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Asymptotic Values of Vector Measure Games

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In honor of L. S. Shapley's eightieth birthday

The asymptotic value, introduced by Kannai in 1966, is an asymptotic approach to the notion of the Shapley value for games with infinitely many players. A vector measure game is a game v where the worth $v(S)$ of a coalition S is a function f of $\mu(S)$ where μ is a vector measure. Special classes of vector measure games are the weighted majority games and the two-house weighted majority games, where a two-house weighted majority game is a game in which a coalition is winning if and only if it is winning in two given weighted majority games. All weighted majority games have an asymptotic value. However, not all two-house weighted majority games have an asymptotic value. In this paper, we prove that the existence of infinitely many atoms with sufficient variety suffice for the existence of the asymptotic value in a general class of nonsmooth vector measure games that includes in particular two-house weighted majority games.

Key words: asymptotic value; weighted majority game; two-house weighted majority game; vector measure game; Shapley value
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1. Introduction. One of the basic solution concepts in cooperative game theory, the Shapley value assigns a unique outcome to each finite transferable utility game. The value of a game can be thought of as a sort of average or expected outcome, or an a priori measure of power. It was introduced by Shapley in 1953 as the unique function (from games to outcomes) satisfying some plausible axioms (efficiency, linearity, symmetry, and the null player axiom).

1.1. Asymptotic value. The Shapley value was initially defined for games with finitely many players. It is most relevant to important game models in economics and political science with many players, where most of the players are "insignificant" on their own, yet important as part of a coalition (e.g., shareholders of a large public company, or a large electorate). In such games there may be, in addition, a set of individually significant players. This suggests the need to analyze the Shapley value of games with a large number of individually insignificant players as well as a group of individually significant players.

There have been two strands in the literature pursuing this issue. One analyzes the asymptotics of values of finite games (e.g., Shapiro and Shapley 1978; Shapley 1962a, 1964b) while the other defines a value for limit games (e.g., Milnor and Shapley 1978, Aumann and Shapley 1974). The asymptotic value, introduced by Kannai in 1966, bridges these two approaches.

The asymptotic value of a limiting game v is defined whenever all the sequences of the Shapley value of finite games that "approximate" v have the same limit. It turns out that the existence of the asymptotic value is not guaranteed and many authors have studied classes of games for which this value exists (e.g., Kannai 1966, Aumann and Shapley 1974, Hart 1977, Fogelman and Quinzii 1980, Dubey 1980, Neyman 1981).

1.2. Vector measure games. A scalar measure game is a game in which the worth, $v(S)$, of a coalition S is a function f of its scalar measure. In a variety of papers (e.g., the above-mentioned studies) the existence results for the asymptotic value turn out to be particular cases of the more general result which states that all games that can be approximated (in the bounded variation norm) by linear combinations of scalar measure games have an asymptotic value. (The other tools used in the existence results are self-duality and diagonality.) In fact, these results follow from the fact that all monotonic scalar measure games (and thus also all scalar measure games of bounded variation) have an asymptotic value (Neyman 1988). ("Continuity" of the game at $\varnothing$ and at the grand coalition is assumed.)

A vector measure game is a game v in which the worth $v(S)$ of a coalition S is a function f of a vector of measures, $\mu_1(S), \ldots, \mu_n(S)$, namely, $v = f \circ (\mu_1, \ldots, \mu_n)$. There are many games that arise in applications and have a representation as vector measure games. Some important examples of such games are coalitional market games of exchange economies with transferable utilities, or market games for short, with finitely many types (see Aumann and Shapley (1974, Chapter VI) (i.e., "markets with money" or "markets with side payments"; cf. Shapley and Shubik 1966, 1969; Shapley 1964b). Market games with smooth utilities can be approximated (in the bounded variation norm) by vector measure games where, in addition, the function f is smooth (see Aumann and Shapley 1974, Chapter VI).

