \( f(0) \) does not have a fixed point.

The graph of a function \( f: C \to D \) is the subset \( \{(x, y) : y = f(x)\} \) of \( C \times D \), and is denoted \( \Gamma_f \). When \( D \subset \mathbb{R}^n \) is a compact subset, the function \( f \) is continuous if and only if \( \Gamma_f \) is a closed subset of \( C \times D \). Indeed, assume that \( f \) is continuous and that the sequence \((x_k, y_k) \in \Gamma_f\) converges as \( k \to \infty \) to a point \((x, y) \in C \times D\). Then \( x_k \to x \) as \( k \to \infty \) and \( y_k = f(x_k) \). As \( f \) is continuous \( y_k = f(x_k) \to f(x) \) and therefore \( y = f(x) \), i.e., \((x, y) \in \Gamma_f \). In the other direction, assume that \( \Gamma_f \) is closed in \( C \times D \). Let \( x_k, x \in C \) and assume that \( x_k \to x \). We have to prove that \( y_k = f(x_k) \to f(x) \) as \( k \to \infty \). Otherwise, as \( D \) is compact, there is a subsequence \((y_{k_m})\) which converges to a point \( y^* \in D \) with \( y^* \neq f(x) \); therefore, \((x_{k_m}, y_{k_m}) \to (x, y^*) \in C \times D \). As \((x_{k_m}, y_{k_m}) \in \Gamma_f \) and \( \Gamma_f \) is closed in \( C \times D \), \((x, y^*) \in C \times D \), i.e \( y^* = f(x) \) a contradiction.

### 4.5 Set-Valued Functions

A set-valued function from a set \( X \) to a set \( Y \) is a function which maps points in \( X \) to subsets of \( Y \). A fixed point of a set-valued function \( f \) from \( X \) to \( X \)
is a point $x \in X$ such that $x \in f(x)$.

Consider the following set-valued function from $[0, 1]$ to $[0, 1]$.

$$
f(x) = \begin{cases} 
\{x + 1/2\} & \text{if } x < 1/2 \\
\{0, 1\} & \text{if } x = 1/2 \\
\{x - 1/2\} & \text{if } x > 1/2 
\end{cases}
$$

The graph of a set valued function $f$ from $X$ to $Y$ is the subset $\{(x, y) : x \in X, y = f(x)\}$ of $X \times Y$. The graph of the above set valued function is depicted in the following figure:

![Graph of set valued function](figure1.png)

Note that this set-valued function does not have any fixed point. We describe next another set-valued function from $[0, 1]$ to $[0, 1]$. 

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The graph of this set-valued function is depicted in the following figure:

\[ f(x) = \begin{cases} 
\{x + 1/2\} & \text{if } x < 1/2 \\
[0, 1] & \text{if } x = 1/2 \\
\{x - 1/2\} & \text{if } x > 1/2 
\end{cases} \]

5 Kaukutani’s Fixed point Theorem

**Definition 6** A set-valued function \( f \) from \( X \) to \( Y \) is called uppersemicontinuous if \( x_n \to x \in X \), \( y_n \to y \in Y \), and \( y_n \in f(x_n) \) for all \( n \) imply \( y \in f(x) \).

The graph of a set-valued function \( f \) from \( C \) to \( D \) is the subset \( \{(x, y) : y \in f(x)\} \) of \( C \times D \), and is denoted \( \Gamma_f \).
The function $f$ is uppersemicontinuous if and only if $\Gamma_f$ is a closed subset of $C \times D$.

Indeed, let $f$ be a set-valued function from $C$ to $D$ and assume that $(x_n, y_n) \in \Gamma_f \subset C \times D$ is a sequence that converges to $(x, y) \in C \times D$. If $f$ is uppersemicontinuous then $y \in f(x)$ and therefore $\Gamma_f$ is closed in $C \times D$, and if $\Gamma_f$ is closed in $C \times D$ then $(x, y) \in \Gamma_f$, i.e., $y \in f(x)$ and thus $f$ is uppersemicontinuous.

If $f$ is an uppersemicontinuous set-valued function from $C$ to $D$, then, for any $x \in C$, $f(x)$ is a closed subset of $D$.

If $f$ is a function from $C$ to $D$, we may identify the function $f$ with the set valued function which maps a point $x \in C$ to the subset $\{f(x)\}$ of $D$. Note that if $f$ is continuous then the associated set-valued function is uppersemicontinuous, and if $D$ is compact and the associated set-valued function is uppersemicontinuous then $f$ is continuous.

Given a subset $A$ of $\mathbb{R}^k$ and a point $x \in \mathbb{R}^k$ we define the distance of $x$ to $A$, $d(x, A)$, as $\inf\{|x - y| : y \in A\}$.

**Claim 1:** Let $f$ be an uppersemicontinuous set-valued function from a
subset $C$ of a Euclidean space into a compact subset $D$ of a Euclidean space. Then, for every $x \in C$ and $\varepsilon > 0$ there is $\delta = \delta(x, \varepsilon) > 0$ such that if $x, x' \in C$ with $||x - x'|| < \delta$ and $z \in f(x')$, then $d(f(x), z) < \varepsilon$.

**Proof.** Otherwise, there are $x \in C$, $\varepsilon > 0$, and two sequences $(x_n)_{n=1}^{\infty}$, and $(z_n)_{n=1}^{\infty}$ such that $||x_n - x|| \to 0$ as $n \to \infty$, $z_n \in f(x_n)$ and $d(z_n, f(x)) > \varepsilon$. W.l.o.g. we assume that $z_n \to z \in D$ as $n \to \infty$ (otherwise we take a subsequence). As $f$ is uppersemicontinuous, $z \in f(x)$. Therefore, $d(z_n, f(x)) \leq ||z_n - z|| \to 0$ as $n \to \infty$. A contradiction.

**Claim 2:** Let $A$ be a convex subset of $\mathbb{R}^m$, $y_1, \ldots, y_m \in \mathbb{R}^m$ and $\alpha_i \geq 0$ with $\sum_{i=1}^{m} \alpha_i = 1$. Then

$$d(\sum_{i=1}^{m} \alpha_i y_i, A) \leq \max_{i=1}^{m} d(y_i, A).$$

**Proof.** Let $\varepsilon > 0$. For every $1 \leq i \leq m$ there is $x_i \in A$ with $||x_i - y_i|| < d(y_i, A) + \varepsilon$. Note that $A$ is convex and therefore $\sum_{i=1}^{m} \alpha_i x_i \in A$ thus $d(\sum_{i=1}^{m} \alpha_i y_i, A) \leq ||\sum_{i=1}^{m} \alpha_i y_i - \sum_{i=1}^{m} \alpha_i x_i||$ which by the triangle inequality is at most $\sum_{i=1}^{m} \alpha_i ||y_i - x_i||$ which is bounded by $\max_{i=1}^{m} ||y_i - x_i|| \leq \max_{i=1}^{m} d(y_i, A) + \varepsilon$.

We are ready now to state Kakutani’s fixed point theorem which is a
generalization of Brouwer’s fixed point theorem.

**Theorem 4 (Kakutani’s fixed point theorem)** Let $f$ be an uppersemi-continuous set-valued function from a nonempty convex compact subset $C$ of $\mathbb{R}^n$ into itself (i.e., $f(x) \subset C$ for all $x \in C$) such that for all $x \in C$ $f(x)$ is a nonempty convex subset of $C$. Then $f$ has a fixed point, i.e., there is a point $x \in C$ with $x \in f(x)$.

**Proof** We prove Kakutani’s fixed point theorem from Brouwer’s fixed point theorem. The idea of the proof is as follows: we construct a sequence of continuous functions $g_n : C \rightarrow C$ such that any limit point of a sequence of points $(x_n, y_n) \in \Gamma_{g_n}$ is in the graph of $f$. By Brouwer’s fixed point theorem it follows that $g_n$ has a fixed point $x_n$, i.e., a point $x_n \in C$ with $x_n = g_n(x_n)$. Let $(x, y) \in C \times C$ be a limit point of $(x_n, g_n(x_n)) = (x_n, x_n)$. As the ‘diagonal,’ $\{(x, x) : x \in C\}$, is closed in $C \times C$, $y = x$ and thus $x$ is a fixed point of $f$. (Alternatively, w.l.o.g. we assume that $x_n \rightarrow x \in C$ as $n \rightarrow \infty$, (otherwise we replace the sequence $(n)_{n=1}^{\infty}$ with a subsequence $(n_k)_{k=1}^{\infty}$), and therefore $(x_n, x_n) \rightarrow (x, x) \in \Gamma_f$ i.e., $x$ is a fixed point of the set-valued function $f$.)

Construction of the functions $g_n$, $n = 1, \ldots$
Fix $n > 0$ and for every $x \in C$ let $B(x, 1/n)$ denote the open ball of radius $1/n$ and center $x$, i.e., $B(x, 1/n) = \{ y \in C : ||y - x|| < 1/n \}$. Recall that $C$ is compact, and observe that $C = \bigcup_{x \in C} B(x, 1/n)$. Therefore there is a finite subset of these balls that covers $C$, i.e., for any $n$ there is a list $x_1^n, \ldots, x_m^n$ $(m = m(n))$ of points in $C$ such that $C = \bigcup_{i=1}^m B(x_i^n, 1/n)$. Define the real valued functions $\delta_i^n : C \to \mathbb{R}$ by

$$\delta_i^n(y) = \max(0, 1/n - ||y - x_i||) .$$

Note that $\delta_i^n(y) > 0$ if $||x_i - y|| < 1/n$ and $\delta_i^n(y) = 0$ if $||y - x_i|| \geq 1/n$, and that $\sum_{j=1}^m \delta_j^n(x) > 0$ for every $x \in C$.

Therefore, for every $i = 1, \ldots, m$, $\frac{\delta_i^n(x)}{\sum_{j=1}^m \delta_j^n(x)}$ is a continuous positive (real valued) function defined on $C$, and

$$\frac{\sum_{i=1}^m \delta_i^n(x)}{\sum_{j=1}^m \delta_j^n(x)} = 1.$$ 

For any $i = 1, \ldots, m$ let $y_i^n \in f(x_i^n)$ and define $g_n : C \to C$ by

$$g_n(x) = \frac{\sum_{i=1}^m \delta_i^n(x)}{\sum_{j=1}^m \delta_j^n(x)} y_i^n .$$

It follows that $g_n$ is a continuous function from $C$ to $C$. Therefore, by Brouwer’s fixed-point theorem, $g_n$ has a fixed point $\bar{x}_n$, i.e., $g_n(\bar{x}_n) = \bar{x}_n$. Let $x$ be a limit point of the sequence $(\bar{x}_n)_{n=1}^\infty$.

We will show that $d(x, f(x)) = 0$. As $f(x)$ is closed, it will follow that $x \in f(x)$ i.e., $x$ is a fixed point of $f$. 

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Fix $\varepsilon > 0$ and let $\delta = \delta(x, \varepsilon)$ be defined as in claim 1. As $x$ is a limit point of the sequence $(\bar{x}_n)_{n=1}^\infty$, there is $n > 2/\delta$ with $\|\bar{x}_n - x\| < \delta/2$. For every $i$ with $\|x^n_i - \bar{x}_n\| \geq \delta/2 > 1/n$, $\delta_i^n(\bar{x}_n) = 0$, and therefore $g_n(\bar{x}_n)$ is a convex combination of finitely many points $y_j$ with $y_j \in f(x^n_j)$ and $\|x^n_j - x\| \leq \|x^n_j - \bar{x}_n\| + \|\bar{x}_n - x\| < \delta$, implying that $d(y_j, f(x)) < \varepsilon$. Thus

$$\bar{x}_n = g_n(\bar{x}_n) = \sum_{j: \|x^n_j - x\| < \delta} \frac{\delta_j^n(\bar{x}_n)}{\sum_j \delta_j^n(\bar{x}_n)} y^n_j$$

with $d(y_j, f(x)) < \varepsilon$ and thus by claim 2,

$$d(\bar{x}_n, f(x)) \leq \sum_{j: \|x^n_j - x\| < \delta} \frac{\delta_j^n(\bar{x}_n)}{\sum_j \delta_j^n(\bar{x}_n)} d(y^n_j, f(x)) \leq \varepsilon,$$

implying that

$$d(x, f(x)) \leq \|x - \bar{x}_n\| + d(\bar{x}_n, f(x)) < 2\varepsilon.$$

As this last inequality holds for every $\varepsilon > 0$, $d(x, f(x)) = 0$, and as $f(x)$ is closed, $x \in f(x)$. 

6 Applications of Kaukutani’s f.p. theorem

We start with few remarks:

(i) If $F_i$ is an upper semicontinuous set-valued function from $X$ to $Y_i$, $i = 1, \ldots, n$, then $(F_1, \ldots, F_n)$ is an upper semicontinuous set-valued function from
$X$ to $Y_1 \times \ldots \times Y_n$.

(ii) If $Y_i$ is a convex subset of a (real) vector space and $F_i$ is a convex valued set-valued function from $X$ to $Y_i$, $i = 1, \ldots, n$, then $(F_1, \ldots, F_n)$ is a convex valued set-valued function from $X$ to $Y_1 \times \ldots \times Y_n$.

(iii) If $g : X \to Y$ is a continuous function, and $F$ is an upper semicontinuous set-valued function from $Y$ to $Z$, then $F \circ g$ is an upper semicontinuous set-valued function from $X$ to $Z$.

(iv) If $F_i$ is an upper semicontinuous set-valued function from $Y_i$ to $X_i$, $i = 1, \ldots, n$, and $\pi_i$ is a continuous function from $X$ to $X_i$, then $(F_1 \circ \pi_1, \ldots, F_n \circ \pi_n)$ is an upper semicontinuous set-valued function from $X$ to $X_1 \times \ldots, X_n$.

