MATHEMATICS OF OPERATIONS RESEARCH
Vol. 34, No. 2, May 2009, pp. 257–262
ISSN 0364-765X | EISSN 1526-5471 | 09 | 3402 | 0257

DOI 10.1287/moor.1080.0372

Absorbing Games with Compact Action Spaces

Jean-François Mertens
CORE, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium,
jfm@core.ucl.ac.be

Abraham Neyman
Institute of Mathematics and Center for the Study of Rationality, The Hebrew University of Jerusalem, Givat Ram,
91904 Jerusalem, Israel, aneyman@math.huji.ac.il

Dinah Rosenberg
Université Paris 13, CNRS, F-93 430 Villetaneuse, France, and
Laboratoire d'Econométrie de l'École Polytechnique, 75005 Paris, France, dinah@math.univ-paris13.fr

We prove that games with absorbing states with compact action sets have a value.

Key words: games; stochastic games; zero-sum; minimax; absorbing states; value
MSC2000 subject classification: Primary: 91A15; secondary: 91A05
OR/MS subject classification: Primary: games-group decisions/stochastic
History: Received July 2, 2007; revised May 7, 2008 and October 27, 2008. Published online in *Articles in Advance* April 10, 2009.

1. Introduction. Stochastic games are Markov decision processes in which the transitions of the state are controlled by the actions of the decision makers. In a stochastic game, the players interact repeatedly. At each stage, the players observe the current state, then choose an action independently, and then are informed of the chosen actions. According to these actions and the current state, the chain moves to a new state that is observed by all players. The stage payoff is a function of the current state and the actions chosen. We focus on two-player zero-sum stochastic games.

Stochastic games were introduced by Shapley [13]. He proved the existence of the value of λ-discounted two-player zero-sum stochastic games with finitely many states and actions.

In the case where the sets of actions and of states are finite, the existence of the limit: say, v, of the values of the λ-discounted games as λ goes to 0 (i.e., as players become more patient) was proved in Bewley and Kohlberg [1], using an algebraic argument. It is proved in Mertens and Neyman [8] that v is a (uniform) value: for each $\varepsilon > 0$, there exists $N(\varepsilon)$ and $\lambda(\varepsilon)$ and each of the players has a strategy that (1) guarantees him a payoff of v up to an error of ε in any n-stage or λ-discounted game provided $n \geq N(\varepsilon)$ or $\lambda \leq \lambda(\varepsilon)$, and (2) the strategy of player 1 (respectively, player 2) guarantees that the expectation of the $\liminf$ (respectively, $\limsup$) of the average payoff in the first n-stages as n goes to infinity is at least $v - \varepsilon$ (respectively, at most $v + \varepsilon$).

In the case of stochastic games with finite-state space but with compact action sets, there is no general result ensuring the convergence of the values of the n-stage or λ-discounted games (and a fortiori none ensuring the existence of the (uniform) value).

An absorbing state of a stochastic game is a state such that for any profile of actions chosen by the players, the state remains the same almost surely. An *absorbing game* is a stochastic game in which all states but one are absorbing. Note that in an absorbing state, the game is reduced to a standard repeated game, where the value exists and is equal to the value of the one-shot game. Therefore, in an absorbing game, there is only one relevant state, the nonabsorbing one. This simplifies the study because in such a game there is at most one transition between states. The first example of such games was studied by Blackwell and Ferguson [2]. Kohlberg [3] proved the existence of the value for (two-person zero-sum) absorbing games with finite-action sets (now seen as a particular case of Mertens and Neyman [8]).

For absorbing games with compact action sets, the algebraic approach of Bewley and Kohlberg [1] does not apply and convergence of the values of the λ-discounted games was proved in Rosenberg and Sorin [11] using an operator approach.

In this paper, we focus on absorbing games and prove that the value of absorbing games with compact action sets exists. The proof relies on Mertens and Neyman [8] and on the characterization of the limit of the values of the λ-discounted games provided in Rosenberg and Sorin [11].

2. The model. Here, a *stochastic game* is a two-player zero-sum game determined by:

• Three sets: S (the set of states), I (the set of actions of player 1), and J (the set of actions of player 2). We will assume throughout that I and J are compact metric; we will see that when focusing on absorbing games, we can assume without loss of generality that S is finite.