Another example of games that are approximated by vector measure games is models of economies where some economic activities need political approval, as in the Aumann and Kurz (1977a, b) models of power and taxation and the power and public goods models of Aumann et al. (1983, 1987). In these models the function f is discontinuous, and in fact is a product of two other functions: a $\{0, 1\}$-valued function h and a smooth function g. The worth $g \circ \mu(S)$ describes the maximal economic output that the coalition S can produce without any political constraint. The game $h \circ \mu$ describes whether or not the coalition is approved to perform the economic activities that yield $g \circ \mu$. In these models of power and taxes or power and public goods the voting game $h \circ \mu$ is a weighted majority voting game.

Many other voting systems can be viewed as more general $\{0, 1\}$-valued vector measure games. Some examples are majority voting coupled with veto power by some members (e.g., the U.N. Security Council) or parliaments with two houses (e.g., the U.K. Parliament or the U.S. Congress), or the method of voting by count and account in some old Jewish communities (see Peleg 1992). The common feature of these voting systems is that the finite vector of values $(h_1 \circ \mu(S), \ldots, h_m \circ \mu(S))$, where $h_j \circ \mu$ is a weighted majority game with a voting measure that is a positive linear combination of the scalar measures μ_i, determines whether the coalition S is winning or not. Whenever such political voting systems specify the economic activities that are permitted, the resulting coalitional game v is defined by means of 2^m coalitional games $g_D \circ \mu(S)$, $D \in \{0, 1\}^m$, where $v(S) = g_D \circ \mu(S)$ if $D = (h_1 \circ \mu(S), \ldots, h_m \circ \mu(S))$. For example, the value of D may specify constraints on the transfer of commodities in an exchange economy or limits of the technology and raw materials in a production economy. In these cases, if the utilities or the production functions are smooth, then (under classical constraints and limitations) the function g_D is smooth.

1.3. Asymptotic values for vector measure games. The existence of asymptotic value in vector measure games is the topic of this research. A partial answer to the existence problem exists for the case where f is a smooth function. In this case the game $v = f \circ (\mu_1, \ldots, \mu_n)$ can be approximated (in the bounded variation norm) by a linear combination of scalar measure games, and thus has an asymptotic value. However, when f is not smooth this does not necessarily hold.

Two examples of vector measure games where the asymptotic value does not exist are:

- The nonatomic three-handed glove market, where μ_1, μ_2, and μ_3 are nonatomic probability measures and $f(x_1, x_2, x_3) = \min(x_1, x_2, x_3)$ (see Aumann and Shapley 1974, Example 19.2).
- Two-house weighted majority games, where a coalition S is winning if it meets the quotas $q_1 = q_2 \neq 1/2$ in two distinct nonatomic probability measures, namely, $f(x_1, x_2) = 1$

if $x_1 \geq q_1$ and $x_2 \geq q_2$ and μ_1, μ_2 are two distinct nonatomic probability measures (see Neyman and Tauman 1979).

Our main result provides sufficient conditions for the existence of an asymptotic value in vector measure games. In particular, it provides sufficient conditions for the existence of an asymptotic value in two-house weighted majority games. The essential assumption that leads to the existence of the asymptotic values of nonsmooth vector measure games is the presence of sufficiently many atoms with sufficient variety; the formal condition is stated in Theorem 1.

In §2, we recall the classical terminology, definitions, and notations, and state the main results of the paper. Section 3 introduces the notion of a stochastic Poisson bridge and derives several properties of such bridges. These properties are used throughout the proof of our main results. Section 4 provides additional background needed for the main proof. In §5, we prove the main result for the particular, and known, case of a smooth vector measure game. Section 6 is the essential part of the proof of the main result for two-dimensional vector measure games, and §7 outlines the proof of the main result for arbitrary vector measure games. Finally, at the heart of our proofs are results on a zero-hitting-probability property for vector measures. In §8 we propose a conjecture for a necessary and sufficient condition for this property. If proven, this condition will provide further insight into the existence of asymptotic values of vector measure games.