(v) If $F_i$ is a convex valued set-valued function from $Y_i$ to $X_i$, where $X_i$ is a convex subset of a real vector space, $i = 1, \ldots, n$, and $\pi_i$ is a continuous function from $X$ to $Y_i$, then $(F_1 \circ \pi_1, \ldots, F_n \circ \pi_n)$ is a convex valued set-valued function from $X$ to $X_1 \times \ldots, X_n$.

**Theorem 5** Let $X_1, \ldots, X_n$ be compact convex subsets of Euclidean spaces, and assume that $f_i$ are nonempty convex valued and upper semicontinuous set-valued functions from $\times_{i=1}^n X_i \to X_i$. The $(f_1, \ldots, f_n)$ has a fixed point.
Proof. By remarks (i) and (ii), $F = (f_1, \ldots, f_n)$ is a convex valued uppersemicontinuous set-valued function. As each $f_i$ in nonempty valued, $F$ is nonempty valued. As each $X_i$ is convex and compact, so is $\times_{i=1}^n X_i$. Therefore the set-valued function $F$ from $\times_{i=1}^n X_i$ to itself obey all the assumptions of Kakutani’s fixed point theorem, and thus has a fixed point.

Lemma 1 Let $X$ and $Y$ be compact subsets of Euclidean spaces, and let $u : X \times Y \to \mathbb{R}$ be a continuous function. Then the set valued function $F$ from $X$ to $Y$ defined by

$$F(x) = \{y \in Y : u(x, y) = \max_{z \in Y} u(x, z)\},$$

is uppersemicontinuous.

Proof. Assume that $x_n \in X$ and $y_n \in Y$, $n = 1, \ldots$ are sequences that converge to $x \in X$ and $y \in Y$ respectively, and that $y_n \in F(x_n)$. Let $z \in Y$. As $y_n \in F(x_n)$,

$$u(x_n, y_n) \geq u(x_n, z).$$

As $u$ is continuous, $u(x_n, y_n) \to u(x, y)$ and $u(x_n, z) \to u(x, z)$ as $n \to \infty$, and therefore

$$u(x, y) = \lim_{n \to \infty} u(x_n, y_n) \geq u(x, z) = \lim_{n \to \infty} u(x_n, z),$$

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and as this holds for any \( z \in Y, \ y \in F(x) \), and thus \( F \) is uppersemicontinuous.

**Definition 7** Let \( X \) be a convex subset of a Euclidean space. A real valued function \( f : X \to \mathbb{R} \) is called quasi concave if for every \( \alpha \in \mathbb{R} \) the set \( \{ x \in X : f(x) \geq \alpha \} \) is convex. If \( Y \) is a set, a real valued function \( f : X \times Y \to \mathbb{R} \) is called quasi concave in \( x \) if for every \( \alpha \in \mathbb{R} \) and every \( y \in Y \), the set \( \{ x \in X : f(x, y) \geq \alpha \} \) is convex.

**Theorem 6** Let \( X_1, \ldots, X_n \) be compact convex subsets of Euclidean spaces, and assume that \( u_1, \ldots, u_n \) are real valued quasi concave continuous function defined on \( \times_{i=1}^n X_i \). Then there is a point \( x = (x_1, \ldots, x_n) \in \times_{i=1}^n X_i \) such that for every \( 1 \leq i \leq n \) and every \( y_i \in X_i \),

\[
  u_i(x_1, \ldots, y_i, \ldots, x_n) \leq u_i(x_1, \ldots, x_i, \ldots, x_n) = u_i(x).
\]

**Proof.** For every \( i = 1, \ldots, n \) define the set valued function \( F_i \) from \( \times_{i=1}^n X_i \) to \( X_i \) by

\[
  F_i(x_1, \ldots, x_n) = \{ y_i \in X_i : u_i(x_1, \ldots, y_i, \ldots, x_n) = \max_{z_i \in X_i} u_i(x_1, \ldots, z_i, \ldots, x_n) \}.
\]

Then, as \( X_1, \ldots, X_n \) are compact and \( u_i \) is continuous, \( F_i \) is non empty valued and uppersemicontinuous (Lemma 1). As \( X_1, \ldots, X_n \) are convex and \( u_i \) is
quasi concave, \( F_i \) is convex valued. By remarks (i) and (ii) \( F = (F_1, \ldots, F_n) \) is a convex valued and uppersemicontinuous, and as each \( F_i \) is nonempty valued \( F \) is also non empty valued. Therefore, by Theorem 4 \( F \) has a fixed point, i.e., there is a point \( x = (x_1, \ldots, x_n) \in \times_{i=1}^{n} X_i \) such that for every \( i \), \( x_i \in F_i(x) \), i.e., for every \( y_i \in X_i \),

\[
u_i(x_1, \ldots, y_i, \ldots, x_n) \leq \nu_i(x_1, \ldots, x_i, \ldots, x_n) = \nu_i(x).
\]

\[
\text{Definition 8} \quad \text{A set valued function} \ f \ \text{from} \ X \ \text{to} \ Y \ \text{is called lowersemicontiuous if for every sequence} \ x_n \ \text{which converges to a point} \ x \ \in \ X \ \text{and every} \ y \ \in \ f(x), \ \text{there is a sequence} \ y_n \ \in \ f(x_n) \ \text{which converges to} \ y. \ \text{It is called continuous if it is uppersemicontinuous and lowersemicontinuous.}
\

\[
\text{Lemma 2} \quad \text{Let} \ X \ \text{and} \ Y \ \text{be compact subsets of Euclidean spaces, and let} \ u : X \times Y \rightarrow \mathbb{R} \ \text{be a continuous function, and} \ G \ \text{a continuous set-valued function from} \ X \ \text{to} \ Y. \ \text{Then the set valued function} \ F \ \text{from} \ X \ \text{to} \ Y \ \text{defined by}
\]

\[
F(x) = \{ y \in G(x) : u(x, y) = \max_{z \in G(x)} u(x, z) \},
\]

\text{is uppersemicontinuous.}
Proof. Assume that $x_n \in X$ and $y_n \in F(x_n) \subset Y$, $n = 1, \ldots$, are sequences that converge to $x \in X$ and $y \in Y$ respectively. Recall that $F(x_n) \subset G(x_n)$ and $G$ is uppersemicontinuous, and therefore $y \in G(x)$. Let $z \in G(x)$. As $G$ is a continuous set valued function, there is a sequence $z_n \in G(x_n)$ that converges to $z$. As $y_n \in F(x_n)$, $u(x_n, y_n) \geq u(x_n, z_n)$. As $u$ is continuous, $u(x_n, y_n) \to u(x, y)$ and $u(x_n, z_n) \to u(x, z)$ as $n \to \infty$, and therefore $u(x, y) \geq u(x, z)$, and as this holds for any $z \in G(x)$, $y \in F(x)$, and thus $F$ is uppersemicontinuous.

Given sets $X_1, \ldots, X_n$, an index $1 \leq i \leq n$, an element $x = (x_1, \ldots, x_n)$ in $\times_{i=1}^n X_i$, and an element $y_i$ in $X_i$, we denote by $(x_{-i}, y_i)$ or by $(x|y_i)$ the element $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ in $\times_{i=1}^n X_i$.

Theorem 7 Let $X_1, \ldots, X_n$ be non-empty convex compact subsets of Euclidean spaces, $X = \times_{i=1}^n X_i$, and assume that $\varphi_1, \ldots, \varphi_n$ are continuous set-valued functions from $X$ to $X_1, \ldots, X_n$ respectively such that $\varphi_i(x)$ is non-empty and convex for every $x \in X$ and $1 \leq i \leq n$. If $u_i : X \to \mathbb{R}$ are continuous and quasi concave in $x_i$, and $F_i$ is the set-valued function from $X$ to $X_i$ defined by

$$F_i(x) = \{y_i \in \varphi_i(x) : u_i(x_{-i}, y_i) \geq u_i(x_{-i}, z_i) \text{ for every } z_i \in \varphi_i(x)\},$$
then, the set-valued function \((F_1, \ldots, F_n)\) from \(X\) to \(X\) has a fixed point.

**Proof.** Note that \(F_i\) is an uppersemicontinuous set-valued function from \(X\) to \(X_i\) with non-empty convex values. Therefore, \((F_1, \ldots, F_n)\) is an uppersemicontinuous set-valued function with non-empty convex values and thus by Kakutani’s fixed point theorem has a fixed point. \(\blacksquare\)

**Lemma 3** Let \(\Delta = \{p = (p_1, \ldots, p_n) : p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i = 1\}, e \in \mathbb{R}^n_+\) with \(e_i > 0\), and let \(Y\) be a convex subset of \(\mathbb{R}^n\) with \(0 \in Y\). Then the set-valued functions \(\varphi\) from \(\Delta\) to \(Y\) that is given by

\[
\varphi(p) = \{x \in Y : px = \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} p_i e_i = pe\}
\]

is continuous and has non-empty convex values.

**Proof.** Assume that \(p^k \to p \in \Delta\) and \(x^k \to x \in Y\) as \(k \to \infty\) and that \(p^k x^k = \sum_{i=1}^{n} p_i^k x_i^k \leq \sum_{i=1}^{n} p_i^k e_i = p^k e\). Then it follows that \(\lim_{k \to \infty} \sum_{i=1}^{n} p_i^k x_i^k = \sum_{i=1}^{n} p_i x_i\) and \(\lim_{k \to \infty} \sum_{i=1}^{n} p_i^k e_i = \sum_{i=1}^{n} p_i e_i\) and therefore

\[
px = \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} p_i e_i = pe, \text{ i.e., } x \in \varphi(p),
\]

proving that \(\varphi\) is uppersemecontinuous. Assume next that \(p^k \to p \in \Delta\) and \(x \in \varphi(p)\). If \(\sum_{i=1}^{n} p_i x_i = px < pe = \sum_{i=1}^{n} p_i e_i\), then for sufficiently large \(k\), \(\sum_{i=1}^{n} p_i^k x_i = p^k x \leq p^k e = \sum_{i=1}^{n} p_i^k e_i\), i.e., \(x \in \varphi(p^k)\), and therefore
by setting \( x^k = x \) we deduce that \( x^k \in \varphi(p^k) \) and \( x^k \to_{k \to \infty} x \). Otherwise, if \( px = pe, p^kx \to_{k \to \infty} pe > 0 \). Therefore, if \( \varepsilon_k \) is defined by

\[
\varepsilon_k = \max(0, (\sum_{i=1}^n p_i^k x_i - \sum_{i=1}^n p_i^k e_i)) / \sum_{i=1}^n p_i^k x_i, 0 \leq \varepsilon_k \to_{k \to \infty} 0.
\]

As \( Y \) is convex and contains the points \( x \) and 0, \( x^k = (1 - \varepsilon_k)x \in Y \). As \( \varepsilon_k \to_{k \to \infty} 0 \), \( x^k \to_{k \to \infty} x \) and by the definition of \( \varepsilon_k \), \( x^k p^k \leq p^k e \) and thus \( x^k \in \varphi(p^k) \).

Thus \( \varphi \) is lowersemicontinuous.

\[\text{Lemma 4}\]

Let \( \Delta = \{p = (p_1, \ldots, p_n) : p_i \geq 0 \text{ and } \sum_{i=1}^n p_i = 1\}, e \in \mathbb{R}_+^n \) with \( e_i > 0 \), and let \( Y \) be a convex subset of \( \mathbb{R}^n \) that contains a neighborhood of \( e \) and with \( 0 \in Y \). Then the set-valued functions \( \varphi \) from \( \Delta \) to \( Y \) that is given by

\[
\varphi(p) = \{x \in Y : px = \sum_{i=1}^n p_i x_i = \sum_{i=1}^n p_i e_i = pe\}
\]

is continuous and has non-empty convex values.

\[\text{Proof.}\] Assume that \( p^k \to p \in \Delta \) and \( x^k \to x \in Y \) as \( k \to \infty \) and that \( p^k x^k = \sum_{i=1}^n p_i^k x_i^k = \sum_{i=1}^n p_i^k e_i = p^k e \). Then it follows that \( \lim_{k \to \infty} \sum_{i=1}^n p_i^k x_i^k = \sum_{i=1}^n p_i x_i \) and \( \lim_{k \to \infty} \sum_{i=1}^n p_i^k e_i = \sum_{i=1}^n p_i e_i \) and therefore

\[
px = \sum_{i=1}^n p_i x_i = \sum_{i=1}^n p_i e_i = pe, \text{ i.e., } x \in \varphi(p),
\]

proving that \( \varphi \) is uppersemicontinuous. Assume next that \( p^k \to_{k \to \infty} p \in \Delta \) and \( x \in \varphi(p) \). Then, \( p^k x \to_{k \to \infty} px = pe > 0 \). \( Y \) contains a neighborhood
of e. Therefore there is $\alpha > 0$ such that $(1 + \alpha)e \in Y$. Let $\theta_k$ be defined as $\inf\{\theta \geq 0 \mid p^k(\theta(1 + \alpha)e + (1 - \theta)x^k) \geq p^ke\}$. For every $\theta > 0$ we have $p^k(\theta(1 + \alpha)e + (1 - \theta)x) \to_{k \to \infty} pe(1 + \alpha) > pe$. Therefore, $\theta_k \to 0$ as $k \to \infty$.

