

The Maximal Variation of Martingales of Probabilities and Repeated Games with Incomplete Information

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Abstract The variation of a martingale $p_0^k = p_0, \dots, p_k$ of probabilities on a finite (or countable) set X is denoted $V(p_0^k)$ and defined by

$$V(p_0^k) = E \left(\sum_{t=1}^k \|p_t - p_{t-1}\|_1 \right).$$

It is shown that $V(p_0^k) \leq \sqrt{2kH(p_0)}$, where $H(p)$ is the entropy function $H(p) = -\sum_x p(x) \log p(x)$, and \log stands for the natural logarithm. Therefore, if d is the number of elements of X , then $V(p_0^k) \leq \sqrt{2k \log d}$. It is shown that the order of magnitude of the bound $\sqrt{2k \log d}$ is tight for $d \leq 2^k$: there is $C > 0$ such that for all k and $d \leq 2^k$, there is a martingale $p_0^k = p_0, \dots, p_k$ of probabilities on a set X with d elements, and with variation $V(p_0^k) \geq C\sqrt{2k \log d}$. An application of the first result to game theory is that the difference between v_k and $\lim_j v_j$, where v_k is the value of the k -stage repeated game with incomplete information on one side with d states, is bounded by $\|G\| \sqrt{2k^{-1} \log d}$ (where $\|G\|$ is the maximal absolute value of a stage payoff). Furthermore, it is shown that the order of magnitude of this game theory bound is tight.

Keywords Maximal martingale variation · Posteriors variation · Repeated games with incomplete information

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1 Introduction

Bounds on the variation of a martingale of probabilities are useful in the theory of repeated games with incomplete information. Such martingales arise as sequences of an uninformed player's posteriors $p_0^k = p_0, \dots, p_k$ of an unknown game parameter. The martingale's variation, $V(p_0^k) := E \sum_{t=1}^k \|p_t - p_{t-1}\|_1$, bounds from above (a positive constant times) the payoff advantage that the more informed player has over the less informed one in a two-person zero-sum k -stage repeated game with incomplete information on one side; see [1, 3–5].

The maximal variation of a martingale p_0^k of probabilities over a finite set depends both on the initial probability $p = p_0$ and on k . It is bounded by a positive constant $C(p)$ times the square root of k . This inequality is used in Aumann and Maschler [1]¹ to prove that the rate of convergence of the minmax value v_k of the k -repeated game with incomplete information on one side and perfect monitoring is $O(1/\sqrt{k})$. Zamir [5] proved the tightness of this bound: there is a repeated game with incomplete information on one side and perfect monitoring for which the error term, $v_k - \lim v_k$, is greater than or equal to $1/\sqrt{k}$.

Mertens and Zamir [3] showed that $C(p)$ is less than or equal to $\sqrt{d-1}$, where d is the number of elements in the support of p , and the error term is less than or equal to $\|G\|\sqrt{d-1}/\sqrt{k}$, where $\|G\|$ is the maximal absolute value of a payoff in one of the possible d single-stage games.

The objective of the present paper is to improve the order of magnitude of the term \sqrt{d} in the above-mentioned bounds. The main result of the paper is that $V(p_0^k) \leq \sqrt{2kH(p)} \leq \sqrt{2k \log d}$, where H is the entropy function. This inequality implies that the error term is less than or equal to $\|G\|\sqrt{2H(p)}/\sqrt{k}$, which is less than or equal to $\|G\|\sqrt{2 \log d}/\sqrt{k}$.

We also provide tightness results for both the variation of a martingale of probabilities and the error term in repeated games with incomplete information on one side: there exists a positive constant C such that for all positive integers k and d with $1 < d \leq 2^k$, there are (1) a martingale of probabilities on a finite set with d elements $p_0^k : p_0, \dots, p_k$ with variation greater than $C\sqrt{k \log d}$ and (2) a repeated game with incomplete information on one side with an error term that is greater than $C\|G\|\sqrt{\log d}/\sqrt{k}$.