2. Model and main result. A *coalitional game* $(v, I, \mathscr{C})$, or game for short, is a real-valued function v on the σ-field $\mathscr{C}$ (of coalitions) of a measurable space I, with $v(\varnothing) = 0$. The game v is *finite* whenever $\mathscr{C}$ is finite. In case Π is a finite subfield of $\mathscr{C}$, the game v naturally induces a finite game on Π, denoted v_Π. A game v is *monotonic* if $v(S) \geq v(T)$ whenever $S, T \in \mathscr{C}$ and $S \supset T$. The game v is of *bounded variation* whenever it is the difference of two monotonic games and its bounded variation norm $\|v\|$ is the supremum of $\sum_{i=1}^n |v(S_i) - v(S_{i-1}|$, where the supremum is over all finite increasing chains of coalition $S_0 \subset \ldots \subset S_n$.

We denote by $A(\mathscr{C})$ the atoms of $\mathscr{C}$. A σ-field $\mathscr{C}$ is *countably generated* if it is generated by a countable family of elements of $\mathscr{C}$; it is *separating* if for any two distinct points in I there is a member of $\mathscr{C}$ containing one but not the other point.

2.1. The Shapley value. The *Shapley value* of a finite game v is the measure on $\mathscr{C}$ given by

$$\psi v(a) = \frac{1}{n!} \sum_{\mathscr{R}} \left(v(\mathscr{P}_a^{\mathscr{R}} \cup \{a\}) - v(\mathscr{P}_a^{\mathscr{R}}) \right) \quad \text{for all } a \in A(\mathscr{C}),$$

where n is the number of atoms of $\mathscr{C}$, the sum runs over all $n!$ orders $\mathscr{R}$ of the players (atoms of $\mathscr{C}$), and $\mathscr{P}_a^{\mathscr{R}}$ is the union of all atoms preceding a in the order $\mathscr{R}$. Thus, $\psi v(a)$ is the expected marginal contribution of a to a random coalition $\mathscr{P}_a^{\mathscr{R}}$. For any coalition $C \in \mathscr{C}$, $\psi v(C) = \sum_{a \subset C} \psi v(a)$.

An alternative formula for the Shapley value, in case $\mathscr{C}$ is finite, could be given by means of a family $\{X_a \mid a \in A(\mathscr{C})\}$ of independent identically distributed (i.i.d.) random variables (r.v.s), uniformly distributed on (0, 1). The values of X_a, $a \in A(\mathscr{C})$, induce, with probability one, an order $\mathscr{R}$ over the atoms of $\mathscr{C}$; a precedes $b \Leftrightarrow X_a < X_b$. As $\{X_a \mid a \in A(\mathscr{C})\}$ are i.i.d. and nonatomic, all orders are equally likely. Thus,

$$\begin{aligned} (1) \quad \psi v(a) &= E[v(\{b \in A(\mathscr{C}) \mid X_b \leq X_a\}) - v(\{b \in A(\mathscr{C}) \mid X_b < X_a\})] \\ &= \int_0^1 E(v(\{b \in A(\mathscr{C}) \mid X_b \leq t\} \cup \{a\}) - v(\{b \in A(\mathscr{C}) \mid X_b \leq t\} \setminus \{a\}))\, dt, \end{aligned}$$

where $\{b \in A(\mathscr{C}) \mid X_b \leq X_a\}$ (respectively, $\{b \in A(\mathscr{C}) \mid X_b \leq t\}$) stands for the union of all $b \in A(\mathscr{C})$ such that $X_b \leq X_a$ (respectively, $X_b \leq t$).

An elementary and useful property of the Shapley value is its weak contractions: if $(v, I, \mathscr{C})$ and $(u, I, \mathscr{C})$ are two finite games, then $\sum_{a \in A(\mathscr{C})} |\psi v(a) - \psi u(a)| \leq \|v - u\|$.