Set $y^k = \theta_k(1 + \alpha)e + (1 - \theta_k)x$. (In other words, $y^k$ is the point on the interval $[x, (1 + \alpha)e]$ with $p^k y^k \geq p^ke$ that is the closest point to $x$.) Note that $y^k \to_{k \to \infty} x$. Let $\varepsilon_k$ be defined by $\varepsilon_k = (\sum_{i=1}^n p_i^k y_i^k - \sum_{i=1}^n p_i^k e_i) / \sum_{i=1}^n p_i^k y_i^k$, $0 \leq \varepsilon_k \to_{k \to \infty} 0$. As $Y$ is convex and contains the points $x$ and $0$, $x^k = (1 - \varepsilon_k)y^k \in Y$. As $\varepsilon_k \to_{k \to \infty} 0$ and $y^k \to_{k \to \infty} x$, $x^k \to_{k \to \infty} x$ and by the definition of $\varepsilon_k$, $x^k p^k \leq p^ke$ and thus $x^k \in \varphi(p^k)$. Thus $\varphi$ is lowersemicontinuous.
Theorem 8 Let $\mathcal{E} = (T; \mathbb{R}^\ell; (X(t))_{t \in T}; (e(t))_{t \in T}; \succ_{t \in T})$ be an exchange economy such that:

1. $\forall t \in T$, $X(t) = [0, M]^\ell$ where $M > \sum_{t \in T} e_i(t)$
2. $\forall t \in T$ and $\forall 1 \leq i \leq \ell$, $e_i(t) > 0$
3. $\forall t \in T$, the preference relation of trader $t$, $\succ_t$, is represented by a continuous and monotonic quasi concave utility function $u^t : \mathbb{R}^\ell \to \mathbb{R}$.

Then, there exists a competitive equilibrium $(p, x) = (p, x(1), \ldots, x(n))$ with $p_i > 0$ for every $1 \leq i \leq \ell$.

Proof. Assume that $T = \{1, \ldots, n\}$. Set $X_0 = \Delta = \{p \in \mathbb{R}^\ell_+ | \sum_{i=1}^\ell p_i = 1\}$, and for $t \in T$, set $X_t = X(t)$. The sets $X_i$, $i = 0, 1, \ldots, n$, are non-empty convex and compact subsets of Euclidean spaces. For every $0 \leq t \leq n$ we define a set-valued function $\varphi_t$ from $X = \times_{i=0}^n X_i$ to $X_t$ as follows:

$$\varphi_0(p, x(1), \ldots, x(n)) = \Delta,$$

and for $1 \leq t \leq n$,

$$\varphi_t(p, x(1), \ldots, x(n)) = \{y \in X_t | py \leq pe(t)\}.$$

By Lemma 3 the set valued functions $\varphi_t$, $1 \leq t \leq n$, are continuous and with non-empty and convex values. Obviously, the set-valued function $\varphi_0$...
is continuous and with non-empty and convex values. Define the set-valued functions $F_i : X \to X_i$, $i = 0, 1, \ldots, n$, by:

\[
F_0(p, x(1), \ldots, x(n)) = \{ q \in \Delta \mid q \sum_{t=1}^{n}(x(t) - e(t)) \geq r \sum_{t=1}^{n}(x(t) - e(t)) \forall r \in \Delta \}
= \operatorname{arg \max}_{q \in \Delta} q \sum_{t=1}^{n}(x(t) - e(t)),
\]

and for $1 \leq i \leq n$,

\[
F_i(p, x(1), \ldots, x(n)) = \{ y \in \varphi_i(p, x(1), \ldots, x(n)) \mid u_i(y) \geq u_i(z) \forall z \in \varphi_i(p, x(1), \ldots, x(n)) \}.\]

Note that by Theorem 7 the set-valued function $F = (F_0, \ldots, F_n)$ from $X$ to $X$ has a fixed point. Let $(p, x(1), \ldots, x(n))$ be a fixed point of $F$. We will show that it is a competitive equilibrium. As $x(i) \in F_i(p, x(1), \ldots, x(n)) \subseteq \varphi_i(p, x)$,

\[
px(i) \leq pe(i) \text{ for every } 1 \leq i \leq n, \tag{1}
\]

and therefore

\[
p\left(\sum_{i=1}^{n}(x(i) - e(i))\right) \leq 0. \tag{2}
\]

Let $f(i)$ denote the $i$-th unit vector in $\mathbb{R}^{\ell}$. As $p \in F_0(p, x(1), \ldots, x(n))$, we deduce that $p(\sum_{i=1}^{n}(x(i) - e(i))) \geq f(j)(\sum_{i=1}^{n}(x(i) - e(i))) = \sum_{i=1}^{n}(x_j(i) - e_j(i))$ for every $1 \leq j \leq \ell$. Therefore, using (5), for every $1 \leq j \leq \ell$,

\[
\sum_{i=1}^{n}(x_j(i) - e_j(i)) \leq 0. \text{ In particular, if } \varepsilon > 0 \text{ with } \varepsilon < M - \sum_{t=1}^{n}e_j(t), \text{ then}
\]

By the monotonicity of \( u^t \) we deduce that \( u^t(x(t) + \varepsilon f(j)) > u^t(x(t)) \). Therefore, as \( x(t) \in F_t(p, x) \), we deduce that

\[
px(t) + \varepsilon p_j = p(x(t) + \varepsilon f(j)) > pe(t) \geq px(t)
\]

and thus we deduce that \( p_j > 0 \), and as (6) holds for all sufficiently small \( \varepsilon > 0 \), we deduce also that \( px(t) \geq pe(t) \) for every \( t \in T \) which together with (4) implies that for every \( t \in T \),

\[
px(t) = pe(t).
\]

Therefore, using \( p > 0 \) and the inequality \( \sum_{t \in T} x(t) \leq \sum_{t \in T} e(t) \), we deduce that \( \sum_{t \in T} x(t) = \sum_{t \in T} e(t) \). As \( x(i) \in F_i(p, x(1), \ldots, x(n)) \) we conclude that \( (p, x(1), \ldots, x(n)) \) is a competitive equilibrium.

The assumption that for every \( t \in T \) and every \( 1 \leq i \leq \ell \) we have \( e_i(t) > 0 \) is restrictive. It assumes that every trader has a strictly positive quantity of every commodity. The next Theorem replaces this assumption by the assumption that every commodity is present in the market, namely, that for every \( 1 \leq i \leq \ell \) we have \( \sum_{t \in T} e_i(t) > 0 \).

**Theorem 9** Let \( \mathcal{E} = \langle T; \mathcal{R}^\ell; (X(t))_{t \in T}; (e(t))_{t \in T}; \succ_{t \in T} \rangle \) be an exchange economy such that:

1. \( \forall t \in T, \ X(t) = [0, M]^\ell \) where \( M > \sum_{t \in T} e_i(t) \)
2. $\forall 1 \leq j \leq \ell, \sum_{t \in T} e_j(t) > 0$

3. $\forall t \in T$, the preference relation of trader $t$, $\succ_t$, is represented by a continuous and monotonic quasi concave utility function $u^t : \mathbb{R}^\ell \to \mathbb{R}$.

Then, there exists a competitive equilibrium $(p, x) = (p, x(1), \ldots, x(n))$ with $p_j > 0$ for every $1 \leq j \leq \ell$.

Proof. Assume that $T = \{1, \ldots, n\}$. Set $X_0 = \Delta = \{p \in \mathbb{R}_+^\ell \mid \sum_{i=1}^\ell p_i = 1\}$, and for $t \in T$, set $X_t = [0, M]^\ell$. The sets $X_i$, $i = 0, 1, \ldots, n$, are non-empty convex and compact subsets of Euclidean spaces. Set $X = \times_{0 \leq i \leq n} X_i$, and $B_i(p) = \{y \in [0, M]^\ell : py \leq pe(t)\}$.

Define the set-valued functions $F_i : X \to X_i$, $i = 0, 1, \ldots, n$, by:

$$F_0(p, x(1), \ldots, x(n)) = \{q \in \Delta \mid q \sum_{i=1}^n (x(t) - e(t)) \geq r \sum_{i=1}^n (x(t) - e(t)) \forall r \in \Delta\}$$

$$= \arg \max_{q \in \Delta} q \sum_{i=1}^n (x(t) - e(t)),$$

and for $1 \leq i \leq n$, if $pe(i) > 0$ then

$$F_i(p, x(1), \ldots, x(n)) = \{y \in B_i(p) \mid u_i(y) \geq u_i(z) \forall z \in B_i(p)\}$$

and

$$F_i(p, x(1), \ldots, x(n)) = B_i(p)$$
if \( pe(i) = 0 \).

The quasi concavity of \( u_t \) implies that \( F_t \) is convex valued for \( 1 \leq t \leq n \). Next we show that \( F_t \) are upper semi continuous with nonempty values. Assume \((p^k, x^k(1), \ldots, x^k(n)) \rightarrow_{k \rightarrow \infty} (p, x(1), \ldots, x(n))\). We have to prove that \( p \in F_0(p, x(1), \ldots, x(n)) \) and that for every \( 1 \leq t \leq n \) we have \( x(t) \in F_t(p, x(1), \ldots, x(n)) \). Fix \( q \in \Delta \). Then \( q(\sum x(t) - \sum e(t)) = \lim_{k \rightarrow \infty} q(\sum x^k(t) - \sum e(t)) \leq \lim_{k \rightarrow \infty} p^k(\sum x^k(t) - \sum e(t)) = p(\sum x(t) - \sum e(t)) \) where the equalities follow from the continuity of the inner product and from the assumptions that \( p^k \rightarrow p \) and \( x^k(t) \rightarrow x(t) \), and the inequality follows from the fact that \( p^k \in F_0(p, x(1), \ldots, x(n)) \). Thus, \( p \in F_0(p, x(1), \ldots, x(n)) \). Next we show that \( x(t) \in F_t(p, x(1), \ldots, x(n)) \). Note that \( px(t) = \lim p^k x^k(t) \leq \lim p^k e(t) = pe(t) \) and thus \( x(t) \in B_t(p) \) and thus \( pe(t) = 0 \) implies that \( x(t) \in F_t(p, \ldots) \). If \( pe(t) > 0 \), set \( y^k = y \) if \( p^k y \leq p^k e(t) \) and \( y^k = \frac{p^k e(t)}{p^k y} y \) if \( p^k y > p^k e(t) \). It follows that \( p^k y^k \leq p^k e(t) \) and \( y^k \rightarrow y \). Therefore, for sufficiently large \( k \) we have \( u_t(y^k) \geq u_t(x^k(t)) \) and \( p^k y^k \leq p^k e(t) \) contradicting the assumption that \( x^k(t) \in F_t(p, \ldots) \). Thus \( F_t \) is upper semi continuous.

Note that by Theorem 7 the set-valued function \( F = (F_0, \ldots, F_n) \) from \( X \) to \( X \) has a fixed point. Let \((p, x(1), \ldots, x(n))\) be a fixed point of \( F \). We will
show that it is a competitive equilibrium. As $x(i) \in F_i(p, x(1), \ldots, x(n)) \subset B_i(p)$,

$$px(i) \leq pe(i) \quad \text{for every } 1 \leq i \leq n, \quad (4)$$

and therefore

$$p\left(\sum_{i=1}^{n}(x(i) - e(i))\right) \leq 0. \quad (5)$$

Let $f(i)$ denote the $i$-th unit vector in $\mathbb{R}^\ell$. As $p \in F_0(p, x(1), \ldots, x(n))$, we deduce that $p(\sum_{i=1}^{n}(x(i) - e(i))) \geq f(j)(\sum_{i=1}^{n}(x(i) - e(i))) = \sum_{i=1}^{n}(x_j(i) - e_j(i))$ for every $1 \leq j \leq \ell$. Therefore, using (5), for every $1 \leq j \leq \ell$, $\sum_{i=1}^{n}(x_j(i) - e_j(i)) \leq 0$. In particular, if $\varepsilon > 0$ with $\varepsilon < M - \sum_{t=1}^{n}e_j(t)$, then $x(t) + \varepsilon f(j) \in X_t$.

Assume that $1 \leq t \leq n$ with $pe(t) > 0$. By the monotonicity of $u^t$ we deduce that $u^t(x(t) + \varepsilon f(j)) > u^t(x(t))$. Therefore, as $x(t) \in F_t(p, x)$, we deduce that

$$px(t) + \varepsilon p_j = p(x(t) + \varepsilon f(j)) > pe(t) > 0 \quad (6)$$

and thus we deduce that for every $1 \leq j \leq \ell$ we have $p_j > 0$. Therefore (w.l.o.g.) $pe(t) > 0$ for every $t$. Thus (6) holds for all $\varepsilon > 0$. 

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As (6) holds for all sufficiently small \( \varepsilon > 0 \), we deduce also that \( px(t) \geq pe(t) \) for every \( t \in T \) which together with (4) implies that for every \( t \in T \),

\[
px(t) = pe(t).
\]

Therefore, using \( p > 0 \) and the inequality \( \sum_{t \in T} x(t) \leq \sum_{t \in T} e(t) \), we deduce that \( \sum_{t \in T} x(t) = \sum_{t \in T} e(t) \). As \( x(i) \in F_i(p, x(1), \ldots, x(n)) \) we conclude that \( (p, x(1), \ldots, x(n)) \) is a competitive equilibrium of the exchange economy \( E = \langle T; I R \ell; (X(t))_{t \in T}; (e(t))_{t \in T}; \succ_i t \rangle \).

The next theorem replaces the assumption that \( X(t) = [0, M] \ell \) where \( M > \sum_{t \in T} e_i(t) \) with the assumption that \( \forall t \in T, X(t) \) is convex and \( I R \ell_+ \supset X(t) \supset [0, M] \ell \) where \( M > \sum_{t \in T} e_i(t) \). We state and prove the result for the the special case \( X(t) = I R_\ell \). However, the same proof applies to the general case.

**Theorem 10** Let \( E = \langle T; (X(t))_{t \in T}; (e(t))_{t \in T}; (\succ_i)_{t \in T} \rangle \) be an exchange economy with \( \ell \) commodities such that:

1. \( \forall t \in T, X(t) = I R_\ell \)
2. \( \sum_{t \in T} e_i(t) > 0 \) for every \( 1 \leq i \leq \ell \)
3. \( \forall t \in T, \) the preference relation of trader \( t, \succ_i, \) is represented by a con-
tinuous and monotonic quasi concave utility function \( u^t : \mathbb{R}^\ell \to \mathbb{R} \). Then, there exists a competitive equilibrium \((p, x)\) with \( p_i > 0 \) for every \( 1 \leq i \leq \ell \).