• A bounded payoff function g: $S \times I \times J \to \mathbb{R}$ that is separately continuous. Note that this implies measurability (see Mertens et al. [9, I.1.Ex.7a]).

• A transition probability q from $S \times I \times J$ to S, where $q(z' \mid z, i, j)$ denotes the probability of reaching state z' from state z given the pair of actions (i, j), where q is separately continuous on $I \times J$. Note that this implies measurability (see Mertens et al. [9, I.1.Ex.7a]).

• An initial state $z_1 \in S$.

The game is played in stages. At each stage $n \in \mathbb{N}$, players 1 and 2 choose an action, $i_n \in I$ and $j_n \in J$, knowing the whole past history, including current state z_n. Then, the current payoff is $g_n = g(z_n, i_n, j_n)$, and $q(z \mid z_n, i_n, j_n)$ is the conditional probability, given $z_1, i_1, j_1, \ldots, z_{n-1}, i_{n-1}, j_{n-1}$, that the next state, z_{n+1}, is z. We denote by h_n the history up to stage n; more precisely $h_n = (z_1, i_1, j_1, \ldots, z_{n-1}, i_{n-1}, j_{n-1}, z_n)$. Let H_n denote the set of histories up to stage n, and H the set of infinite-length histories. Let $\mathcal{H}_n$ be the σ-algebra on H induced by histories h_n (of length n) and $\mathcal{H}_\infty$ be the σ-algebra spanned by $\bigcup_n \mathcal{H}_n$.

Thus, a player's behavioral strategy (σ of player 1, τ of player 2) specifies a probability distribution over his actions at each stage conditional on the current state and the past history. A pair (σ, τ) induces a probability $\mathrm{P}_{\sigma,\tau}$ on histories $(H, \mathcal{H}_\infty)$. The corresponding expectation is denoted by $\mathrm{E}_{\sigma,\tau}$.

NOTATION 2.1. *We use* $\|\cdot\|$ *for the* sup*-norm, and let* $A = \|g\| := \sup_{z,i,j} |g(z, i, j)|$.

We are interested in the existence of infinite-game strategies that guarantee a given payoff in all sufficiently long games, as well as in the infinite undiscounted game.

DEFINITION 2.1. Player 1 can guarantee $v \in \mathbb{R}^S$ if for every $\delta > 0$ and $z_1 \in S$, there is a strategy σ of player 1 and $N > 0$ such that for any strategy τ of player 2,

$$\mathrm{E}_{\sigma,\tau}\left[\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} g_k\right] \geq v(z_1) - \delta$$

$$\forall\, n \geq N, \quad \mathrm{E}_{\sigma,\tau}\left[\frac{1}{n}\sum_{k=1}^{n} g_k\right] \geq v(z_1) - \delta.$$

Player 2 can guarantee $v \in \mathbb{R}^S$ if for every $\delta > 0$ and $z_1 \in S$, there is a strategy τ of player 2 and $N > 0$ such that for any strategy σ of player 1,

$$\mathrm{E}_{\sigma,\tau}\left[\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} g_k\right] \leq v(z_1) + \delta$$

$$\forall\, n \geq N, \quad \mathrm{E}_{\sigma,\tau}\left[\frac{1}{n}\sum_{k=1}^{n} g_k\right] \leq v(z_1) + \delta.$$

A stochastic game has a value $v \in \mathbb{R}^S$ if both players can guarantee v.

All stochastic games with finite state space S and finite action sets I and J have a value.[1] It is unknown whether all stochastic games with finite state space and compact action sets have a value. This paper focuses on the special class of absorbing games. It proves that all absorbing games have a value.

DEFINITION 2.2. An absorbing state is a state $z \in S$ such that for all $i \in I$ and all $j \in J$, $q(z \mid z, i, j) = 1$. An absorbing game is a stochastic game in which all but one of the states are absorbing.

Absorbing games with finite-action sets I and J were studied in Kohlberg [3]. We are going to study absorbing games with compact action sets.