2.2. The asymptotic value. Given a coalition, $C \in \mathscr{C}$, a *C-admissible sequence* is an increasing sequence $(\Pi_1, \Pi_2, \dots)$ of finite fields such that $C \in \Pi_1$ and $\bigcup_{i=1}^{\infty} \Pi_i$ generates $\mathscr{C}$. A finitely additive measure φv on $\mathscr{C}$ is said to be the *asymptotic value* of v if for all $C \in \mathscr{C}$ and for all C-admissible sequences $(\Pi_1, \Pi_2, \dots)$,

$$\exists \lim_{i\to\infty} \psi v_{\Pi_i}(C) = \varphi v(C),$$

where $\psi v_{\Pi_i}(C)$ denotes the Shapley value of C in the finite game v_{Π_i} and v_{Π} is the restriction of v to Π.

In studying the existence of an asymptotic value we can assume w.l.o.g. that the measurable space of players $(I, \mathscr{C})$ is countably generated and separating (and thus isomorphic to a subset I^* of $[0, 1]$ with the σ-field of coalitions consisting of all intersections of a Borel set and I^*). Indeed, if $\mathscr{C}$ is not countably generated, there is no I-admissible sequence. Let $B_i \in \mathscr{C}$ be a sequence of coalitions that generate $\mathscr{C}$. Let ξ: $I \to \{0, 1\}^{\infty}$ be defined by $\xi_i(x) = \mathbb{1}(x \in B_i)$ (where $\mathbb{1}$ is the indicator function taking on the value 1 if its argument is true and 0 otherwise). Set $I^* := \xi(I)$ and $\mathscr{C}^* := \xi(\mathscr{C})$. A coalitional game $(v, I, \mathscr{C})$ is mapped to the coalitional game $(v^*, I^*, \mathscr{C}^*)$, where $v^*(D) = v(\xi^{-1}(D))$. The game v has an asymptotic value iff v^* has an asymptotic value and $\mathscr{C}^*$ is countably generated and separating.

The set of all games of bounded variation that have an asymptotic value is denoted *ASYMP*.

2.3. Vector measure games. Here, we first define vector measures, thereafter k-house weighted majority games, and then vector measure games.

2.3.1. Vector measures. Let $(I, \mathscr{C})$ be a measurable space. By a positive scalar measure (measure for short) on $(I, \mathscr{C})$ we mean a countably additive function from $\mathscr{C}$ to $\mathbb{R}_+$. A finite-dimensional vector measure (vector measure for short) is a countably additive function from $\mathscr{C}$ to $\mathbb{R}^n_+$. Any vector measure μ: $\mathscr{C} \to \mathbb{R}^n_+$ is a vector of positive scalar measures $(\mu_1, \dots, \mu_n)$.

A vector measure $\mu = (\mu_1, \dots, \mu_n)$ is nonatomic iff for every $S \in \mathscr{C}$ with $\mu(S) \neq 0$, there is a measurable subset $T \subset S$ with $\mu(T) \neq 0$ and $\mu(T) \neq \mu(S)$. Equivalently, μ is nonatomic iff for every $1 \le i \le n$ the scalar measure μ_i is nonatomic.

A set $F \in \mathscr{C}$ is an *atom* of the vector measure $\mu = (\mu_1, \dots, \mu_n)$ if $\mu(F) \neq 0$, and if $E \in \mathscr{C}$, $E \subset F$, then either $\mu(E) = 0$ or $\mu(E) = \mu(F)$. A vector measure μ can have at most a countable family of disjoint atoms. A vector measure μ is *nonatomic* if it has no atoms. A vector measure μ is *purely atomic* if it has a countable family of atoms F_i such that $\mu(S) = \mu(S \cap (\bigcup_i F_i))$ for every $S \in \mathscr{C}$.

Every vector measure μ is a sum of a purely atomic vector measure μ^A and a nonatomic vector measure μ^{NA}.