2 The Results

Let X be a finite (or countable) set. For $x \in X$, the x th coordinate of an element $q \in \mathbb{R}^X$ is denoted $q(x)$, and $\ell_1(X)$ is the (Banach) space of all elements $q \in \mathbb{R}^X$ with $\sum_{x \in X} |q(x)| < \infty$. Obviously, if X is a finite set, then $\ell_1(X)$ equals \mathbb{R}^X . The ℓ_1

¹This book is based on reports by Robert J. Aumann and Michael Maschler which appeared in the 1960s in *Report of the U.S. Arms Control and Disarmament Agency*. See "Game theoretic aspects of gradual disarmament" (1966, ST-80, Chap. V, pp. V1–V55), "Repeated games with incomplete information: a survey of recent results" (1967, ST-116, Chap. III, pp. 287–403), and "Repeated games with incomplete information: the zero-sum extensive case" (1968, ST-143, Chap. III, pp. 37–116).

norm of $q \in \ell_1(X)$ is $\|q\|_1 := \sum_{x \in X} |q(x)|$, and (thus) the ℓ_1 distance between two elements $p, q \in \ell_1(X)$ is the sum $\|p - q\|_1 = \sum_{x \in X} |p(x) - q(x)|$. A k -step $\ell_1(X)$ -valued martingale is a stochastic process $p_0^k = p_0, \dots, p_k$ where $p_t, 0 \leq t \leq k$, takes values in $\ell_1(X)$ and $E(p_t | p_0, \dots, p_{t-1}) = p_{t-1}$. Let $\Delta(X)$ denote all probabilities on X , i.e., all elements $p \in \mathbb{R}_+^X$ with $\sum_{x \in X} p(x) = 1$, and for $p \in \Delta(X)$ and a positive integer k , we denote by $\mathcal{M}_k(X, p)$ the set of all martingales p_0^k with $p_t \in \Delta(X)$ and $p_0 = p$.

The variation of the martingale p_0^k is denoted $V(p_0^k)$ and is defined by $V(p_0^k) = E(\sum_{t=1}^k \|p_t - p_{t-1}\|_1)$. Set

$$V(k, p) := \sup\{V(p_0^k) : p_0^k \in \mathcal{M}_k(X, p)\} \tag{1}$$

and

$$V(k, d) := \sup\{V(k, p) : p \in \Delta(X) \text{ and } |X| = d\}. \tag{2}$$

A trivial inequality is $V(k, p) \leq 2k$. A classical bound (that is used in the theory of repeated games with incomplete information; see [1, 3]) of $V(k, d)$ is

$$V(k, d) \leq \sqrt{k(d - 1)}.$$

This classical bound improves the trivial bound only for $d \leq 4k$. Our objective is to derive a meaningful bound that (1) is applicable also to $d > 4k$, and (2) such that its order of magnitude is the best possible one for large d . We have the following:

Theorem 1

$$V(k, p) \leq \sqrt{2kH(p)}$$

and thus

$$V(k, d) \leq \sqrt{2k \log d},$$

where $H(p) = -\sum_x p(x) \log p(x)$ is the entropy function, and \log stands for the natural logarithm.

As $V(k, p) \leq 2k$, the results of Theorem 1 are of interest for $H(p) \leq 2k$ and for $d \leq e^{2k}$. For large values of $d \leq e^{2k}$, the bound $\sqrt{2k \log d}$ is a significant improvement over the classical bound $\sqrt{k(d - 1)}$. Moreover, as there are probabilities p over a countable set X with finite entropy, the bound $\sqrt{2kH(p)}$ is applicable independently of the size of the set X .

One may wonder if the order of magnitude of each of the bounds, $\sqrt{2k \log d}$ and $\sqrt{2kH(p)}$, is the best possible. For $X = \{0, 1\}$ and $p(\alpha) = (\alpha, 1 - \alpha) \in \Delta(X)$, we have $V(k, p(\alpha)) \leq \sqrt{k\alpha(1 - \alpha)}$. Since $\alpha(1 - \alpha) = o(H(p(\alpha)))$ as $\alpha \rightarrow 0+$, the order of magnitude of the bound $\sqrt{kH(p)}$ is not tight. The next result demonstrates the tightness of the order of magnitude of the bound $\sqrt{2k \log d}$ for large values of d .² We have the following:

²I wish to thank Benjamin Weiss for raising the question of the tightness of the factor $\sqrt{\log d}$ in the bound and demonstrating for each positive ℓ the existence of a simple martingale of probabilities p_0^ℓ on a set

Theorem 2 *There is a positive constant $C > 0$ such that for all k and $d \leq 2^k$, there is $p_0^k \in \mathcal{M}_k(X, p_0)$ with $|X| = d$ such that*

$$V(p_0^k) \geq C\sqrt{k \log d}.$$

Bounds of the variation of martingales of probabilities are useful in the study of repeated games with incomplete information [1]. In a two-person zero-sum repeated game with incomplete information on one side (henceforth, RGII-OS) the players play repeatedly the same stage game G . However, the game depends on a state $x \in X$ known only to player 1 (P1), and x is chosen according to a probability $p \in \Delta(X)$ that is commonly known. In the course of the game player 2 (P2) may learn information about x only from past actions of player 1.