**Proof.** Let \( \mathcal{E}^* \) be the economy obtained from \( \mathcal{E} \) by modifying the consumption set \( \mathbb{R}^\ell \) of each trader to the consumption set \([0, M]^{\ell}\) where \( M = 2 \max_i \sum_{t \in T} e_i(t) \). By the previous theorem, \( \mathcal{E}^* \) has a competitive equilibrium \((p, x)\) with \( p > 0 \). We claim that the competitive equilibrium \((p, x)\) of \( \mathcal{E}^* \) is a competitive equilibrium of \( \mathcal{E} \). As \( x \) is an allocation of \( \mathcal{E}^* \) it is also an allocation of \( \mathcal{E} \). It remains to show that if \( t \in T \) and \( y \in \mathbb{R}^\ell_+ \) with \( u^t(y) > u^t(x(t)) \) then \( py > pe(t) \). Indeed, if \( u^t(y) > u^t(e(t)) \), the continuity of \( u^t \) implies that there is a sufficiently small \( 1 > \varepsilon > 0 \) such that \( u^t((1 - \varepsilon)y) > u^t(x(t)) \). Set \( y^\varepsilon = (1 - \varepsilon)y \). If \( py \leq pe(t) \) then \( py^\varepsilon = (1 - \varepsilon)py < pe(t) \) and for any \( 1 \geq \delta > 0 \) the point \( y^{\varepsilon, \delta} = \delta y^\varepsilon + (1 - \delta)x(t) \) obeys \( py^{\varepsilon, \delta} < pe(t) \), and if \( \delta > 0 \) is sufficiently small, \( y^{\varepsilon, \delta}_i < M \). By the quasi concavity of \( u^t \), \( u^t(y^{\varepsilon, \delta}) \geq u^t(x(t)) \). As \( u^t \) is monotonic \( u^t(y^{\varepsilon, \delta} + \theta e_j) > u^t(y^{\varepsilon, \delta}) \geq u^t(x(t)) \) for every \( \theta > 0 \) and every \( 1 \leq j \leq \ell \). On the other hand, for sufficiently small \( \theta > 0 \), \( y^{\varepsilon, \delta} + \theta e_j \in [0, M]^\ell \) contradicting the assumption that \((p, x)\) is a competitive equilibrium of \( \mathcal{E}^* \).

Consider the following examples of exchange economies.

The set of traders is \( T = \{1, 2\} \) and the number of commodities is \( \ell = 2 \).
The utilities of the traders are \( u_1(x, y) = x + y \) and \( u_2(x, y) = \sqrt{x} + 2\sqrt{y} \).

What is the set of competitive equilibrium when \( e(1) = (1, 2) \) and \( e(2) = (2, 1) \)?

Show that when \( e(1) = (1, 0) = e(2) \) there is no competitive equilibrium.

Show that there is no competitive equilibrium in the case \( e(1) = e(2) = (1, 1) \), \( u_1(x, y) = x^2 + y^2 \), and \( u_2(x, y) = \sqrt{x} + \sqrt{y} \).

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Definition 9 An abstract social system consists of:

1. A finite set \( N \)
2. \( \forall i \in N \) a nonempty set \( A_i \); set \( A = \times_{i \in N} A_i \) and \( A_{-i} = \times_{j \neq i} A_j \)
3. \( \forall i \in N \) a set-valued function \( \varphi_i : A \rightarrow A_i \)
4. \( \forall i \in N \) a set-valued function \( P_i : A \rightarrow A_i \).

The set \( A_i \) is interpreted as the possible actions of agent \( i \), (e.g., in an exchange economy it may stand for the consumption set). The set-valued function \( \varphi_i \) represents the constrained feasible actions of agent \( i \) as a function of the actions of all agents or all the other agents, (e.g., if we associate to
an exchange economy an additional agent whose action set is the simplex of prices, the feasible actions of a trader is a consumption bundle in his budget set). The set-valued function \( P_i \) is interpreted as the wishful actions of agent \( i \) as a function of the \( N \)-tuple of actions.

**Definition 10** An equilibrium of an abstract social system is an element \( a \in A \) such that

\[
P_i(a) \cap \varphi_i(a) = \emptyset
\]

**Theorem 11** Let \( (N; (A_i)_{i \in N}; (\varphi_i)_{i \in N}; (P_i)_{i \in N}) \) be an abstract social system such that:
1. \( \forall i \in N, A_i \) is a non-empty convex compact subset of a Euclidean space.
2. \( \forall i \in N, \varphi_i \) is a continuous set-valued function with nonempty convex values.
3. \( \forall a \in A, a_i \notin \text{co}P_i(a), \) and
4. \( \forall i \in N, \) the graph of \( P_i, \)

\[
\Gamma(P_i) = \{(a, b) \in A \times A_i \mid b \in P_i(a)\}
\]

is an open subset of \( A \times A_i \). Then the abstract social system has an equilibrium.

We start with a lemma.
Lemma 5 Let $K \subset \mathbb{R}^m$. Then any point in $coK$ is a convex combination of $m + 1$ points in $K$.

proof. Assume that $x \in coK$. Then $x$ is a convex combination of finitely many points in $K$, i.e., $x$ can be written as $x = \sum_{i=1}^{k} a_i x_i$ where $x_i \in K$, $a_i > 0$ and $\sum_{i=1}^{k} a_i = 1$. Let $k$ be the smallest positive integer for which $x$ can be written as a convex combination of $k$ points from $K$. Assume that $k > n + 1$, and let $x = \sum_{i=1}^{k} a_i x_i$ where $x_i \in K$, $a_i > 0$ and $\sum_{i=1}^{k} a_i = 1$. It is sufficient to prove that any other representation of $x$ of the form, $x = \sum_{i=1}^{k} c_i x_i$ where $x_i \in K$, $c_i \geq 0$ and $\sum_{i=1}^{k} c_i = 1$, obeys $c_i > 0$ for every $1 \leq i \leq k$. Define the linear transformation $T: \mathbb{R}^k \to \mathbb{R}^{m+1}$ by

$$T(b_1, \ldots, b_k) = (\sum_{i=1}^{k} b_i x_i, \sum_{i=1}^{k} b_i).$$

The kernel of $T$, $\{b \in \mathbb{R}^k \mid T b = 0\}$, has dimension at least $k - (m + 1) \geq 1$. Therefore there is $b \in \mathbb{R}^k \setminus \{0\}$ with $\sum_{i=1}^{k} b_i x_i = 0$ and $\sum_{i=1}^{k} b_i = 0$. Set $a = (a_1, \ldots, a_k)$ and $\Delta_k = \{c \in \mathbb{R}^k \mid \sum_{i=1}^{k} c_i = 1 \text{ and } \forall i \ c_i \geq 0\}$. Set $J = \{j \mid b_j < 0\}$ and note that $J$ is not empty. For every $j \in J$ set $t_j = -a_j/b_j$. Then $t_j > 0$, $a_j + t_j b_j = 0$, and for every $0 < t < t_j$, $a_j + t b_j > a_j + t_j b_j = 0$. Therefore, if $t = \min t_j$, $a + t b \in \Delta_k$, at least one of the coordinates of $a + tb$ vanishes, and $\sum_{i=1}^{k} (a + tb_i)x_i = x$ contradicting
the minimality of $k$. (Alternatively, consider the half line starting in $a$ in the direction of $b$ and let $t = \sup\{s \geq 0 \mid a + sb \in \Delta_k\}$. Note that $\Delta_k$ is convex and bounded and therefore $t < \infty$. As $\Delta_k$ is closed $a + tb \in \Delta_k$ and as $a_i > 0$ for every $i$ it follows that $t > 0$. If $(a + tb)_i > 0$ for every $1 \leq i \leq k$ then there is $t' > t$ with $a + t'b \in \Delta_k$ contradicting the definition of $t$. Thus $\sum_{i=1}^k(a + tb)_i x_i$ is a representation of $x$ as a convex combination of less than $k$ points from $K$.)

**Lemma 6** Let $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^m$ be convex compact subsets, and assume that $F$ is an upper semicontinuous set-valued function from $X$ to $Y$. Then the set-valued function $G$ from $X$ to $Y$ defined by $G(x) = \text{co}F(x)$ is upper semicontinuous.

**Proof.** Assume that $x_n \to x \in X$, $y_n \in G(x_n)$ and $y_n \to y$. We have to prove that $y \in G(x)$. By the previous lemma, each $y_n$ is a convex combination of $m + 1$ elements of $F(x_n)$. Therefore, for every $n$ we can write $y_n = \sum_{i=1}^{m+1} a_i(n) z_i(n)$ where $a_i(n) \geq 0$, $\sum_{i=1}^{m+1} a_i(n) = 1$ and $z_i(n) \in F(x_n)$. The sequence $(a_1(n), \ldots, a_{m+1}(n), z_1(n), \ldots, z_{m+1}(n))$, $n = 1, \ldots$, is a bounded sequence in $\mathbb{R}^{m+1+(m+1)m}$, and therefore has a subsequence that converges, i.e. there is a sequence $n_k \to_{k \to \infty} \infty$ such that $\lim_{k \to \infty} a_i(n_k) = a_i$ and $\lim_{k \to \infty} z_i(n_k) = z_i$. It follows that $a_i \geq 0$ and $\sum_{i=1}^{m+1} a_i = 1$, and
\[ \lim_{k \to \infty} \sum_i a_i(n_k)z_i(n_k) = \sum_i a_i z_i. \] As \( F \) is upper semicontinuous \( z_i \in F(x) \) and therefore \( \sum_i a_i z_i \in G(x) \).

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We return now to the proof of the theorem. For every \( i \in N \) we define a continuous function \( g_i : A \times A_i \to \mathbb{R}_+ \) such that:

\[ \forall x \in P_i(a), \quad g_i(a, x) > 0 \]

\[ \forall x \notin P_i(a), \quad g_i(a, x) = 0 \]

E.g. we define \( g_i(a, x) = \text{dist}((a, x), A \times A_i \setminus \Gamma(P_i)) \). Note that the graph of \( P_i \) is open and thus its complement in \( A \times A_i \) is closed and therefore the distance of \((a, x)\) to \( A \times A_i \setminus \Gamma(P_i) \) is positive (\( > 0 \)) iff \( x \in P_i(a) \). For every \( i \in N \) define a set-valued function \( F_i \) from \( A \) to \( A_i \) by:

\[ F_i(a) = \{ x \in \varphi_i(a) \mid g_i(a, x) = \max\{g_i(a, y) \mid y \in \varphi_i(a)\}\}. \]

By lemma 2, \( F_i \) is upper semicontinuous. Next we define a set-valued function \( G_i \) from \( A \) to \( A_i \) by \( G_i(a) = \text{co}F_i(a) \). Again by a previous result \( G_i \) is upper semicontinuous. Also for every \( a \in A \), \( F_i(a) \) is nonempty and thus also \( G_i(a) \) is nonempty. Therefore letting \( G(a) = \times_{i \in N} G_i(a) \), \( G \) is an upper
semicontinuous set-valued function with non empty convex values from $A$ to itself and therefore has (by Kakutani’s fixed point theorem) a fixed point, i.e., there is $a^* \in A$ with $a^* \in G(a^*)$. We will show next that this point $a^*$ is an equilibrium of the abstract social system. Thus we have to prove that $a^*_i \in \varphi_i(a^*)$ and that $\varphi_i(a^*) \cap P_i(a^*) = \emptyset$. Indeed,

$$a^*_i \in G_i(a^*) = \text{co} F_i(a^*) \subset \text{co} \varphi_i(a^*) = \varphi_i(a^*)$$

and thus $a^*_i \in \varphi_i(a^*)$. If $\varphi_i(a^*) \cap P_i(a^*) \neq \emptyset$, there is $x \in \varphi_i(a^*) \cap P_i(a^*)$ and therefore $g_i(a^*, x) > 0$ implying that $F_i(a^*) \subset P_i(a^*)$ and therefore $G_i(a^*) \subset \text{co} P_i(a^*)$ which together with $a^*_i \in G_i(a^*)$ imply that $a^*_i \in \text{co} P_i(a^*)$ which contradicts the assumption of the theorem.

**Definition 11** An exchange economy with externalities consists of:

1. A set of traders $T$,
2. A positive integer $\ell$, representing the number of commodities.
3. For every $t \in T$ a subset $X(t) \subset \mathbb{R}^\ell$ — the consumption set of trader $t$.
4. For every $t \in T$, a element $e(t) \in X(t)$ — the initial endowment of trader $t$.
5. For every $t \in T$ a set-valued function $P_i$ from $\Delta \times \times_{t \in T} X(t)$ to $X(t)$ — the preference of trader $t$ given the vector of prices and the consumption of
each agent.

**Definition 12** A competitive equilibrium of an exchange economy with externalities, \( E = \langle T, (X(t))_{t \in T}, (e(t))_{t \in T}, (P_t)_{t \in T} \rangle \), is an ordered pair \((x^*, p^*)\) where \( p^* \in \Delta \) and \( x^* : T \to \mathbb{R}^\ell \) are such that:

1) \( \forall t \in T, \ x^*(t) \in X(t) \),
2) \( \sum_{t \in T} x^*(t) \leq \sum e(t) \),
3) \( \forall t \in T, \ p^* x^*(t) = p^* e(t) \),
4) \( \forall t \in T \ P_t(x^*, p) \cap \{ y \in X(t) \mid p^* y = p^* e(t) \} = \emptyset \).