The *range* of a vector measure μ, denoted $\mathscr{R}(\mu)$, is the set of all vectors $\mu(S)$, where $S \in \mathscr{C}$, i.e.,

$$\mathscr{R}(\mu) = \{\mu(S): S \in \mathscr{C}\}.$$

The range of a vector measure is compact. The range of a nonatomic vector measure is convex (Lyapunov's Theorem). The range of any vector measure is symmetric around $\mu(I)/2$; i.e., for every $x \in \mathscr{R}(\mu)$, there is a vector $y \in \mathscr{R}(\mu)$ with $(x+y)/2 = \mu(I)/2$ (e.g., $y = \mu(I \setminus S)$ if $x = \mu(S)$).

2.4. *k*-house weighted majority games. A game is called a *weighted majority game* (w.m.g.) if there exist a finite positive measure μ on $(I, \mathscr{C})$ and a quota $q \in (0, \mu(I))$ such that $v(C) = \mathbb{1}(\mu(C) \ge q)$. For a weighted majority game with finitely many players, i.e.,

$|\mathscr{C}| < \infty$, the Shapley value is given by

$$\begin{aligned}(2)\qquad \psi v(a) &= E\left[\mathbb{1}\left(q \le \sum_{b\in A(\mathscr{C})} \mu(b)\cdot \mathbb{1}(X_b \le X_a) < q+\mu(a)\right)\right]\\ &= \int_0^1 E\left[\mathbb{1}\left(q-\mu(a) \le \sum_{b\in A(\mathscr{C})-\{a\}} \mu(b)\cdot \mathbb{1}(X_b \le t) < q\right)\right] dt,\end{aligned}$$

and for a coalition $C \subset \mathscr{C}$,

$$\begin{aligned}(3)\qquad \psi v(C) &= \sum_{a\in C} \psi v(a)\\ &= E\left[\sum_{a\in C} \mathbb{1}\left(q-\mu(a) \le \sum_{b\in A(\mathscr{C})-\{a\}} \mu(b)\cdot \mathbb{1}(X_b \le X_a) < q\right)\right].\end{aligned}$$

If there are two finite positive measures, μ_1 and μ_2, on $(I, \mathscr{C})$, and two quotas, $q_1 \in (0, \mu_1(I))$ and $q_2 \in (0, \mu_2(I))$, such that $v(C) = \mathbb{1}(\mu_1(C) \ge q_1)\mathbb{1}(\mu_2(C) \ge q_2)$, then the game v is called a *two-house w.m.g.* It is denoted $[(q_1, q_2); (\mu_1, \mu_2)]$. A *k-house w.m.g.* can be defined in a similar spirit for all $k \in \mathbb{N}$.

2.4.1. Vector measure games. A vector measure game v is a game of the form $v = f \circ \mu$, where μ is a vector measure and f is a function defined on the range of μ and such that $f(0) = 0$ and f continuous at $0 = \mu(\varnothing)$ and at $\mu(I)$.

All scalar measure games with bounded variation have an asymptotic value (Neyman 1988). A vector measure game $p \circ \mu$, where p is a polynomial, is a linear combination of polynomials of scalar measures and thus has an asymptotic value. A vector measure game $f \circ \mu$ has an asymptotic value whenever f is smooth on the range of μ. (A function f defined on a subset B of a Euclidean space is *smooth* if it is the restriction of a continuously differentiable function g defined on a neighborhood of the closure of B.) Indeed, if f is smooth on the range of μ, then f can be approximated in C^1 by a polynomial p, i.e., $\forall\, \varepsilon > 0$ there is a polynomial p such that

$$\left|\frac{\partial f}{\partial x_j}(x) - \frac{\partial p}{\partial x_j}(x)\right| < \varepsilon \qquad \text{for all } x \in \mathscr{R}(\mu);$$

hence, $\|f \circ \mu - p \circ \mu\| < \varepsilon \sum_j \mu_j(I)$. The set *ASYMP* of all games of bounded variation that have an asymptotic value is closed in the bounded variation norm and thus $f \circ \mu$ has an asymptotic value.