Formally, an RGII-OS Γ is defined by a state space X , a probability $p \in \Delta(X)$, finite sets of stage actions, I for P1 and J for P2, and for every $x \in X$, we have a two-person zero-sum $I \times J$ matrix game G^x . We write $\Gamma = \langle X, p, I, J, G \rangle$, where G stands for the list of matrix games $(G^x)_{x \in X}$. The (i, j) th entry of G^x , denoted $G^x_{i,j}$, is the payoff from P2 to P1 when in state x the players play the action pair (i, j) .

The k -stage repeated game, denoted $\Gamma_k(p)$, or Γ_k for short, is played as follows. Nature chooses $x \in X$ according to the probability p . P1 is informed of nature’s choice x , but P2 is not. At stage $1 \leq t \leq k$, P1 chooses $i_t \in I$, and simultaneously P2 chooses $j_t \in J$ (and these choices are observed by the players following the play in stage t). The choice of i_t may depend on $x, i_1, j_1, \dots, i_{t-1}, j_{t-1}$ (which is the information of P1 before the play at stage t), and the choice of j_t may depend on $i_1, j_1, \dots, i_{t-1}, j_{t-1}$ (which is the information of P2 before the play at stage t).

A pair of strategies σ of P1 and τ of P2 (together with the initial probability p) defines a probability distribution $P_{\sigma,\tau}^p$, or $P_{\sigma,\tau}$ for short, on the space of plays $x, i_1, j_1, \dots, i_k, j_k$, and thus on the stream of payoffs $g_t := G^x_{i_t, j_t}$. The (normalized) payoff of the k -stage repeated game is the average of the payoffs in the k -stages of the game, namely, $\bar{g}_k = \frac{1}{k} \sum_{t=1}^k g_t$. The minmax value of $\Gamma_k(p)$ is $v_k(p) := \max_{\sigma} \min_{\tau} E_{\sigma,\tau} \bar{g}_k$, where $E_{\sigma,\tau}$ stands for the expectation with respect to the probability $P_{\sigma,\tau}^p$, the maximum is over all mixed (or behavioral) strategies σ of P1, and the minimum is over all mixed (or behavioral) strategies τ of P2.

For fixed components $\langle X, I, J, G \rangle$, the minmax value of the matrix game $\sum_x q(x)G^x$ is a function of $q \in \Delta(X)$ and is denoted $u(q)$. The least concave function on $\Delta(X)$ that is greater than or equal to u (“smallest concave majorant”) is denoted $\text{cav } u$. Aumann and Maschler [1] proved that $v_k(p) \geq (\text{cav } u)(p)$ and that $v_k(p)$ converges to $(\text{cav } u)(p)$ as $k \rightarrow \infty$. Moreover, [1] shows that the bound of the variation of the martingale of probabilities bounds the (nonnegative) difference $v_k(p) - (\text{cav } u)(p)$. Explicitly, if $\|G\| := \max_{x,i,j} |G^x_{i,j}|$, we have

$$v_k(p) - (\text{cav } u)(p) \leq \|G\|V(k, p)/k. \tag{3}$$

with 2^ℓ elements and with variation ℓ . Specifically, starting with the uniform probability, in each stage half of the nonzero probabilities (each half equally likely) move to zero, and the other half double their probabilities. Therefore, for each fixed $\alpha > 0$, there is a positive constant $0 < C(\alpha) (\rightarrow_{\alpha \rightarrow 0+} 0)$ such that for k and d with $\alpha \leq \frac{\log_2 d}{k} \leq 1$, $V(k, d) \geq \lceil \log_2 d \rceil \geq C(\alpha)\sqrt{k \log d}$.

Inequality (3) yields on the one hand a rate of convergence of $v_k(p)$ and on the other hand enables us to approximate the value $v_k(p)$ for a specific k and a specific game. The classical bound of $V(k, p)$ that is used in [1, 3] and in subsequent works is

$$V(k, p) \leq V(k, d) \leq \sqrt{k(d - 1)}.$$

For $d > k$, this bound is not useful. Theorem 1 provides an effective bound when d is subexponential in k , namely, when $\log d = o(k)$, or, more generally, when $H(p)/k$ is small. Applying the bound in Theorem 1 to inequality (3) implies that

$$v_k(p) - (\text{cav } u)(p) \leq \|G\| \sqrt{\frac{2 \log d}{k}}. \tag{4}$$

One may wonder if the order of magnitude of the bound in (4) is tight. We have the following:

Theorem 3 *There is a positive constant C such that for all k and $d \leq 2^k$, there is a repeated game Γ with incomplete information on one side with $(\|G\| > 0)$ and d states such that*

$$v_k(p) - (\text{cav } u)(p) \geq C \|G\| V(k, d)/k. \tag{5}$$

3 Proofs

Proof of Theorem 1 Let p_0^k be $(\mathcal{H}_t)_t$ -adapted; that is, p_t is measurable with respect to the σ -algebra $\mathcal{H}_t \subset \mathcal{H}_{t+1}$. Without loss of generality we can assume that \mathcal{H}_t are finite, namely, algebras. (Indeed, if p_t is measurable with respect to the σ -algebra $\mathcal{H}_t \subset \mathcal{H}_{t+1}$, one replaces \mathcal{H}_t with an algebra $\mathcal{H}_t^* \subset \mathcal{H}_{t+1}^*$ such that $\|E(p_t | \mathcal{H}_t^*) - p_t\| \leq \varepsilon/k$ and replaces p_t with $\hat{p}_t := E(p_t | \mathcal{H}_t^*) (= E(p_k | \mathcal{H}_t^*))$. Note that $\sum_{t=1}^k \|\hat{p}_t - \hat{p}_{t-1}\| + 2\varepsilon \geq \sum_{t=1}^k \|p_t - p_{t-1}\|$.) In that case we can assume that: (1) P is a probability on the product $X \times (\times_{t=0}^k A_t)$, where A_t are finite sets (e.g., the atoms of the algebra \mathcal{H}_t); (2) $(x, a_0, a_1, \dots, a_k)$ is a vector of random variables having distribution P ; and (3) p_t is the conditional distribution of x given a_0, \dots, a_t . Let P_t be the conditional (joint) distribution of (x, a_t) given (a_0, \dots, a_{t-1}) , P_{tX} its marginal on X , and P_{tA_t} its marginal on A_t . Let $P_{tX} \otimes P_{tA_t}$ denote the product distribution on $X \times A_t$, i.e., $P_{tX} \otimes P_{tA_t}(x, a_t) = P_{tX}(x)P_{tA_t}(a_t)$. By Pinsker’s inequality (see, e.g., [2, p. 300]), we have

$$\|P_t - P_{tX} \otimes P_{tA_t}\| \leq \sqrt{2} \sqrt{D(P_t \| P_{tX} \otimes P_{tA_t})},$$

where for two probabilities P and Q on a finite (or countable) set Y , $\|P - Q\| = \sum_y |P(y) - Q(y)|$ and $D(P \| Q) = \sum_{y \in Y} P(y) \log \frac{P(y)}{Q(y)}$ (where \log denotes the natural logarithm, and $0 \log 0 = 0$).

Let $H_{P_t}(x) := -\sum_x P_t(x) \log P_t(x)$, $H_{P_t}(a_t)$, and $H_{P_t}(x, a_t)$ denote the entropies of the random variables x , a_t , and (x, a_t) , where (x, a_t) has distribution

P_t , and $H_{P_t}(x | a_t) := H_{P_t}(x, a_t) - H_{P_t}(a_t)$. A straightforward computation yields $D(P_t \| P_{tX} \otimes P_{tA_t}) = H_{P_t}(x) - H_{P_t}(x | a_t)$. Therefore,

$$\|P_t - P_{tX} \otimes P_{tA_t}\| \leq \sqrt{2} \sqrt{H_{P_t}(x) - H_{P_t}(x | a_t)}. \tag{6}$$

Note that P_t is a random variable, which is a function of a_0, \dots, a_{t-1} , and therefore, by the properties of conditional entropy, $E_P H_{P_t}(x) = H_P(x | a_0, \dots, a_{t-1})$ (where E_P denotes the expectation with respect to the probability distribution P) and $E_P H_{P_t}(x | a_t) = H_P(x | a_0, \dots, a_{t-1}, a_t)$. Therefore,

$$E_P (H_{P_t}(x) - H_{P_t}(x | a_t)) = H_P(x | a_0, \dots, a_{t-1}) - H_P(x | a_0, \dots, a_{t-1}, a_t).$$

Since the square root is a concave function, we have, by Jensen’s inequality,

$$E_P \|P_t - P_{tX} \otimes P_{tA_t}\| \leq \sqrt{2} \sqrt{H_P(x | a_0, \dots, a_{t-1}) - H_P(x | a_0, \dots, a_{t-1}, a_t)}.$$