**Theorem 12** Let \( E = \langle T, (X(t))_{t \in T}, (e(t))_{t \in T}, (P_t)_{t \in T} \rangle \) be an exchange economy with externalities, such that:

(0) \( T \) is finite,
(1) \( X(t) = \mathbb{R}_+^\ell \),
(2) \( 0 < e(t) \in \mathbb{R}_+^\ell \), with \( e_i(t) > 0 \ \forall 1 \leq i \leq \ell \),
(3) the graph of the set-valued function \( P_t : \times_{t \in T} X(t) \times \Delta \to X(t) \) is open,
(4) \( \forall ((x(t))_{t \in T}, p) \in \times_{j \in T} X(t) \times \Delta, \ x(t) \notin \text{co} P_t((x(t)), p) \).

Then there is \( p^* \in \Delta \) and \( x^* \in X = \times_{t \in T} X(t) \) such that:

(a) \( \sum x(t)^* \leq \sum e(t) \),
(b) \( \forall t \in T, \ p^* x^*(t) = p^* e(t) \),
(c) \( \forall t \in T, P_t(x^*, p) \cap \{ y \in X(t) \mid p^* y = p^* e(t) \} = \emptyset \).

**Proof.** We construct an abstract social system whose equilibrium points correspond to those of the economy \( \mathcal{E} \). Assume that \( T = \{1, \ldots, n\} \), then \( N = \{0, 1, \ldots, n\} \), where we imagine player 0 as “the price maker”. \( A_0 = \Delta \), and for \( 1 \leq i \leq n \), \( A_i = [0, M]^{\ell} \) where \( M \) is a sufficiently large constant, e.g., \( M > \max_{j=1}^n \sum_{i=1}^n e_j(i) \). For \( i = 0 \) we define \( \varphi_0 \) by

\[
\varphi_0(*) = \Delta,
\]

and for \( i = 1, \ldots, n \),

\[
\varphi_i(x_1, \ldots, x_n, p) = \{ y \in A_i \mid py = pe(i) \}.
\]

Define \( \hat{P}_i \), \( i = 0, 1, \ldots, n \), by

\[
\hat{P}_0(x_1, \ldots, x_n, p) = \{ q \in \Delta \mid q(\sum x_i - \sum e(i)) > p(\sum x_i - \sum e(i)) \},
\]

and for \( i = 1, \ldots, n \),

\[
\hat{P}_i(x_1, \ldots, x_n, p) = P_i(x_1, \ldots, x_n, p) \cap A_i.
\]

The abstract social system \( S(M) := (N; (A_i)_{i \in N}; (\varphi_i); (\hat{P}_i)) \) satisfies:

a) \( N \) is finite,

b) \( \forall 0 \leq i \leq n, A_i \) is a convex and compact subset of a Euclidean space,
c.1) $\varphi_0$ is a set-valued function with a constant nonempty compact convex value ($\Delta$), and thus is a continuous set-valued function with convex values,
c.2) $\forall 1 \leq i \leq n, \varphi_i$ is a continuous set-valued function with nonempty convex values (by lemma 4)
d.1) as $\hat{P}_i(a) \subset P_i(a)$ and $a_i \notin \text{co}P_i(a)$, we have $a_i \notin \text{co}\hat{P}_i(a)$
d.2) as $\hat{P}_0(x_1, \ldots, x_n, p)$ is convex and $p \notin \hat{P}_0(x_1, \ldots, x_n, p)$ we deduce that $p \notin \text{co}\hat{P}_0(x_1, \ldots, x_n, p)$,
e) $\forall 0 \leq i \leq n, \Gamma(\hat{P}_i)$ is open in $A \times \Delta \times A_i$ (in the relative topology).

Therefore, the abstract social system $\langle N; (A_i)_{i \in N}; (\varphi_i); (\hat{P}_i) \rangle$ satisfies all the conditions of the previous theorem and thus has a fixed point $(x^*, p^*)$.

We show that this is an equilibrium of $S(M) = \langle T, ([0, M]^\ell), (e_i), (P_i)_{i \in T} \rangle$. As $x^*(i) \in \varphi_i(x^*, p^*)$, $p^*x_i^* = p^*e(i)$. Therefore $\sum_i p^*x_i^* = \sum_i p^*e(i)$ and thus $p^*(\sum_i x_i^* - \sum_i e(i)) = 0$. If there is $j$ with $(\sum_i x_i^*)_j > (\sum_i e_i)_j$, setting $\delta_j = (0, \ldots, 1, 0, \ldots, 0)$, $\delta_j(\sum_i x_i^* - \sum_i e_i) > 0 = p^*(\sum_i x_i^* - \sum_i e_i)$ and therefore $\delta_j \in \varphi_0(x^*, p^*) \cap P_0(x^*, p^*) = \emptyset$, a contradiction. Therefore, $\sum_i x_i^* \leq \sum_i e(i)$. Obviously, $x_i^* \in \varphi_i(x^*, p)$ implies that $x_i^* \in [0, M]^\ell$, and as $\varphi_i(x^*, p) \cap \hat{P}_i(X^*) = \emptyset$ we deduce that for every $i \in T$, $P_i(x^*, p^*) \cap \{y \in [0, M]^\ell \mid p^*y = p^*e(i)\} = \emptyset$. Thus $(x^*, p^*)$ is an equilibrium of $E(M)$.
Let \((x(M), p(M))\) be competitive equilibria of the economy \(E(M)\). Then \(x(M) = (x^1(M), \ldots, x^n(M)) \in [0, M]^{\ell_n}\) and \(p(M) \in \Delta\). As both sets, \([0, M]^{\ell_n}\) and \(\Delta\) are compact, there is a sequence \((M_k)^{\infty}_{k=1}\) such that

\[ M_k \to \infty \text{ as } k \to \infty, \]

\[ x(M_k) \to x \in \mathbb{R}^{\ell_n} \text{ as } k \to \infty, \]

and

\[ p(M_k) \to p \in \Delta \text{ as } k \to \infty. \]

As \(\sum_{i=1}^{n} e(i) \geq \sum_{i=1}^{n} x^i(M_k) \to k\to\infty \sum_{i=1}^{n} x^i\), \(x\) is a suballocation.

As \(px^i = \lim_{k\to\infty} p(M_k)x^i(M_k) = \lim_{k\to\infty} p(M_k)e(i) = pe(i)\), we deduce that \(px^i = pe(i)\). In order to prove that indeed \((x, p)\) is a competitive equilibrium of \(E\) we have to show further that there is no \(i \in T\) and \(y^i \in \mathbb{R}_+^{\ell}\) such that \(py^i = pe(i)\) and \(y^i \in P_i(x, p)\). Assume that \(y^i \in \mathbb{R}_+^{\ell}\) with \(py^i = pe(i)\) and \(y^i \in P_i(x, p)\). Let \(y^i(k)\) be a sequence such that \(y^i(k) \to y^i\) as \(k \to \infty\) and \(p(M_k)y^i(k) = p(M_k)e(i)\). E.g., \(y^i(k) = \frac{p(M_k)e(i)}{p(M_k)p^i} y^i\). As \(P_i\) is continuous, i.e., \(\Gamma(P_i)\) is open, it will follow that there is a sufficiently large \(k\), such that \(y^i(k) \in P_i(x(M_k), p(M_k)) \cap [0, M_k]^\ell = \hat{P}_i(x(M_k), p(M_k))\) contradicting the fact that \((x(M_k), p(M_k))\) is an equilibrium of \(S(M_k)\).
10 Sard’s Lemma

A rectangular parallelepiped in $\mathbb{R}^n$ is a set of the form $I = \times_{i=1}^n [a_i, b_i]$. Its volume, $\text{vol } I$, is defined as $\prod_{i=1}^n (b_i - a_i)$. A subset $V$ of $\mathbb{R}^n$ is a set of measure 0 if for every $\varepsilon > 0$ there is a sequence of rectangular parallelepipeds, $I_k = \times_{i=1}^n [a_i^k, b_i^k]$ which cover $V$, i.e., such that $V \subset \bigcup_{k=1}^{\infty} I_k$ and the sum of their volumes is less then $\varepsilon$, i.e., $\sum_{k=1}^{\infty} \prod_{i=1}^n (b_i^k - a_i^k) < \varepsilon$.

A countable union of sets of measure 0 is a set of measure 0. Indeed, if $V_j$, $j = 1, \ldots$ is a sequence of subsets of $\mathbb{R}^n$ having measure 0, for every $\varepsilon > 0$ and every $j \geq 1$, there is a sequence of rectangular parallelepipeds $I_{j,k}$ such that $V_j \subset \bigcup_{k=1}^{\infty} I_{j,k}$ and $\sum_{k=1}^{\infty} \text{vol } I_{j,k} = \sum_{k=1}^{\infty} \prod_{i=1}^n (b_i^{j,k} - a_i^{j,k}) < \varepsilon 2^{-j}$. Then $\bigcup_{j=1}^{\infty} V_j \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} I_{j,k}$ and $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \text{vol } I_{j,k} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \prod_{i=1}^n (b_i^{j,k} - a_i^{j,k}) < \varepsilon$, and therefore $\bigcup_{j=1}^{\infty} V_j$ is a set of measure 0.

A function $f : U \to \mathbb{R}^n$ where $U$ is an open subset of $\mathbb{R}^n$ is called $C^1$ if it has continuous partial derivatives. The Jacobian of $f$ at $x \in U$ is the determinant of the $n \times n$ matrix $J(x)$ where $J_{i,j}(x) = \frac{\partial f_i}{\partial x_j}(x)$. A critical point (of $f$) is a point $x \in U$ with $\det J(x) = 0$. A critical value is a point of the form $f(x)$ where $x$ is a critical point. A regular value is a point $y \in \mathbb{R}^n$ which is not a critical value.
Theorem 13 (Sard’s Lemma). Let \( f : U \to \mathbb{R}^n \) be a \( C^1 \) function where \( U \) is an open subset of \( \mathbb{R}^n \). Then the set of critical values of \( f \) has measure 0.

Proof. For every \( x \in \mathbb{R}^n \) we denote \( \max_{i=1}^{n} |x_i| \) by \( \| x \| \). Assume first that \( [0, 1]^n \subset U \), and we will prove that the set, \( \{ f(x) : x \in [0, 1]^n \text{ with } \det J(x) = 0 \} \) has measure 0.

For every \( x \in \mathbb{R}^n \) and \( \alpha > 0 \) we denote by \( I(x, \alpha) \) the cube \( \{ y \in \mathbb{R}^n : \forall 1 \leq i \leq n, |y_i - x_i| \leq \alpha \} \).

Fix \( \varepsilon > 0 \).

For every \( 1 \leq i \leq n \), \( f_i(y) - f_i(x) = \sum_j J_{ij}(x + \theta_i(y - x))(y_j - x_j) \) for some \( 0 < \theta_i < 1 \) and therefore, as \( J(x) \) is continuous in the compact set \( [0, 1]^n \) it is uniformly continuous there, and thus there is \( \delta_i > 0 \) such that

\[
|J_{ij}(x + \theta(y - x)) - J_{ij}(x)| < \varepsilon/n \quad \text{whenever } \| x - y \| < \delta_i \text{ and } 0 < \theta < 1 \text{ and therefore } |f_i(y) - f_i(x) - J_i(x)(y - x)| < \varepsilon \| y - x \| \quad \text{whenever } \| x - y \| < \delta_i.
\]

Therefore, there is \( \delta > 0 \) (e.g. \( \delta = \min \delta_i \)) such that for every \( x, y \in [0, 1]^n \) with \( \| x - y \| < \delta \), \( \| f(x) - f(y) - J(x)(x - y) \| < \varepsilon \| x - y \| \).

There is a sufficiently large constant \( K \), such that for every \( x, y \in [0, 1]^n \), \( \| f(x) - f(y) \| < K \| x - y \| \). Let \( k \) and \( \ell \) be two positive integers with \( k > \delta^{-1} \) and \( K/\varepsilon < \ell \leq 1 + K/\varepsilon \). Note that for any two points \( x, y \in [0, 1]^n \)
with \( \|x - y\| \leq 1/(k\ell) \), \( \|f(x) - f(y)\| < \varepsilon/k \). Define for every \( 1 \leq i \leq n \), \( A_i = \{x \in [0,1]^n : kx_i \in \mathbb{Z} \text{ and } \forall 1 \leq j \leq n, k\ell x_j \in \mathbb{Z} \} \) and \( A = \bigcup A_i \).

Note that the number of points in \( A_i \) is \( \leq (k + 1)(k\ell + 1)^{n-1} \leq 2^n\ell^{n-1}k^n \) and thus the number of points in \( A \) is \( \leq n(k + 1)(k\ell + 1)^{n-1} \leq n2^n\ell^{n-1}k^n \).

Assume that \( x^0 \in [0,1]^n \) with \( \det J(x^0) = 0 \). As \( \det J(x^0) = 0 \), the kernel of the linear transformation \( J(x^0) : \mathbb{R}^n \to \mathbb{R}^n \) is nonempty, i.e., there is \( z \in \mathbb{R}^n \) with \( z \neq 0 \) and \( J(x^0)z = 0 \). W.l.o.g. assume that \( x^0_j \leq 1/k \).

Therefore for any point \( y \) on the line \( \{x^0 + tz : t \in \mathbb{R}\} \), \( J(x^0)(y - x^0) = 0 \) and therefore, there is \( x^1 \) in the boundary of \([0,1/k]^n\) with \( J(x^0)(x^1 - x^0) = 0 \), and therefore in particular \( \|f(x^1) - f(x^0)\| \leq \varepsilon/k \), and there is \( x^2 \in A \) with \( \|x^2 - x^1\| \leq 1/(k\ell) \), and therefore in particular \( \|f(x^2) - f(x^1)\| \leq K/(k\ell) \).