The class of two-house w.m.g.s is an important class of simple monotonic vector measure games. Theorem 1 provides conditions on the vector measure $\mu = (\mu_1, \mu_2)$ that guarantee that for every quota $0 < q = (q_1, q_2) < \mu(I)$ the two-house w.m.g. $[(q_1, q_2); \mu]$ has an asymptotic value. The conditions on $\mu = (\mu_1, \mu_2)$ in Theorem 1 guarantee that the vector measure $\mu = (\mu_1, \mu_2)$ satisfies a property, termed the zero-hitting-probability property, which is described below. To every vector measure μ we associate (see §3) a right continuous and monotonic stochastic process $Z^\mu\colon [0, 1] \to \mathscr{R}(\mu)$. The vector measure $\mu = (\mu_1, \mu_2)$ satisfies the zero-hitting-probability property if for every vector q with $0 < q < \mu(I)$, the probability that there is t s.t. $Z^\mu(t) = q$, equals zero.

The vector measure $\mu = (\mu_1, \mu_2)$ has the q-zero-hitting-probability property if the probability that the stochastic process Z^μ hits the quota q, namely, that there is t s.t. $q \in \{Z^\mu(t), Z^\mu(t-)\}$ (where $Z^\mu(t-) = \lim_{s\to t-} Z^\mu(s)$) equals zero. This property of the pair μ and q enables us to prove that $[q; \mu]$ has an asymptotic value. Therefore, a corollary of the proof of Theorem 1 is that for every pair consisting of a two-dimensional measure μ and a quota $q=(q_1, q_2)$ such that μ has the q-zero-hitting-probability property, the two-house w.m.g $[q;\mu]$ has an asymptotic value.

If the two-dimensional vector measure $\nu=(\nu_1,\nu_2)$ has the zero-hitting-probability property, then also the vector measure $\mu=(\mu_1,\mu_2)$, where μ_1 and μ_2 are two distinct convex combinations of ν_1 and ν_2, has the zero-hitting-probability property.

The set of games having an asymptotic value is a linear space. Therefore, our result for two-house w.m.g.s implies that every game in the linear span of these two-house w.m.g.s has an asymptotic value. We now comment on the linear span.

Let $\nu=(\nu_1,\nu_2)$ be a vector of two probability measures with the zero-hitting-probability property. We define the algebra of coalitions $\mathscr{B}(\nu)$ as the algebra (closed under finite unions and complements) of sets generated by the sets of the form $\{S\in\mathscr{C}\mid \lambda\nu_1(S)+(1-\lambda)\nu_2(S)\geq\theta\}$ or $\{S\in\mathscr{C}\mid \lambda\nu_1(S)+(1-\lambda)\nu_2(S)>\theta\}$, where $0\leq\lambda\leq 1$ and $0<\theta<1$. A game v is *$\mathscr{B}(\nu)$-piecewise constant* if there is a finite partition of $\mathscr{C}$, $\mathscr{C}=\bigcup_{\ell=1}^{k}B_\ell$, s.t. $B_\ell\in\mathscr{B}(\nu)$ and the restriction of v to each element B_ℓ of the partition is a constant function. The linear span of the two-house w.m.g.s $[q;\mu]$, where μ_1 and μ_2 are two distinct convex combinations of ν_1 and ν_2 and $0<q_i<1$, is the space of all games that are $\mathscr{B}(\nu)$-piecewise constant. Therefore, Theorem 1 is equivalent to the more general result that every $\mathscr{B}(\nu)$-piecewise-constant game has an asymptotic value.

A game v is *$\mathscr{B}(\nu)$-piecewise affine* (respectively, *$\mathscr{B}(\nu)$-piecewise smooth*) if there is a finite partition of $\mathscr{C}$, $\mathscr{C}=\bigcup_{\ell=1}^{k}B_\ell$, s.t. $B_\ell\in\mathscr{B}(\nu)$ and the restriction of v to each element B_ℓ of the partition is an affine function of ν (respectively, a smooth function of ν).