Since $E_P(\|p_t - p_{t-1}\| | \mathcal{H}_{t-1})$ equals $\sum_{a \in A_t} P_{tA_t}(a) \sum_x | \frac{P_t(x,a)}{P_{tA_t}(a)} - P_{tX}(x) | = \sum_{a \in A_t} \sum_x |P_t(x, a) - P_{tA_t}(a) P_{tX}(x)| = \|P_t - P_{tX} \otimes P_{tA_t}\|$, we deduce that $E_P \|p_t - p_{t-1}\| = E_P \|P_t - P_{tX} \otimes P_{tA_t}\|$, and therefore by substituting $E_P \|p_t - p_{t-1}\|$ for $E_P \|P_t - P_{tX} \otimes P_{tA_t}\|$ we get

$$E_P \|p_t - p_{t-1}\| \leq \sqrt{2} \sqrt{H_P(x | a_0, \dots, a_{t-1}) - H_P(x | a_0, \dots, a_{t-1}, a_t)}.$$

Since the square root is a concave function, using Jensen’s inequality and the equality and inequality $\sum_{t=1}^k (H(x | a_0, \dots, a_{t-1}) - H(x | a_0, \dots, a_t)) = H(x) - H(x | a_0, \dots, a_k) \leq H(x)$, we have

$$E \sum_{t=1}^k \|p_t - p_{t-1}\| \leq \sqrt{2k} \sqrt{H(x)} \leq \sqrt{2k} \sqrt{\log d}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2 Note that $V(k, d)$ is increasing in d and k , and there is a positive constant $C_1 > 0$ such that $V(k, 2) \geq C_1 \sqrt{k}$.

If $p_0^{k_1}$ and $q_0^{k_2}$ are two martingales with total variation V_1 and V_2 , respectively, then $p_0 \otimes q_0, \dots, p_{k_1} \otimes q_0$ is a martingale with total variation V_1 , and $p_{k_1} \otimes q_0, p_{k_1} \otimes q_1, \dots, p_{k_1} \otimes q_{k_2}$ is a martingale with total variation V_2 , and therefore $p_0 \otimes q_0, \dots, p_{k_1} \otimes q_0, p_{k_1} \otimes q_1, \dots, p_{k_1} \otimes q_{k_2}$ is a martingale with total variation $V_1 + V_2$. Therefore,

$$V(k_1, p) + V(k_2, q) \leq V(k_1 + k_2, p \otimes q), \tag{7}$$

from which it follows that

$$V(k_1, d_1) + V(k_2, d_2) \leq V(k_1 + k_2, d_1 d_2). \tag{8}$$

Inequality (8) implies that if k is a multiple of ℓ , we have $V(k, 2^\ell) \geq \ell V(k/\ell, 2) \geq \ell C_1 \sqrt{k/\ell} = C_1 \sqrt{k} \ell$. Note that for all k and $2 \leq d \leq 2^k$, there is $k \geq k_1 > k/2$ that

is a multiple of $\ell = \lceil \log_2 d \rceil \geq (\log_2 d)/2$ (where $\lceil x \rceil$ is the largest integer that is less than or equal to x), and therefore $V(k, d) \geq V(k_1, 2^\ell) \geq C_1 \sqrt{k_1 \ell} \geq C_1/2 \sqrt{k \log_2 d}$. This completes the proof of Theorem 2. \square

Proof of Theorem 3 Given two repeated games with incomplete information on one side, $\Gamma^1 = \langle X_1, p_1, I_1, J_1, G^1 \rangle$ and $\Gamma^2 = \langle X_2, p_2, I_2, J_2, G^2 \rangle$, we define the game $\Gamma = \Gamma_1 \otimes \Gamma_2$ by

$$\Gamma = \langle X = X_1 \times X_2, p = p_1 \otimes p_2, I = I_1 \times I_2 \times \{1, 2\}, J = J_1 \times J_2, G \rangle,$$

where for $x = (x_1, x_2)$, $i = (i^1, i^2, b)$, and $j = (j^1, j^2) \in J$,

$$G_{i,j}^x = G_{i^b, j^b}^{x_b},$$

where G^{x_b} stands for the more explicit G^{b, x_b} . Note that $\|G\| = \max(\|G^1\|, \|G^2\|)$.

A possible helpful interpretation of Γ is that nature chooses a pair $x_1 \in X_1$ and $x_2 \in X_2$, equivalently a pair of games G^{x_1} and G^{x_2} , according to the product probability $p_1 \otimes p_2$. P1 is informed of the choice (G^{x_1}, G^{x_2}) of nature, but P2 is not. In each stage of the repeated game, both players select strategies for the first and for the second game, and P1 chooses in addition which one of the two games determines the stage payoff.