REMARK 2.1. Both players know the current state at each stage of the game, and so they know as soon as an absorbing state z is hit. By the definition of an absorbing state, the absorbing state will never be left. The rest of the game is therefore a repeated zero-sum game, with value $v(z)$. Let us now define a new absorbing game with three states that has the same value as the original and the same optimal strategy in the nonabsorbing state. The new absorbing game has one nonabsorbing state z^0, with the same payoff function as the original game, and two absorbing states z^+ and z^-, with constant payoff A and $-A$. The transitions from the nonabsorbing state z^0 are modified by replacing the transition to an absorbing state z in the original game by transitions with probabilities $(1 \pm A^{-1}v(z))/2$ to z^+ and z^-. Let (σ, τ) be a pair of strategies of the players in the original game that play

[1] This result is not a consequence of the one proved in Maitra and Sudderth [5, 6] for the weaker lim sup value; see Neyman [10, §§1.2–1.4]. In fact, there are stochastic games where the lim sup value and lim inf value exist and the uniform value does not exist, and there are stochastic games where all these different values exist and differ from each other.

repeatedly an optimal strategy in any one-shot game corresponding to an absorbing state. The payoff associated to (σ, τ) is the same as the payoff in the auxiliary game obtained by playing in z^0 $(\sigma(z^0), \tau(z^0))$. Similarly, for any pair of strategies (σ', τ') in the auxiliary game, define (σ, τ) to be the pair of strategies in the original game that follow (σ', τ') in the nonabsorbing state and that play repeatedly the one-shot game optimal strategy in any absorbing state. Again, the payoff induced by (σ', τ') in the auxiliary game is equal to the payoff induced by (σ, τ) in the original game. Thus, without loss of generality, we need at most 3 states in an absorbing game: z^0 (the nonabsorbing state), z^+, and z^-.

This note proves the following result.

THEOREM 2.1. *Absorbing games have a value.*

3. The proof.

3.1. Reminder of λ-discounted games.

NOTATION 3.1. *For a compact set X, $\Delta(X)$ denotes the set of probability distributions over X.*

For $\lambda \in (0, 1)$, the λ-discounted payoff function is

$$\mathrm{E}_{\sigma,\tau}\left[\sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} g_k\right].$$

The value $v_\lambda(z_1)$ and stationary optimal strategies (x_λ, y_λ) (of the λ-discounted game) exist (Mertens et al. [9, Chapter VI, Proposition 1.4]), and v_λ is characterized by

$$v_\lambda = T(\lambda, v_\lambda), \tag{1}$$

where for $u \in \mathbb{R}^S$,

$$T(\lambda, u)(z) = \max_{x\in\Delta(I)} \min_{y\in\Delta(J)} \mathrm{E}_{x,y,q}[\lambda g(z, i, j) + (1-\lambda)u(\tilde{z})], \tag{2}$$

where $\mathrm{E}_{x,y,q}$ is the expectation operator where, independently, i and j are distributed according to x and y, and then $\tilde{z}$ is distributed according to $q(\tilde{z} \mid z, i, j)$.

3.2. Reminder of the Mertens-Neyman [8] theorem. The proof in §2 of Mertens and Neyman [8] proves the following[2] theorem.

THEOREM 3.1. *If $\lambda \mapsto w_\lambda \in \mathbb{R}^S$ is a function defined on $(0, 1)$ with*

$$\|w_\lambda - w_{\bar\lambda}\| \le \int_\lambda^{\bar\lambda} \psi(x)\,dx \quad \textit{for all } 0 < \lambda, \bar\lambda < 1, \tag{3}$$

where ψ: $(0, 1] \to \mathbb{R}_+$ is integrable, and for every $\lambda \in (0, 1]$ sufficiently small, we have

$$T(\lambda, w_\lambda) \ge w_\lambda, \tag{4}$$

then player 1 can guarantee $\lim_{\lambda\to 0+} w_\lambda$.

[2] In Mertens and Neyman [8, §2], the function v_λ indeed stands for the value of the λ-discounted game, and thus condition (4) of Theorem 3.1 (equivalently, Mertens and Neyman [8, Inequality 2.1]) and Mertens and Neyman [8, Inequality 2.2] follow. These two conditions are the only use in Mertens and Neyman [8, §2] of the fact that v_λ is the value of the λ-discounted game. Other examples where the proof in Mertens and Neyman [8] is applied to functions v_λ that are not necessarily the values of the discounted games that appear in Neyman [10] and Rosenberg et al. [12].