For a general monotonic game v, it is known that v has an asymptotic value whenever the simple games v^α defined by $v^\alpha(S)=\mathbb{1}(v(S)\geq\alpha)$ have an asymptotic value for every $\alpha>0$ (Neyman 2002, p. 2139). If v is a monotonic $\mathscr{B}(\nu)$-piecewise-affine vector measure game, then the simple game v^α is $\mathscr{B}(\nu)$-piecewise constant. Therefore, Theorem 1 implies that all monotonic $\mathscr{B}(\nu)$-piecewise-affine vector measure games have an asymptotic value.

Theorem 1 and its above-mentioned generalization will follow from Theorem 2, which states that every $\mathscr{B}(\nu)$-piecewise-smooth game has an asymptotic value.

2.4.2. Essentially scalar measure games. A game v has a *co-finite scalar measure game presentation* if there is a finite set of players $F=\{c_1,\ldots,c_k\}$, a probability measure ν on $(I,\mathscr{C})$, and functions $f_G:[0,1]\to\mathbb{R}$, $G\subset F$, with $f_\varnothing$ continuous at $\nu(\varnothing)=0$ and f_F continuous at $\nu(I)=1$, such that

$$v(S)=f_{S\cap F}(\nu(S)).$$

Note that if $v=f\circ\nu$ is a vector measure game, where $\nu=(\nu_1,\ldots,\nu_m)$ is a vector of probability measures such that $(\nu_2,\ldots,\nu_m)$ has finite support and $f\colon\mathscr{R}(\nu)\to\mathbb{R}$ continuous at $\nu(\varnothing)$ and $\nu(I)$, then v has a co-finite scalar measure game presentation.

Note also that every co-finite scalar measure game is a scalar measure game. Define a measure μ by $\mu(c_j)=3^{-j}$ and $\mu(S)=\sum_{j:\,c_j\in S}3^{-j}+\nu(S\setminus F)3^{-k}/2$. As there are disjoint intervals J_G, $G\subset F$, such that $\mu(S)\in J_{S\cap F}$, we can define a function g: $[0,\mu(I)]\to\mathbb{R}$ continuous at 0 and at $\mu(I)$ such that $v=g\circ\mu$. It follows that every co-finite scalar measure game of bounded variation is a scalar measure game of bounded variation and thus has an asymptotic value.

2.5. Main result. Our results are stated for the special case of two-dimensional vector measure games as well as for the general case of m-dimensional vector measure games. For simplicity of exposition, the detailed proof is given for the case of two-dimensional vector measure games. Section 7 outlines the proof of the general case of m-dimensional vector measure games.

Our first result, which is a special case of Theorem 2, identifies families of two-house w.m.g.s that have an asymptotic value. Recall that not all nonatomic two-house w.m.g.s have an asymptotic value (Neyman and Tauman 1979), and, moreover, for every two distinct nonatomic voting measures μ_1 and μ_2 there are quotas q_1 and q_2 such that the two-house w.m.g. $[(q_1, q_2); (\mu_1, \mu_2)]$ does not have an asymptotic value. Our result states conditions on the pair μ_1 and μ_2 of scalar measures that guarantee that for every quota q_1 and q_2 the two-house w.m.g. $[(q_1, q_2); (\mu_1, \mu_2)]$ has an asymptotic value. Informally, the result asserts that whenever the vector measure μ has sufficiently many atoms, then for every pair of quotas q_1 and q_2 the w.m.g. $[(q_1, q_2); \mu]$ has an asymptotic value.