Therefore,

\[
\|f(x^0) - f(x^2)\| \leq \|f(x^0) - f(x^1)\| + \|f(x^1) - f(x^2)\| \leq \varepsilon/k + K/(k\ell) < 2\varepsilon/k.
\]

Therefore, \( f(x^0) \in I(f(x^2), 2\varepsilon/k) \). Therefore the set of critical values (in \( f([0,1]^n) \)) is contained in \( \cup_{y \in A} I(y, 2\varepsilon/k) \) and \( \sum_{y \in A} \text{vol}I(y, 2\varepsilon/k) \leq |A|(4\varepsilon/k)^n \leq n2^n\ell^{n-1}k^n(4\varepsilon/k)^n \leq n8^n\ell^{n-1}(\varepsilon)^n \leq n8^nK^n\varepsilon \).

Next assume that \( x \in U \) and \( \delta > 0 \) are such that \( I(x, \delta) = \{y \in \mathbb{R}^n : \|y - x\| \leq \delta \} \subset U \). The function \( g : U \to \mathbb{R}^n \) defined by \( g_i(z) = (z - x)/2\delta + 1/2 \) maps \( I(x, \delta) \) onto \([0,1]^n\) and has a nonvanishing Jaco-
bian. Therefore, \( \{ f(z) : z \in I(x, \delta) \text{ with } \det J(z) = 0 \} = \{ f \circ g^{-1}(y) : y \in [0, 1]^n \text{ with } \det J(f \circ g^{-1})(y) = 0 \} \) has measure zero by the first part of our proof.

Next we show that any open set \( U \) is the (countable) union of all (rational) cubes \( I(x, \delta) \) with \( x \in \mathbb{Q}^n \) and \( \delta \in \mathbb{Q} \) which are subsets of \( U \). Indeed, given \( x \in U \) there is \( \alpha > 0 \) such that \( I(x, \alpha) \subset U \). Let \( \delta \in \mathbb{Q} \) with \( \delta < \alpha/2 \) and \( y \in \mathbb{Q}^n \) with \( \| y - x \| < \delta \). Then \( x \in I(y, \delta) \subset I(x, \alpha) \subset U \) which proves our claim. As the set of all rational cubes which are subsets of \( U \), is a countable collection of sets which cover \( U \), the set of critical values has measure 0.

\[ \Box \]

## 11 Generic finiteness of Competitive Equilibria

In the previous sections the models of an exchange economy included as a basic ingredient the preference relations of the individuals, and the resulting income as a function of prices \( (pe(t)) \) and the demand set-valued functions of each individual were derived. In the present section we consider a model in which demand functions are part of the description of the exchange economy.

We consider here a model of an exchange economy which consists of:
1. A finite set of traders, \( T = \{1, \ldots, n\} \),

2. A finite number of commodities, \( \ell \),

3. For every \( t \in T \), an initial bundle \( e(t) \in \mathbb{R}^{\ell}_{++} \),

4. For every \( t \in T \), a demand function \( \delta_t : \Delta \times \mathbb{R}^{\ell}_{++} \rightarrow \mathbb{R}^{\ell}_{++} \) which satisfies \( p\delta_t(p, y) = y \).

The demand function \( \delta_t \) represents the demand of trader \( t \) as a function of the price vector \( p \in \Delta \) and his income \( y \in \mathbb{R}^{\ell}_{++} \). The condition \( p\delta_t(p, y) = y \) asserts that trader \( t \) balances his budget.

A **Competitive equilibrium price vector**, or an **equilibrium price vector** for short, of the economy \( \mathcal{E} = (T; \ell; (e(t))_{t \in T}; (\delta_t)_{t \in T}) \) is a price vector \( p \in \Delta \) such that

\[
\sum_{t \in T} \delta_t(p, pe(t)) = \sum_{t \in T} e(t).
\]

We will state and prove a result that asserts that given demand functions which satisfy suitable conditions (e.g., continuously differentiable), for almost all initial endowments, \( (e(t))_{t \in T} \), the exchange economy has finitely many competitive equilibria.

**Theorem 14** Assume that the demand functions \( \delta_t \) are continuously differentiable and for every \( y_0 > 0 \), \( \|\delta_1(p, y)\| \rightarrow \infty \) as \( p \rightarrow \partial \Delta \) uniformly in \( y \geq y_0 \), i.e., for every \( K > 0 \) there is \( \varepsilon > 0 \) such that for every \( y \geq y_0 \) and ev-


\[ \forall p \in \Delta \text{ with } \min_{i=1}^{\ell} p_i < \varepsilon, \|\delta_1(p, y)\| > K. \text{ Then, the set of } e(1), \ldots, e(n) \text{ (viewed as a subset of } \mathbb{R}^{n\ell}) \text{ for which there are infinitely many competitive equilibria price vectors } p \in \text{ int}\Delta \text{ is a subset of a closed set of measure } 0. \]

**Proof.** The idea of the proof is as follows. We construct a \( C^1 \) function \( F : \text{ int}\Delta \times \mathbb{R}_{++} \times \mathbb{R}_{++}^{(n-1)\ell} \to \mathbb{R}^{n\ell} \) such that \( p \) is a competitive equilibrium price vector of \( (e(1), \ldots, e(n)) \) if and only if

\[
F(p, pe(1), e(2), \ldots, e(n)) = (e(1), \ldots, e(n)).
\]

The set \( U = \text{ int}\Delta \times \mathbb{R}_{++} \times \mathbb{R}_{++}^{(n-1)\ell} \) is an open subset of \( \mathbb{R}^{n\ell} \) and \( F : U \to \mathbb{R}^{n\ell} \) is continuously differentiable and therefore by Sard’s lemma the set of critical values has measure zero. If \( (e(1), \ldots, e(n)) \) is a regular value and \( p \) is a competitive equilibrium price vector for \( (e(1), \ldots, e(n)) \), then

\[
F(p, pe(1), e(2), \ldots, e(n)) = (e(1), \ldots, e(n)),
\]

and

\[
\det J(F)(p, pe(1), e(2), \ldots, e(n)) \neq 0.
\]

Therefore, in a neighborhood of \( (p, pe(1), e(2), \ldots, e(n)) \), \( F \) is \( 1 - 1 \), and therefore for any price vector \( q \) which is sufficiently close to the price vector \( p \), \( F(q, qe(1), e(2), \ldots, e(n)) \neq (e(1), \ldots, e(n)) \) and thus any price vector \( q \),
which is sufficiently close to the price vector \( p \), is not an equilibrium price vector. We turn to the formal prove.

Set \( U = \text{int}\Delta \times \mathbb{R}_{++} \times \mathbb{R}^{(n-1)\ell}_{++} \) and observe that \( U \) is an open subset of \( \mathbb{R}^{n\ell} \).

Define \( F : U \to \mathbb{R}^{n\ell} \) by: \( F = (F_1, \ldots, F_n) \) where \( F_i : U \to \mathbb{R}^{\ell} \) is given by

\[
F_1(p, y, z_2, \ldots, z_n) = \delta_1(p, y) + \sum_{t=2}^{n} \delta_t(p, pz_t) - \sum_{t=2}^{n} z_t,
\]

and for \( t = 2, \ldots, n, \)

\[
F_t(p, y, z_2, \ldots, z_n) = z_t.
\]

Claim 1: the price vector \( p \in \text{int}\Delta \) is a competitive equilibrium price vector of the economy \( ((\delta_t)_{t \in T}; (e(t))_{t \in T}) \) iff

\[
F_1(p, pe(1), e(2), \ldots, e(n)) = e(1),
\]

and this holds iff

\[
F(p, pe(1), e(2), \ldots, e(n)) = (e(1), \ldots, e(n))
\]

(and this holds iff there is \( y \in \mathbb{R}_{++} \) such that

\[
F(p, y, e(2), \ldots, e(n)) = (e(1), \ldots, e(n))
\])

Proof: \( p \) is an equilibrium price vector iff \( \sum_{t=1}^{n} \delta_t(p, pe(t)) = \sum_{t=1}^{n} e(t) \) which holds iff \( F_1(p, pe(1), e(2), \ldots, e(n)) = \sum_{t=1}^{n} \delta_t(p, pe(t)) - \sum_{t=2}^{n} e(t) = e(1). \)
The second iff is obvious. If $F(p, y, e(2), \ldots, e(n)) = (e(1), \ldots, e(n))$, then

$$pF_1(p, y, e(2), \ldots, e(n)) = p\delta_1(p, y) + \sum_{t=2}^{n} p\delta_t(p, pe(t)) - \sum_{t=2}^{n} pe(t) = pe(1).$$

As $p\delta_t(p, pe(t)) = pe(t)$ and $p\delta_1(p, y) = y$ we deduce that $y = pe(1)$. This completes the proof of claim 1. We prove next that if $(e(1), \ldots, e(n))$ is a regular value, any competitive equilibrium price vector $p \in \text{int} \Delta$ is an isolated equilibrium price vector. Otherwise, for any $\varepsilon > 0$ there is a price vector $q \neq p$ with $\|q - p\| < \varepsilon$ and

$$F(p, pe(1), e(2), \ldots, e(n)) = (e(1), \ldots, e(n)) = F(q, qe(1), e(2), \ldots, e(n))$$

which contradicts the fact that $F$ is $1 - 1$ in a sufficiently small neighborhood of $p$.

Next we prove that there is $\varepsilon > 0$ such that any equilibrium price vector $p$ obeys $\min_{i=1}^{\ell} p_i > \varepsilon$. Indeed, if $y_0 = \min_{i=1}^{\ell} e_i(1)$, then $pe(1) \geq y_0$ for every $p \in \Delta$. Thus, if $\varepsilon > 0$ is sufficiently small so that $\|\delta_1(p, y)\| > \sum_{t=1}^{\ell} \|e(t)\|$ whenever $y \geq y_0$ and $\min_{i=1}^{\ell} p_i \leq \varepsilon$, there is no equilibrium price vector $p \in \Delta$ with $\min_{i=1}^{\ell} p_i \leq \varepsilon$.

Finally, we show that the set of critical values of $F$ is a closed subset of $\mathbb{R}^{n\ell}_{++}$. Let $e^k = (e^k(1), \ldots, e^k(n)) \in \mathbb{R}^{n\ell}_{++}$ be a sequence of critical values which converges as $k \to \infty$ to $(e(1), \ldots, e(n)) \in \mathbb{R}^{n\ell}_{++}$. Thus, there is a
sequence of \((p^k, y^k)\) such that

\[
F(p^k, y^k, e^k(2), \ldots, e^k(n)) = (e^k(1), \ldots, e^k(n)) \to_{k \to \infty} (e(1), \ldots, e(n)) \in \mathbb{R}^{n^2}_{++}
\]

and

\[
\det J(F)(p^k, y^k, e^k(2), \ldots, e^k(n)) = 0.
\]

As the price simplex \(\Delta\) is compact, we may assume W.l.o.g. that the sequence \((p^k)_{k=1}^\infty\) converges in \(\Delta\); (otherwise we take a subsequence). Assume that \(p^k \to p \in \Delta\) as \(k \to \infty\). We show now that the limiting price vector \(p\) is in \(\text{int}\Delta\). On the one hand, as

\[
F_1(p^k, y^k, e^k(2), \ldots, e^k(n)) = e^k(1),
\]

\[
p^k F_1(p^k, y^k, e^k(2), \ldots, e^k(n)) = p^k e^k(1).
\]

On the other hand, as

\[
F_1(p^k, y^k, e^k(2), \ldots, e^k(n)) = \delta_1(p^k, y^k) + \sum_{t=2}^n \delta_t(p^k, p^k e(t)) - \sum_{t=2}^n e^k(t),
\]

and

\[
p^k \sum_{t=2}^n \delta_t(p^k, p^k e(t)) = p^k \sum_{t=2}^n e^k(t),
\]

\[
p^k F_1(p^k, y^k, e^k(2), \ldots, e^k(n)) = p^k \delta_1(p^k, y^k) = y^k.
\]

Therefore, \(y^k = p^k e^k(1) \to_{k \to \infty} p e(1) \geq \min_{i=1}^n e_i(1) > 0\). Setting \(y_0 = \frac{\min_{i=1}^n e_i(1)}{2} > 0\) we deduce that for sufficiently large \(k\), \(y^k > y_0\). Setting \(K = 2 \sum_{t \in T} \|e(t)\|\) we deduce that for sufficiently large \(k\), \(\sum_{t \in T} \|e^k(t)\| < K\).
By assumption, there is $\varepsilon > 0$, such that if $\min_{i=1}^{n} p_i^k < \varepsilon$, and $y_k \geq y_0$, $\|\delta_1(p^k, y^k)\| > K$ which proves that $\min_{i=1}^{n} p_i^k \geq \varepsilon$ for sufficiently large $k$ and thus $\min_{i=1}^{n} p_i \geq \varepsilon$. Therefore, $p \in \text{int } \Delta$. Therefore,

$$(p^k, y^k, e^k(2), \ldots, e^k(n)) \rightarrow_{k \to \infty} ((p, pe(1)), e(2) \ldots, e(n)) \in \mathbb{R}^m_{++},$$

$$F(p, pe(1), e(2), \ldots, e(n)) = (e(1), \ldots, e(n)),$$

and

$$0 = \det J(F)(p^k, y^k, e^k(2), \ldots, e^k(n)) \rightarrow_{k \to \infty} \det J(F)(p, pe(1), \ldots, e(n))$$

and thus $(e(1), \ldots, e(n))$ is a critical value of $F$. 

12 The Equivalence Principle

The equivalence principle asserts that in large economies any core allocation is ‘almost’ a competitive equilibrium.