3.3. The auxiliary function w_λ. Recall that we can assume w.l.o.g. that the absorbing game has three states: z^0, z^+, and z^-. Fixing in (2) $u(z^+)$ and $u(z^-)$ at the values A and $-A$, one gets a map $T(\lambda, \cdot)\colon \mathbb{R} \to \mathbb{R}$; this map is the restriction of $T(\lambda, .)$ to vectors that take values A and $-A$ at their z^+ and z^- coordinates; unless otherwise specified, $T(\lambda, .)$ will now denote this restriction. Then, by Rosenberg and Sorin [11], v_λ converges to v; note that $v(z^+) = A$, $v(z^-) = -A$, and therefore v is characterized by the real value $v(z^0)$. In the following w is a function on S such that $w(z^+) = A$, $w(z^-) = -A$. Then, v is characterized by the following equations:

$$
\begin{aligned}
&T(0, v) = v \qquad \text{and}\\
&\lim_{\lambda\to 0}(T(\lambda, w)(z^0) - w(z^0))/\lambda < 0 \quad \text{for } w(z^0) > v(z^0)\\
&\lim_{\lambda\to 0}(T(\lambda, w)(z^0) - w(z^0))/\lambda > 0 \quad \text{for } w(z^0) < v(z^0).
\end{aligned} \tag{5}
$$

Note that by definition of w, it is always the case that $T(\lambda, w)(z^+) = w(z^+)$ and similarly for z^-.

Take $\varepsilon > 0$. Our goal is to apply the previous theorem with $w_\lambda = v_\varepsilon$ (for any λ) defined by $v_\varepsilon(z) = v(z) - \varepsilon \mathbf{1}_{z=z^0}$.

3.4. Proof of Theorem 2.1. First, we prove that w_λ $(=v_\varepsilon)$ satisfies the conditions of Theorem 3.1. As w_λ is independent of λ the function $\lambda \mapsto w_\lambda$ satisfies condition (3) with the function $\psi(\lambda) = 0$. Condition (4) holds trivially for $z_1 = z^+$ and for $z_1 = z^-$. It remains to prove that the condition holds for $z_1 = s^0$. However, Equation (5) implies that for λ small enough,

$$T(\lambda, v_\varepsilon)(z^0) \ge v_\varepsilon(z^0). \tag{6}$$

Therefore, for every $\varepsilon > 0$, player 1 can guarantee v_ε, and therefore player 1 can guarantee v. This completes the proof that continuous absorbing games have a value.

4. An explicit strategy. For the sake of completeness, in this section, we construct an explicit strategy, based on the construction in Mertens and Neyman [8, §2] and using the auxiliary function $w_\lambda = v_\varepsilon$ that obeys inequality (6).

Fix $\varepsilon > 0$, and let $\lambda_0 > 0$ be sufficiently small, so that $T(\lambda, v_\varepsilon) \ge v_\varepsilon$ for all $0 < \lambda < \lambda_0$. The correspondence (from $(0, \lambda_0)$ to $\Delta(I)$)

$$\lambda \mapsto \bigcap_{y\in\Delta(J)} \{x \in \Delta(I)\colon\ \mathrm{E}_{x,y,q}(\lambda g(z, i, j) + (1-\lambda) v_\varepsilon(\tilde{z})) \ge v_\varepsilon(z)\}$$

has nonempty and closed values, and therefore (by e.g., Kuratowski and Ryll-Nardzewski [4]) it has a measurable[3] selection $\lambda \mapsto x(\lambda)$.

We define a sequence $(\lambda_k)_{k=1}^\infty$, where $0 < \lambda_k < \lambda_0$ is a function of the past history, i.e., measurable w.r.t. the σ-algebra $\mathscr{H}_k$ of all events preceding time k (including the choice of a new state z_k after the play at time $k-1$). The $(\lambda_k)_{k=1}^\infty$-strategy of player 1 is to play on time k the mixed action $x_k = x(\lambda_k)$. It follows that for the $(\lambda_k)_{k=1}^\infty$-strategy (of player 1), σ, inequality (2.1) of Mertens and Neyman [8],

$$\mathrm{E}_{\sigma,\tau}\big[w_{\lambda_k}(z_{k+1}) - w_{\lambda_k}(z_k) + \lambda_k(g_k - w_{\lambda_k}(z_{k+1}))\,\big|\,\mathscr{H}_k\big] \ge 0 \tag{7}$$

holds for any strategy τ of player 2 and with $w_{\lambda_k}(z) := v(z) - \varepsilon \mathbf{1}_{z=z_1}$, where $z_1 = z^0$ is the initial nonabsorbing state. We now define precisely how the sequence λ_k is to be defined.