As a function of $i = (i^1, i^2, b)$, for each fixed $b = 1, 2$, the payoff function $G_{i,j}^x$ does not depend on the coordinate i^c for $c \neq b$. Therefore we can replace the set I (which has $2|I_1||I_2|$ elements) of stage actions of P1 in the repeated game Γ with the disjoint union of I_1 and I_2 .

Note that if v_k^b and v_k stand for the (normalized) values of the k -stage repeated games Γ^b and Γ , then

$$v_{k_1+k_2} \geq \frac{k_1 v_{k_1}^1 + k_2 v_{k_2}^2}{k_1 + k_2}.$$

Indeed, P1 can play $b_t = 1$ in stages $t = 1, \dots, k_1$ and $b_t = 2$ in stages $t = k_1 + 1, \dots, k_1 + k_2$, and the first coordinates i_t^1 of i_t follow, in stages $t = 1, \dots, k_1$, an optimal strategy of P1 in $\Gamma_{k_1}^1(p_1)$, and the second coordinates i_t^2 of i_t follow, in stages $t = k_1 + 1, \dots, k_1 + k_2$, an optimal strategy of P1 in $\Gamma_{k_2}^2(p_2)$.

For $\ell > 2$ and a sequence $\Gamma^1 = \langle X_1, p_1, I_1, J_1, G^1 \rangle, \dots, \Gamma^\ell = \langle X_\ell, p_\ell, I_\ell, J_\ell, G^\ell \rangle$ of RGII-OS, we define by induction on ℓ the game $\Gamma = \bigotimes_{b=1}^\ell \Gamma^b$ by $\Gamma = (\bigotimes_{b=1}^{\ell-1} \Gamma^b) \otimes \Gamma^\ell$.

If v_k^b , respectively v_k , denotes the normalized value of the k -stage repeated game $\Gamma_k^b(p_b)$, respectively $\Gamma_k(\bigotimes_{b=1}^\ell p_b)$, and $k = k_1 + \dots + k_\ell$, then

$$v_k \geq \frac{\sum_{b=1}^\ell k_b v_{k_b}^b}{k}.$$

Note that a stage action of P1 in Γ is a list of stage actions i^1, \dots, i^ℓ (with $i^b \in I_b$) and a number b (with $1 \leq b \leq \ell$). However, given b , the payoff depends only on the

coordinate i^b of the stage actions. Therefore we can replace the stage actions of P1 in Γ with the disjoint union of the action sets I_b , and so with a set of size $\sum_b |I_b|$.

Consider the example of the RGII-OS $\Gamma^z = \langle X = \{0, 1\}, (1/2, 1/2), I, J, G \rangle$, introduced by Zamir [5, Sect. 3]. The set of states is $X = \{0, 1\}$, and players' action sets are $I = \{0, 1\}$ for P1 and $J = \{0, 1\}$ for P2. The two payoff matrices are G^0 and G^1 : $G_{0,0}^0 = 3, G_{0,1}^0 = -1, G_{0,0}^1 = 2 = -G_{0,1}^1$, and $G_{i,j}^* = -G_{1-i,j}^*$. Let v_k^z denote the normalized value of the k -stage repeated game Γ^z . Zamir [5] shows that $\lim_n v_n^z = 0$ and $v_k^z \geq C_1/\sqrt{k}$, where $C_1 > 0$ is a positive constant.

Consider the RGII-OS $\Gamma = \otimes_{b=1}^\ell \Gamma^z$, and let v_k denote the normalized value of the k -stage repeated game Γ . It follows that

$$v_k \geq \max \left\{ \frac{\sum_{b=1}^\ell k_b v_{k_b}^z}{k} : k_b \geq 0 \text{ and } \sum_{b=1}^\ell k_b = k \right\},$$

and therefore if k is a multiple of ℓ , we can take $k_b = k/\ell$, and therefore

$$v_k \geq v_{k/\ell}^z \geq C_1 \sqrt{\ell/k}.$$

For arbitrary k and $d \leq 2^k$, there is $k \geq k_1 > k/2$ that is a multiple of $\ell := \lceil \log_2 d \rceil$ ($\geq (\log_2 d)/2$). As P1 can play $(1/2, 1/2)$ in the last $k - k_1$ stages, $kv_k \geq k_1 v_{k_1}$, and thus $kv_k \geq C_1 \sqrt{k_1} \sqrt{\ell}$. Therefore, $v_k \geq \frac{C_1}{2\sqrt{k}} \sqrt{\log d}$.