Before stating a theorem to that effect, we introduce few preliminaries from convexity. Given subsets $A$ and $B$ of $\mathbb{R}^n$ their sum $A + B$ is defined by $A + B = \{a + b \mid a \in A, b \in B\}$. Note that if both $A$ and $B$ are convex so is $A + B$. Indeed, if $a, a' \in A$ and $b, b' \in B$ and $0 \leq \alpha \leq 1$, then $\alpha(a + b) + (1 - \alpha)(a' + b') = \alpha a + (1 - \alpha)a' + \alpha b + (1 - \alpha)b'$ and thus
\[ \alpha(a + b) + (1 - \alpha)(a' + b') \in A + B \] whenever \( A \) and \( B \) are convex. Similarly, for any subsets \( A_1, \ldots, A_k \) of \( \mathbb{R}^n \) we define \( \sum_{i=1}^k A_i = \{ \sum_{i=1}^k a_i \mid a_i \in A_i \} \), or equivalently, \( \sum_{i=1}^k A_i = \sum_{i=1}^{k-1} A_i + A_k \). Thus by induction \( \sum_{i=1}^k A_i \) is convex whenever each of the sets \( A_i \) is convex. Therefore, for any subsets \( A_1, \ldots, A_k \) of \( \mathbb{R}^n \), \( \sum_{i=1}^k \text{co}A_i \) is convex. On the other hand \( \sum_{i=1}^k A_i \subset \sum_{i=1}^k \text{co}A_i \), and therefore \( \text{co} \sum_{i=1}^k A_i \subset \sum_{i=1}^k \text{co}A_i \). On the other hand, any point \( z \in \text{co}A_1 + \text{co}A_2 \) has a representation \( z = x_1 + x_2 \) with \( x_i = \sum_{j=1}^{n_i} \alpha_{ij} x_j \), where \( x_j \in A_i \), \( i = 1, 2 \), and \( \alpha^i = (\alpha_{i,1}, \ldots, \alpha_{i,n_i}) \in \Delta^{n_i} \). Therefore, \[ z = x_1 + x_2 = \sum_{j=1}^{n_1} \sum_{l=1}^{n_2} \alpha_{ij}^1 \alpha_{jl}^2 (x_j^1 + x_l^2) \] is a convex combination of points from \( A_1 + A_2 \), and therefore \( \text{co}(A_1) + \text{co}(A_2) \subset \text{co}(A_1 + A_2) \) and by induction on \( k \), \( \sum_{i=1}^k \text{co}A_i \subset \text{co} \sum_{i=1}^k A_i \), and thus \[ \text{co} \sum_{i=1}^k A_i = \sum_{i=1}^k \text{co}A_i. \]

**Theorem 15 (Shapley Folkman)** Let \( A_1, \ldots, A_k \) be subsets of \( E^n \). Then any vector \( z \in \sum_{i=1}^k \text{co}A_i \) is a sum \( z = \sum_{i=1}^k y_i \) where \( \forall 1 \leq i \leq k, y_i \in \text{co}A_i \) with \( y_i = \sum_{j=1}^{n_i} \alpha_{ij} a_j^i \) where \( a_j^i \in A_i, \ \alpha_{ij} \geq 0 \) and \( \sum_{j=1}^{n_i} \alpha_{ij} = 1 \), such that \( \sum_{i=1}^k n_i \leq n + k \), and thus in particular, \( |\{ i : y_i \notin A_i \}| \leq n. \)
Proof. As \( z \in \sum_{i=1}^{k} \text{co} A_i \), \( z \) is a sum \( z = \sum_{i=1}^{k} z(i) \) where \( \forall 1 \leq i \leq k \), \( z(i) \in \text{co} A_i \), and thus each \( z(i) \) is a convex combination of finitely many elements from \( A_i \), i.e., there are \( \alpha_i^j > 0 \) with \( \sum_{j=1}^{n_i} \alpha_i^j = 1 \), and \( a_i(j) \in A_i \), such that \( z(i) = \sum_{j=1}^{n_i} \alpha_i^j a_i(j) \). Assume that this representation minimizes \( \sum_{i=1}^{k} n_i \).

It is sufficient to prove that then \( \sum_{i=1}^{k} n_i \leq k + n \). Define \( T : \mathbb{R}^{\sum_{i=1}^{k} n_i} \to \mathbb{R}^n \) by

\[
T(b_1^1, \ldots, b_1^{n_1}, \ldots, b_k^1, \ldots, b_k^{n_k}) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} b_i^j a_i(j).
\]

The kernel of \( T \), \( \{b \in \mathbb{R}^{\sum_{i=1}^{k} n_i} \mid T b = 0 \} \) has dimension at least \( \sum_{i=1}^{k} n_i - n \), and \( V = \{b \in \mathbb{R}^{\sum_{i=1}^{k} n_i} \mid \forall 1 \leq i \leq k, \sum_{j=1}^{n_i} b_i^j = 0 \} \) is a subspace of dimension \( \sum_{i=1}^{k} n_i - k \). Therefore, if \( \sum_{i=1}^{k} n_i > k + n \), there is \( b \in V \setminus \{0\} \) with \( T b = 0 \).

Set \( J = \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq n_i \text{ and } b_i^j < 0 \} \) and note that \( J \) is not empty. For every \((i, j) \in J \) set \( t_i^j = -\alpha_i^j/\lambda_i^j \). Then \( t_i^j > 0 \), \( \alpha_i^j + t_i^j b_i^j = 0 \), and for every \( 0 < t < t_i^j \), \( \alpha_i^j + t b_i^j > 0 \). Therefore, if \( t = \min \{t_i^j \mid (i, j) \in J\} \), for every \( 1 \leq i \leq k \), \( \alpha_i^j + t b_i^j \in \Delta^n \), and at least one of the coordinates of \( \alpha + t b \) vanishes, and \( z = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\alpha_i^j + t b_i^j) a_i(j) \), contradicting the minimality of \( \sum_{i=1}^{k} n_i \).

The Shapley-Folkman theorem implies in particular that the sum of a large number of subsets of \([0,1]^n\) is almost convex, by asserting that any point in the convex hull of such a sum is within a fixed bounded distance.
from a point of the sum. Moreover,

**Corollary 1** Let $K_i, i = 1, \ldots, k$, be subsets of $\mathbb{R}^n$ and denote by $d_i$ the diameter of $K_i$, i.e., $d_i = \max\{\|x - y\| | x, y \in K_i\}$. Then, if $\sum_{j=1}^{n} d_{i_j} \leq M$, for every $i_1 < i_2 < \ldots < i_n$, then for every $z \in \text{co} \sum_{i=1}^{k} K_i$ there is $y \in \sum_{i=1}^{k} K_i$ with $\|y - z\| \leq M$.

We need an additional classical result, regarding the separation of convex sets. Recall that for two vectors, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the inner product of $x$ and $y$ is defined as $x y = \sum_{i=1}^{n} x_i y_i$, and that the inner product obeys the following properties:

$$\forall x, y, z \in \mathbb{R}^n \quad (x + y)z = xz + yz$$

and

$$\forall x, y \in \mathbb{R}^n \quad \forall \alpha \in \mathbb{R} \quad (\alpha x)y = \alpha(xy).$$

**Theorem 16** Let $K$ and $C$ be two nonempty convex closed subsets of $\mathbb{R}^n$ where one of them, say $K$, is compact, and with empty intersection, i.e., $K \cap C = \emptyset$. Then there is $p \in \mathbb{R}^n$ with

$$\sup\{px : x \in C\} < \inf\{py : y \in K\}$$
Proof. Let $\alpha = \inf\{(y-x)(y-x) : y \in K \text{ and } x \in C\}$. For every positive integer $k$, let $y(k) \in K$ and $x(k) \in C$ with

$$\alpha \leq (y(k) - x(k))(y(k) - x(k)) \leq \alpha + 1/k.$$ 

As $C$ is compact, there is a subsequence $(n_k)_{k=1}^\infty$ of the sequence of positive integers, for which

$$x(n_k) \to x \in C \text{ as } k \to \infty.$$ 

It follows that $y(n_k) \in K$ is a bounded sequence ($|y_i(n_k)| \leq |y_i(n_k) - x_i(n_k)| + |x_i(n_k) - x_i| + |x_i|$ and thus $\limsup_{k \to \infty} |y_i(n_k)| \leq |x_i| + \alpha^{1/2}$) and as $K$ is closed we may assume without loss of generality, by possible taking a subsequence, that

$$y(n_k) \to y \in K \text{ as } k \to \infty.$$ 

Therefore, $(y - x)(y - x) = \lim_{k \to \infty}(y(n_k) - x(n_k))(y(n_k) - x(n_k))$, and $\alpha \leq (y(n_k) - x(n_k))(y(n_k) - x(n_k)) \leq \alpha + 1/n_k \to_{k \to \infty} \alpha$, and thus

$$(y - x)(y - x) = \alpha.$$ 

As $K \cap C = \emptyset$, $x \neq y$ and thus $\alpha > 0$. Assume that $z \in K$. Then for every $0 \leq t \leq 1$, $tz + (1-t)y \in K$. Therefore, the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(t) = (tz + (1-t)y - x)(tz + (1-t)y - x) = (t(z-y)+y-x)(t(z-y)+y-x)$$

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\[(y - x)(y - x) + 2t(z - y)(y - x) + t^2(z - y)(z - y)\]

is differentiable at 0, with \(f'(0) = 2(z - y)(y - x)\). Therefore, as \(f(0) = \alpha \leq f(t)\), \((tz + (1 - t)y \in K)\), the derivative at 0 is nonnegative, i.e.,

\[(z - y)(y - x) = z(y - x) - y(y - x) \geq 0,\]

implying that

\[z(y - x) \geq y(y - x).\]

Similarly, for every \(z \in C\),

\[z(y - x) \leq x(y - x).\]

As \(y(y - x) - x(y - x) = (y - x)(y - x) > 0\),

\[y(y - x) > x(y - x)\]

and therefore

\[\sup\{px : x \in C\} \leq x(y - x) < y(y - x) \leq \inf\{py : y \in K\}.\]

which completes the proof of the theorem.

\[\text{Theorem 17 (Anderson, 1979)}\]

Let \(\mathcal{E} = \langle T; \mathbb{R}_+^\ell; (e(t))_{t \in T}; (\succ_t)_{t \in T} \rangle\) be an exchange economy with \(\ell\) commodities with weakly monotonic preferences, i.e., \(x \succ_t y\) whenever \(x \succ y\) and if \(x \succ_t y\) and \(z \succ 0\), then \(x + z \succ_t y\).
Then, setting

\[ M = \max \{ \sum_{t \in S} \| e(t) \| : S \subset T \text{ with } |S| \leq \ell \}, \]

for any core allocation \( x \) of \( E \), there is a price vector \( p \in \Delta \), for which:

\[ \frac{1}{|T|} \sum_{t \in T} \left| p(x(t) - e(t)) \right| \leq \frac{2M}{|T|} \]

(7)

and

\[ \frac{1}{|T|} \sum_{t \in T} - \inf \{ p(y - x(t)) : y \succ_t x(t) \} \leq \frac{2M}{|T|} \]

(8)

**Proof.** Assume that \( x \) is a core allocation. For every \( t \in T \), set

\[ V(t) = \{ y - e(t) : y \succ_t x(t) \} \cup \{0\}, \]

\[ V = \sum_{t \in T} V(t) \]

Let \( u = (1, \ldots, 1) \) denote the vector in \( \mathbb{R}^\ell \) with all of its coordinates are equal to 1. We will prove that

\[ -Mu - \mathbb{R}^\ell_{++} \cap \text{co}V = \emptyset, \]

(9)

i.e., that for every \( v \in \text{co}V \) there is \( 1 \leq i \leq n \) with \( v_i \geq -M \). Assume that \( v \in \text{co}V \). By the Shapley Folkman theorem, there are elements \( z(t) \in \text{co}V(t) \) such that

\[ v = \sum_{t \in T} z(t), \]
and the number of elements in the set $\hat{T} = \{t \in T : z(t) \notin V(t)\}$ it less than or equal to $\ell$. Set $\bar{T} = T \setminus \hat{T}$. Note that for every $z \in V(t)$, $z \geq -e(t)$, i.e., $V(t) \subset R^\ell_+ - e(t)$. As $R^\ell_+ - e(t)$ is convex it contains also $\text{co}V(t)$, i.e., for every $z \in \text{co}V(t)$, $z \geq -e(t)$, and therefore

$$\sum_{t \in \hat{T}} z(t) \geq \sum_{t \in \hat{T}} -e(t) \geq -Mu. \quad (10)$$

Therefore,

$$v = \sum_{t \in \bar{T}} z(t) + \sum_{t \in \hat{T}} z(t) \geq \sum_{t \in \bar{T}} z(t) - Mu.$$ 

We prove next that

$$\sum_{t \in \bar{T}} z(t) \notin -R^\ell_+. \quad (11)$$

Indeed, if $\sum_{t \in \bar{T}} z(t) \in -R^\ell_+$, the set $S = \{t \in \bar{T} : z(t) \neq 0\}$ is nonempty, (otherwise $\sum_{t \in \bar{T}} z(t) = 0 \notin -R^\ell_+$) and

$$\sum_{t \in \bar{T}} z(t) = \sum_{t \in S} z(t) \in -R^\ell_+$$

and therefore there is a vector $w$ in $R^\ell_+$ with

$$\sum_{t \in S} z(t) + w = 0.$$

For every $t \in S$, let $y(t) \in R^\ell_+$ with $y(t) \succ_t x(t)$ and $z(t) = y(t) - e(t)$.

Then,

$$\sum_{t \in S} (y(t) + \frac{w}{|S|}) = \sum_{t \in S} e(t),$$

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i.e., \( y(t) + \frac{w}{|S|} \), \( t \in S \), is an \( S \)-allocation, and as \( \succ \) is weakly monotonic,
\[
y(t) + \frac{w}{|S|} \succ x(t)
\]
contradicting the assumption that \( x \) is a core allocation.