Let $M > 1/\varepsilon$ be a sufficiently large constant such that (6) holds for $\lambda < 1/M^2$ and $(6A)^2/M < \varepsilon$, and thus for all $s_k, s_{k+1} \ge M$ with $|s_{k+1} - s_k| \le 6A$, we have

$$\left|(s_{k+1} - s_k)\left(\frac{s_k}{s_{k+1}} - 1\right)\right| < 2\varepsilon. \tag{8}$$

The computation of λ_k in stage k is done inductively as follows. Define inductively (as in Mertens and Neyman [8, §2]), $s_{k+1} = \max[M, s_k + g_k - w_{\lambda_k}(z_{k+1}) + 4\varepsilon]$ and $\lambda_k = 1/s_k^2$, starting with $s_1 \ge M$ arbitrary.

[3] We thank John Levy for pointing out the omission of this measurability requirement in an earlier draft.

We now prove that such a strategy is indeed optimal. Fix any strategy τ of player 2. All expectations are taken with respect to the probability induced on histories by the previously defined strategy of player 1, τ, and the transition probabilities. Set $Y_k = w_{\lambda_k}(z_k) - 1/s_k$ $(=v_\varepsilon - 1/s_k)$. We have

$$\begin{aligned} Y_{k+1} - Y_k &= w_{\lambda_k}(z_{k+1}) - w_{\lambda_k}(z_k) + \lambda_k(g_k - w_{\lambda_k}(z_{k+1})) \\ &\quad - 1/s_{k+1} + 1/s_k - \lambda_k(g_k - w_{\lambda_k}(z_{k+1})), \end{aligned}$$

and therefore, using (7), we have

$$\begin{aligned} \mathrm{E}(Y_{k+1} - Y_k \mid \mathcal{H}_k) &\geq \mathrm{E}\big(1/s_k - 1/s_{k+1} - \lambda_k(g_k - w_{\lambda_k}(z_{k+1})) \mid \mathcal{H}_k\big) \\ &\geq \mathrm{E}\left(\frac{s_{k+1} - s_k}{s_k s_{k+1}} - \lambda_k(s_{k+1} - s_k - 4\varepsilon) \,\middle|\, \mathcal{H}_k\right) \\ &= \mathrm{E}\left(4\varepsilon\lambda_k - \lambda_k(s_{k+1} - s_k)\left(\frac{s_k}{s_{k+1}} - 1\right) \,\middle|\, \mathcal{H}_k\right) \\ &\geq 2\varepsilon\lambda_k, \end{aligned}$$

where the second inequality follows from $s_{k+1} - s_k \geq g_k - w_{\lambda_k}(z_{k+1}) + 4\varepsilon$, and the last inequality follows from (8).

Because $\lambda_k \geq 0$, the inequality $\mathrm{E}(Y_{k+1} - Y_k \mid \mathcal{H}_k) \geq 2\varepsilon\lambda_k$ implies that Y_k is a submartingale. Obviously, Y_k is bounded and thus it converges a.s., say to Y_∞, with $\mathrm{E}(Y_\infty \mid \mathcal{H}_1) > Y_1$. It follows that $4A \geq 2A + 2/M \geq \mathrm{E}(Y_k - Y_1) \geq 2\varepsilon\,\mathrm{E}(\sum_{1 \leq l < k} \lambda_l)$. Thus, by the monotone convergence theorem,

$$\mathrm{E}\left(\sum_{k<\infty} \lambda_k\right) < 2A/\varepsilon, \quad \text{and thus} \tag{9}$$

$$\mathrm{E}(\#\{k \mid \lambda_k \geq \eta\}) \leq \frac{2A}{\varepsilon\eta}. \tag{10}$$

Thus, a.s. $\lambda_k \to 0$, $s_k \to \infty$ (and hence $\mathbf{1}_{s_k=M} \to 0$), as $k \to \infty$, and therefore

$$w_{\lambda_k}(z_{k+1}) \to Y_\infty \quad \text{with } \mathrm{E}(Y_\infty \mid \mathcal{H}_1) > Y_1 \geq v(z_1) - \varepsilon - 1/M. \tag{11}$$