Finally, the existence of an optimal strategy of P2 in the infinitely repeated game Γ^z (or a direct computation of the function $u(p)$, the minmax value of the game $\sum_x p(x)G^x$, for the game Γ) yields $\lim_n v_n = 0$. Note that the stage payoffs of the RGII-OS Γ are bounded by 3 (independent of the number of factors ℓ). Altogether, we have constructed for any k and $d \leq 2^k$ a repeated game $\Gamma = \langle X, p, I, J, G \rangle$ with $|X| \leq d$, equivalently $|X| = d$ ($|I| \leq 2 \log d$) and $\|G\| = 3$, and

$$v_k - \lim_{n \rightarrow \infty} v_n \geq C_1/2\sqrt{\log d/\sqrt{k}}.$$

This completes the proof of Theorem 3. □

4 Remarks

4.1 Comments on the Proof of Theorem 1

Our proof of Theorem 1 relies on Pinsker's inequality, and it uses information-theoretic tools. In fact, the information-theoretic intuition has led us to the result and its proof. However, readers unfamiliar with the information-theoretic concepts may find the proof obscure. The following is an alternative derivation (which disguises the use of the information-theoretic techniques) and uses classical martingale theory techniques. First, note that Pinsker's inequality implies that if Z and Y are two nonnegative random variables with $EZ = EY$, then $E|Z - Y| \leq \sqrt{2EZ} \sqrt{EZ \log Z - EZ \log Y}$. In particular, if $E(Z | Y) = Y$ (e.g., when Y is

the constant random variable $Y = EZ$, then $EZ \log Y = E(E(Z \log Y | Y)) = EY \log Y$, and therefore

$$E|Z - Y| \leq \sqrt{2EZ} \sqrt{EZ \log Z - EZ \log Y} = \sqrt{2EZ} \sqrt{EZ \log Z - EY \log Y}. \tag{9}$$

Inequality (9), which is equivalent to Pinsker’s inequality, can obviously be proved directly.³ The continuation of the proof avoids the (explicit) use of information-theoretic techniques.

It follows from inequality (9) that if Z_0, \dots, Z_k is a martingale of nonnegative random variables, then

$$\sum_{t=1}^k |Z_t - Z_{t-1}| \leq \sqrt{2EZ_0} \sum_{t=1}^k \sqrt{EZ_t \log Z_t - EZ_{t-1} \log Z_{t-1}},$$

which by Jensen’s inequality, the concavity of the square root, and the telescopic feature of the series $EZ_t \log Z_t - EZ_{t-1} \log Z_{t-1}$, is less than or equal to

$$\sqrt{2kEZ_0} \sqrt{EZ_k \log Z_k - EZ_0 \log Z_0}.$$

Therefore, if p_0, \dots, p_k is a martingale with values in \mathbb{R}_+^X , we have

$$\sum_{t=1}^k E \|p_t - p_{t-1}\|_1 \leq \sum_{x \in X} \sqrt{2kE p_0(x)} \sqrt{E p_k(x) \log p_k(x) - E p_0(x) \log p_0(x)}.$$

By the Schwarz inequality we obtain that

$$\sum_{t=1}^k E \|p_t - p_{t-1}\| \leq \sqrt{2kE \sum_{x \in X} p_0(x)} \sqrt{E \sum_{x \in X} (p_k(x) \log p_k(x) - p_0(x) \log p_0(x))}.$$

If $p_k(x) \leq M(x)$, then by the convexity of $q \log q$ we have $E p_k(x) \log p_k(x) \leq E p_0(x) \log M(x)$, and then

$$\sum_{t=1}^k E \|p_t - p_{t-1}\|_1 \leq \sqrt{2kE \sum_{x \in X} p_0(x)} \sqrt{\sum_{x \in X} -E p_0(x) \log(p_0(x)/M(x))}.$$

If $p_k(x) \leq 1$, then $p_k(x) \log p_k(x) \leq 0$, and therefore

$$\sum_{t=1}^k E \|p_t - p_{t-1}\|_1 \leq \sqrt{2kE \sum_{x \in X} p_0(x)} \sqrt{\sum_{x \in X} -E p_0(x) \log p_0(x)}.$$

We conclude that if $\sum_x p_0(x) = 1$, then $\sum_{t=1}^k E \|p_t - p_{t-1}\|_1 \leq \sqrt{2k} \sqrt{EH(p_0)}$.