**Lemma 7** There is \( p \in \Delta \) such that \( \inf \{ pz : z \in \text{co}V \} \geq -M \).

**Proof.** For every \( k > 0 \) let \( K_k = \text{cl} \{ v \in \text{co}V : \|v\|_\infty < k \} \) (where for a subset \( A \) of \( IR^n \), \( \text{cl}A \) denotes the closure of \( A \) in \( IR^n \)) and \( C_k = \{ w \in IR^\ell : -k \leq w_i \leq -M - 1/k \}. \) The closure of a convex set is convex (indeed, if \( A \) is convex and \( x, y \in \text{cl}A \) and \( 0 < \alpha < 1 \), there are sequences \( x_n \in A \) and \( y_n \in A \) with \( x_n \to x \) and \( y_n \to y \) as \( n \to \infty \), and therefore \( \alpha x_n + (1 - \alpha)y_n \to \alpha x + (1 - \alpha)y \) as \( n \to \infty \). As \( a \) is convex \( \alpha x_n + (1 - \alpha)y_n \in A \) and thus \( \alpha x + (1 - \alpha)y \in \text{cl}A \), proving that \( \text{cl}A \) is also convex). The set \( \{ v \in \text{co}V : \|v\|_\infty < k \} \) is the intersection of two convex sets and thus convex, and therefore the set \( K_k = \text{cl} \{ v \in \text{co}V : \|v\|_\infty < k \} \) is convex closed and bounded. Thus the sets \( K_k \) and \( C_k \) are convex and compact, and therefore there is a sequence of vectors \( p^k \in IR^\ell \) with
\[
\sup \{ p(k)w : w \in C_k \} < \inf \{ p(k)v : v \in K_k \} \leq 0.
\]
In particular \( p(k) \neq 0 \) and therefore we may assume without loss of generality that \( \|p(k)\|_1 = 1 \) (where for a vector \( x \) we denote \( \|x\|_1 = \sum_i |x_i| \); otherwise
we replace the vectors $p(k)$ with $p(k)/\|p(k)\|_1$. W.l.o.g. assume that $p(k) \to p$ as $k \to \infty$. Then $\|p\|_1 = 1$. We prove next that $p \geq 0$. Let $u_i$ denote the $i$-th unit vector in $\mathbb{R}^\ell$. Note that for every $k$, $0 \in K_k$, and for sufficiently large values of $k$, $-2Mu - ku_i/2 \in C_k$. Therefore, $p(k)(-2Mu - ku_i/2) < p(k)0 = 0$, i.e., $-kp_i(k)/2 < p(k)(2Mu)$, or equivalently, $p_i(k) > -2Mp(k)u/k \to 0$ as $k \to \infty$ and therefore for every $1 \leq i \leq \ell$, $p_i = \lim_{k \to \infty} p_i(k) \geq 0$. Therefore, $p \geq 0$ and as $\sum_{i=1}^\ell p_i = 1$, the limiting vector $p$ is in the price simplex $\Delta$. It follows that $\inf\{pv : v \in \text{co}V\} \geq -M$.

Let $S = \{t \in T : p(x(t) - e(t)) < 0\}$. Note that it follows from weak monotonicity of $\succ_t$ that for every $\varepsilon > 0$, $x(t) - e(t) + \varepsilon u \in V(t)$ and thus $\sum_{t \in S}(x(t) - e(t) + \varepsilon u) \in V$ and therefore $\sum_{t \in S} p(x(t) - e(t) + \varepsilon u) \geq -M$. As $\sum_{t \in S} p(x(t) - e(t) + \varepsilon u) \to \sum_{t \in S} p(x(t) - e(t))$ as $\varepsilon \to 0$, we deduce that

$$\sum_{t \in S} p(x(t) - e(t)) \geq -M.$$ 

Note also that as $\sum_{t \in T} p(x(t) - e(t)) = 0$,

$$\sum_{t \in T} |p(x(t) - e(t))| = 2 \sum_{t \in S} |p(x(t) - e(t))| \leq 2M,$$

which proves that the price vector $p$ obeys (7).

Observe that for every $\varepsilon > 0$, $x(t) + \varepsilon u \succ_t x(t)$ and therefore,

$$\inf\{p(y - x(t)) : y \succ_t x(t)\} \leq p\varepsilon u = \varepsilon \to 0 \text{ as } \varepsilon \to 0+,$$

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and therefore the \( \inf \) is nonpositive. Assume that for every \( t \in T \) we select \( y(t) \in \mathbb{R}_+^\ell \) with \( y(t) \succ_t x(t) \). Then \( y(t) - e(t) \in V(t) \) and thus \( \sum_{t \in T}(y(t) - e(t)) \in V \subset \text{co}V \) implying that

\[
\sum_{t \in T} p(y(t) - e(t)) \geq -M.
\]

As \( x \) is an allocation, \( \sum_{t \in T} p(y(t) - x(t)) = \sum_{t \in T} p(y(t) - e(t)) \), and therefore

\[
\inf\{\sum_{t \in T} p(y(t) - x(t)) : y(t) \succ_t x(t)\} \geq -M
\]

and as \( \inf\{\sum_{t \in T} p(y(t) - x(t)) : y(t) \succ_t x(t)\} = \sum_{t \in T} \inf\{p(y(t) - x(t)) : y(t) \succ_t x(t)\} \), the price vector \( p \) obeys (8).

\[\sqrt{\text{ }}\]

### 12.1 Replicated economies

Let \( \mathcal{E} = \langle T; \mathbb{R}_+^\ell; (e(t))_{t \in T}; (\succ)_{t \in T} \rangle \) be an exchange economy with \( \ell \) commodities and \( m \) consumers. The \( r \) replicated exchange economy, \( \mathcal{E}^r \), is the exchange economy with \( mr \) consumers, where each consumer is indexed by a pair \( (t, q) \), with \( t \in T \) and \( q = 1, \ldots, r \). The first index is refers to the type of the consumer and the second distinguishes between different consumers of the same type. Two consumers of the same type \( t \in T \) have the same preference relation \( \succ_t \) and the same initial endowment \( e(t) \). A type-symmetric allocation of \( \mathcal{E}^r \) is an allocation \( x : T \times \{1, \ldots, r\} \to \mathbb{R}_+^\ell \) for which there is
a function $y : T \to \mathbb{R}^\ell_+$ with $x(t, q) = y(t)$. It follows that $y$ is an allocation in $\mathcal{E}$. Given an allocation $y$ of $\mathcal{E}$ it induces a type symmetric allocation $x$ defined by $x(t, q) = y(t)$.

**Corollary 2** Let $\mathcal{E} = \langle T; \mathbb{R}^\ell_+; (e(t))_{t \in T}; (\succ)_{t \in T} \rangle$ be an exchange economy with $\ell$ commodities and $m$ consumers, with strictly positive initial endowments, i.e., $e(t) > 0$, and with weakly monotonic and continuous preferences. Then any allocation $y$ of $\mathcal{E}$ such that for every $r$ the induced type symmetric allocation in $\mathcal{E}^r$ is in the core of $\mathcal{E}^r$ is a competitive allocation.

**Proof.** Let $M = \ell \max_{t \in T} \|e(t)\|$. By Anderson’s theorem, for every positive integer $r$, there is a price vector $p(r) \in \Delta$ with

$$\frac{1}{mr} \sum_{t \in T} r|p(r)(y(t) - e(t))| \leq \frac{2M}{mr}$$

and

$$\frac{1}{mr} \sum_{t \in T} -r \inf \{p(r)(z - y(t)) : z \succ_t y(t) \leq \frac{2M}{mr} \}.$$ 

Therefore, for every $t \in T$,

$$|p(r)(y(t) - e(t))| \leq \sum_{t \in T} |p(r)(y(t) - e(t))| \leq \frac{2M}{r}$$

which implies that

$$\limsup_{r \to \infty} |p(r)(y(t) - e(t))| = 0,$$

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and for every $t \in T$ and $z \in \mathbb{R}^\ell_+$ with $z \succ_t y(t)$,

$$-p(r)(z - y(t)) \leq \sum_{t \in T} -\inf \{p(r)(z - y(t)) : z \succ_t y(t) \leq \frac{2M}{r}\},$$

which implies that for every $z \in \mathbb{R}^\ell_+$ with $z \succ_t y(t)$,

$$\liminf_{r \to \infty} p(r)(z - y(t)) \geq 0.$$

W.l.o.g. assume that $p(r) \to p \in \Delta$ as $r \to \infty$. (Otherwise we take a subsequence.) It follows that for every $t \in T$, $py(t) - pe(t) = 0$, and if $z \in \mathbb{R}^\ell_+$ with $z \succ_t y(t)$ and $pz \geq p(y(t)) = p(e(t))$. By continuity of the preference relation $\succ_t$, for sufficiently small $\varepsilon > 0$, $(1 - \varepsilon)z \succ_t y(t)$ and therefore also $(1 - \varepsilon)pz = (1 - \varepsilon)pe(t) \geq pe(t)$, which is possible only if $pe(t) \leq 0$. However, as the initial endowments are strictly positive, and $p \in \Delta$, $pe(t) > 0$. Therefore, $pz > pe(t)$ whenever $z \succ_t y(t)$, i.e., $(p, y)$ is a competitive equilibrium of $E$.

\textbf{Corollary 3} Let $E = \langle T; \mathbb{R}^\ell_+; (e(t))_{t \in T}; (\succ)_{t \in T} \rangle$ be an exchange economy with $\ell$ commodities and $m$ consumers, with strictly positive initial endowments, i.e., $e(t) > 0$, and with weakly monotonic and continuous preferences defined by strictly concave utility functions. Then any limit point $y$ of a sequence of allocation $y^r$ of $E$ such that for every $r$ the induced type symmetric allocation in $E^r$ is in the core of $E^r$ is a competitive allocation.
Proof. Let \( M = \ell \max_{t \in T} \|e(t)\| \). By Anderson’s theorem, for every positive integer \( r \), there is a price vector \( p(r) \in \Delta \) with

\[
\frac{1}{mr} \sum_{t \in T} r|p(r)(y^r(t) - e(t))| \leq \frac{2M}{mr}
\]

and

\[
\frac{1}{mr} \sum_{t \in T} -r \inf \{p(r)(z - y^r(t)) : z \succ_t y^r(t)\} \leq \frac{2M}{mr}.
\]

Therefore, for every \( t \in T \),

\[
|p(r)(y(t) - e(t))| \leq \sum_{t \in T} |p(r)(y^r(t) - e(t))| \leq \frac{2M}{r}
\]

which implies that

\[
\limsup_{r \to \infty} |p(r)(y^r(t) - e(t))| = 0.
\]

Let \( r_1 < r_2 < \ldots \) with \( y^{r_k}(t) \to_{k \to \infty} y(t) \) for every \( t \in T \). Then,

\[
\lim_{k \to \infty} |p(r_k)(y(t) - e(t))| = 0.
\]

For every \( t \in T \) and \( z \in \mathbb{R}^\ell_+ \) with \( z \succ_t y^r(t) \),

\[
-p(r)(z - y^r(t)) \leq \sum_{t \in T} -\inf \{p(r)(z - y^r(t)) : z \succ_t y^r(t)\} \leq \frac{2M}{r}.
\]

If \( z \in \mathbb{R}^\ell_+ \) with \( z \succ_t y(t) \) then for sufficiently large \( k \) we have \( z \succ_t y^{r_k}(t) \) which implies that

\[
\liminf_{k \to \infty} p(r_k)(z - y^{r_k}(t)) \geq 0,
\]
and thus

$$\liminf_{r \to \infty} p(r_k)(z - y(t)) \geq 0.$$  

Let \( p \in \Delta \) be a limit point of the sequence \( p(r_k), k \geq 1 \). W.l.o.g. we may assume that \( p(r_k) \to_k \infty p \) (Otherwise we replace the sequence \( r_k \) with a subsequence). It follows that for every \( t \in T \), \( py(t) - pe(t) = 0 \), and if \( z \in IR^\ell_+ \) with \( z_t \succ_t y(t) \) then \( pz \geq p(y(t)) = p(e(t)) \). By continuity of the preference relation \( \succ_t \), for sufficiently small \( \varepsilon > 0 \), \( (1 - \varepsilon)z \succ_t y(t) \) and therefore also \( (1 - \varepsilon)pz \geq pe(t) \). However, as the initial endowments are strictly positive, and \( p \in \Delta \), \( pe(t) > 0 \). Therefore, \( pz > pe(t) \) whenever \( z \succ_t y(t) \), i.e., \( (p, y) \) is a competitive equilibrium of \( \mathcal{E} \).

\textbf{Lemma 8}  Let \( \mathcal{E} = (T; IR^\ell_+; (e(t))_{t \in T}; (\succ)_{t \in T}) \) be an exchange economy with \( \ell \) commodities and \( m \) consumers, with strictly positive initial endowments, i.e., \( e(t) >> 0 \), and with weakly monotonic and continuous preferences defined by strictly concave utility functions. Then any core allocation of \( \mathcal{E}^r \) is an equal treatment allocation.

\textbf{Proof.} Let \( y^r \) be a core allocation of \( \mathcal{E}^r \) and define \( y(t) = \frac{1}{r} \sum_j y^r(t, j) \).

Note that \( y \) is an allocation in \( \mathcal{E} \). We have to prove that \( y^r(t, j) = y(t) \) for every \( t \) and every \( 1 \leq j \leq r \). Otherwise, there is \( t \in T \) s.t. \( u_t(y(t)) > \)
\[ \frac{1}{r} \sum_{1 \leq j \leq r} u_t(y^r(t, j)) \] and therefore there is \( t \in T \) and \( j \) with \( u_t(y(t)) > u_t(y^r(t, j)). \)

For every \( s \in T \) and every \( j \) we have \( u_s(y(s)) > \frac{1}{r} \sum_{1 \leq j \leq r} u_s(y^r(s, j)). \) Altogether we can find a sequence \( 1 \leq j_s \leq r \) such that \( u_s(y^r(s, j_s)) \leq u_s(y(s)) \) with a strict inequality for at least one \( t \in T \). This enables us to construct an \( S \)-allocation \( z \) where \( S = \{(s, j_s) : s \in T\} \) s.t. for every \((s, j_s) \in S\) we have \( z(s, j_s) \succ_{s, j_s} y^r(s, j_s). \)