Also

$$\mathrm{E}(w_{\lambda_k}(z_{k+1})) \geq v(z_1) - \varepsilon - 1/M. \tag{12}$$

From the definition of s_{k+1}, it follows that

$$g_k \geq w_{\lambda_k}(z_{k+1}) + s_{k+1} - s_k - 4\varepsilon - 2A\mathbf{1}_{s_{k+1}=M}.$$

Summing these inequalities over $1 \leq k < n$, we have

$$\sum_{k<n} g_k \geq \sum_{k<n} w_{\lambda_k}(z_{k+1}) + s_n - s_1 - 4\varepsilon n - \sum_{k<n} 2A\mathbf{1}_{s_{k+1}=M}, \tag{13}$$

implying that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k<n} g_k \geq Y_\infty - 4\varepsilon,$$

and thus

$$\mathrm{E}\left(\liminf_{n\to\infty} \frac{1}{n} \sum_{k<n} g_k\right) \geq \mathrm{E}(Y_\infty \mid \mathcal{H}_1) - 4\varepsilon \geq v(z_1) - 5\varepsilon - 1/M$$

and

$$\mathrm{E}\left(\frac{1}{n} \sum_{k<n} g_k\right) \geq v(z_1) - 2\varepsilon - 4\varepsilon - s_1/n.$$

Altogether, we deduce that the $(\lambda_k)_{k=1}^\infty$-strategy of player 1 guarantees $v(z_1) - 7\varepsilon$. Thus, player 1 can guarantee $v(z_1)$. Similarly, player 2 can guarantee $v(z_1)$, and therefore $v(z_1)$ is the value of the absorbing game.

The $(\lambda_k)_{k=1}^\infty$-strategy of player 1 (is a constant mixed action on the absorbing states, and) has a simplified form on the nonabsorbing state. Indeed, as $w_\lambda(z) = v(z) - \varepsilon\mathbf{1}_{z=z^0}$, we can define $s_{k+1} = \max[M, s_k + g_k - v(z^0) + 5\varepsilon]$.

Acknowledgments. Research was supported, in part, by the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Prime Minister's Science Policy Office; Israel Science Foundation Grants 263/03 and 1123/06, the Center for the Study of Rationality of the Hebrew University of Jerusalem; and European TMR Network FMRX CT 960055.

References

[1] Bewley, T., E. Kohlberg. 1976. The asymptotic theory of stochastic games. *Math. Oper. Res.* **1** 197–208.
[2] Blackwell, D., T. S. Ferguson. 1968. The big match. *Ann. Math. Statist.* **39** 159–168.
[3] Kohlberg, E. 1974. Repeated games with absorbing states. *Ann. Statist.* **2** 724–738.
[4] Kuratowski, K., C. Ryll-Nardzewski. 1965. A general theorem on selectors. *Bull. Polish Acad. Sci.* (*Ser. Math.*) **13** 379–403.
[5] Maitra, A., W. D. Sudderth. 1992. An operator solution for stochastic games. *Israel J. Math.* **78** 33–49.
[6] Maitra, A., W. D. Sudderth. 1993. Borel stochastic games with limsup payoff. *Ann. Probab.* **21** 861–885.
[7] Mertens, J. F., A. Neyman. 1980. Stochastic games. Core Discussion Paper 8001, Université Catholique de Louvain-la-Neuve, Belgium.
[8] Mertens, J. F., A. Neyman. 1981. Stochastic games. *Internat. J. Game Theory* **10** 53–66.
[9] Mertens J. F., S. Sorin, S. Zamir. 1994. Repeated games. Core Discussion Papers 9420, 9421, 9422, Université Catholique de Louvain-la-Neuve, Belgium.
[10] Neyman, A. 2003. Stochastic games: Existence of the minmax. A. Neyman, S. Sorin, eds. *Stochastic Games and Applications*, Vol. 570. *Mathematical and Physical Sciences*. NATO Science Series C, Kluwer Academic Publishers, Dordrecht, The Netherlands, 57–75.
[11] Rosenberg, D., S. Sorin. 2001. An operator approach to stochastic games. *Israël J. Math.* **121** 221–246.
[12] Rosenberg, D., E. Solan, N. Vieille. 2003. The maxmin value of stochastic games with imperfect monitoring. *Internat. J. Game Theory* **32** 133–150.
[13] Shapley, L. 1953. Stochastic games. *Proc. National Acad. Sci.* **39** 1095–1100.