³I wish to thank Stanislaw Kwapien for suggesting a proof that avoids the information-theoretic techniques and communicating a simple analytical proof of the above displayed version of Pinsker’s inequality.

4.2 The Variation of a Martingale of Probabilities Over a Countable Set

It is of interest to find a necessary and sufficient condition for a distribution p on a countable set X for $\sup_k \frac{1}{\sqrt{k}} V(k, p) < \infty$. We remark here on the sufficient conditions derived from the classical method and our method of bounding the variation of martingales of probabilities.

The classical bound of the variation of a martingale p_0^k is obtained by bounding, for each fixed $x \in X$, the expectation variation $E\|y(x)\|_1$, where $y(x) \in \mathbb{R}^k$ is the vector of martingale differences $(p_1(x) - p_0(x), \dots, p_k(x) - p_{k-1}(x))$ (thus $\|y(x)\|_1 = \sum_{t=1}^k |p_t(x) - p_{t-1}(x)|$), and summing over all $x \in X$. Assuming without loss of generality that p_0 is a constant $p \in \Delta(X)$, we have (by the Cauchy–Schwarz inequality) $\|y(x)\|_1 \leq \sqrt{k}\|y(x)\|_2$, and, therefore, by Jensen’s inequality, $E\|y(x)\|_1 \leq \sqrt{k}\sqrt{E\|y(x)\|_2^2}$. By the martingale property, $E\|y(x)\|_2^2 = E((p_k(x))^2 - (p_0(x))^2) = E((p_k(x) - p_0(x))^2)$, which is less than or equal to $p_0(x) - (p_0(x))^2$. Therefore,

$$E\|y(x)\|_1 \leq \sqrt{k}\sqrt{p_0(x) - (p_0(x))^2}.$$

Therefore, if $p \in \Delta(X)$ and X is countable, the classical method yields that $\sup_k \frac{1}{\sqrt{k}} V(k, p) < \infty$ whenever $\sum_x \sqrt{p(x)} < \infty$. Since $-q \log q = o(\sqrt{q})$ as $q \rightarrow 0+$, the condition $\sum_x \sqrt{p(x)} < \infty$ implies that $H(p) = -\sum_x p(x) \log p(x) < \infty$. Obviously, there are probabilities p over a countable set X such that $H(p) < \infty$ but $\sum_x \sqrt{p(x)} = \infty$. Therefore our bound provides a strictly sharper sufficient condition, $H(p) < \infty$, for $\sup_k \frac{1}{\sqrt{k}} V(k, p) < \infty$, compared to the one derived by using the classical method.

4.3 The Asymptotic Behavior of $V(k, d)$

The asymptotic behavior of $V(k, d)$ deserves further study. Mertens and Zamir [4] prove that $V(k, 2)/\sqrt{k}$ converges as $k \rightarrow \infty$ to $\sqrt{\frac{2}{\pi}}$. It is of interest to find a corresponding limit theorem for $V(k, d)/\sqrt{k \log d}$ as $2^k \geq d \rightarrow \infty$. The above-mentioned result of [4], together with our construction in the proof of Theorem 2, yields that the \liminf of $V(k, d)/\sqrt{k \log_2 d}$ is $\geq \sqrt{\frac{2}{\pi}}$ as $\frac{\log d}{k} + 1/d \rightarrow 0$ (namely, as $\log d = o(k)$ and $d \rightarrow \infty$).

4.4 Repeated Games with Incomplete Information

The proof of Theorem 3 constructs for each $d \leq 2^k$ an RGII-OS $\Gamma = \langle X, p, I, J, G \rangle$ with $|X| = d$ and $v_k \geq \lim_n v_n + C\sqrt{k^{-1} \log d}$, where $0 < C = O(\|G\|)$, and, in addition, $|I| = O(\log d)$ and $|J| = O(d)$. We have not tried to minimize the order of magnitude of the number of elements of I and J . It is however impossible to construct such an example with bounded $|I|$ and $|J|$. Indeed, in a forthcoming note we will show that for every RGII-OS $\Gamma = \langle X, p, I, J, G \rangle$, we have $v_k \leq \lim_n v_n + \|G\|_* \sqrt{\frac{2|I \times J| \log k}{k}}$, where $\|G\|_* := 2E_p(\max_{i,j} G_{i,j}^x - \min_{i,j} G_{i,j}^x)$. Therefore the inequality $v_k \geq \lim_n v_n + C\sqrt{k^{-1} \log d}$ is possible only if $|I \times J| \geq C \frac{\log d}{2 \log k}$.

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