THEOREM 1. *Let v be a two-house w.m.g. induced by the measures μ_1, μ_2 and quotas q_1, q_2. Assume that each one of the measures μ_i is a convex combination of two measures ν_1 and ν_2. If the cardinality of $A(\nu_1) \cup A(\nu_2)$ is infinite and $A(\nu_1) \cap A(\nu_2) = \varnothing$, then v has an asymptotic value.*

Let $\nu = (\nu_1, \ldots, \nu_m)$ be a vector of probability measures. Let $\mathcal{B}^\nu$, or $\mathcal{B}$ for short, be the algebra of subsets of $\operatorname{co}\mathcal{R}(\nu)$ generated by the sets $B = \{(x_1, \ldots, x_m) \in \operatorname{co}\mathcal{R}(\nu) \mid \sum_{i=1}^m \alpha_i x_i < q\}$ and their closure $\bar{B}$ and where $\sum_{i=1}^m \alpha_i = 1$, $0 \le \alpha_i \le 1$, and $0 < q < 1$.

A real-valued function f: $\operatorname{co}\mathcal{R}(\nu) \to \mathbb{R}$ is $\mathcal{B}$-piecewise smooth if there is a finite partition $\bigcup_{j=1}^k B_j$ of $\operatorname{co}\mathcal{R}(\nu)$ with $B_j \in \mathcal{B}$ such that the restriction of f to each element B_j of the partition is smooth. (A function g: $B \to \mathbb{R}$ is smooth if it has an extension to a C^1 function defined on a neighborhood of the closure of B.)

The set of all games of the form $f \circ \nu$, where f is $\mathcal{B}$-piecewise smooth, is denoted $LPS(\nu)$. The set $LPS(\nu)$ is a linear space, and the main results provide a condition on the vector measure ν such that every game in $LPS(\nu)$ (and thus every game in its bounded variation closure) has an asymptotic value.

THEOREM 2 (MAIN RESULT: TWO-DIMENSIONAL VECTOR MEASURE GAMES). *Let $\nu = (\nu_1, \nu_2)$ be a vector of two probability measures for which the cardinality of $A(\nu_1) \cup A(\nu_2)$ is infinite and $A(\nu_1) \cap A(\nu_2) = \varnothing$. If f: $\operatorname{co}\mathcal{R}(\nu) \to \mathbb{R}$ is $\mathcal{B}$-piecewise smooth and continuous at $\mu(I)$ and at $\mu(\varnothing)$ and $f(0) = 0$, then $f \circ \nu$ has an asymptotic value.*

Theorem 2 is a special case of the following theorem.

THEOREM 3 (MAIN RESULT: m-DIMENSIONAL VECTOR MEASURE GAMES). *Let $\nu = (\nu_1, \ldots, \nu_m)$ be a vector of probability measures for which the cardinality of $A(\nu_i)$, $1 \le i < m$, is infinite and $A(\nu_i) \cap A(\nu_j) = \varnothing$ for all $1 \le i < j \le m$. If f: $\operatorname{co}\mathcal{R}(\nu) \to \mathbb{R}$ is $\mathcal{B}$-piecewise smooth and continuous at $\mu(I)$ and at $\mu(\varnothing)$ and $f(0) = 0$, then $f \circ \nu$ has an asymptotic value.*

3. The stochastic Poisson bridge associated with a vector measure.

3.1. The auxiliary stochastic bridge. Let μ be an $\mathbb{R}^k$-valued vector measure defined on the measurable space $(I, \mathcal{C})$. For simplicity, we assume that I is infinite and $\mathcal{C}$ is countably generated and separating; thus, there is a countable infinite set $A(\mu) \subset I$, or A for short, such that A is a support of μ^A. Assume that $A(\mu) = \{a_j \mid j \in \mathbb{N}\}$ with $a_i \neq a_j$ whenever $i \neq j$. Let X_j, $j = 1, 2, \ldots$, be a sequence of i.i.d. r.v.s uniformly distributed on $[0, 1]$. W.l.o.g. we assume that $X_i \neq X_j$ whenever $i \neq j$. The stochastic bridge Z^μ, or Z for short, associated with the vector measure μ is the continuous-time stochastic process Z: $[0, 1] \to \mathcal{R}(\mu)$ defined by

$$Z(t) = t\mu^{NA}(I) + \sum_{j=1}^{\infty} \mu(a_j)\mathbb{1}(X_j \le t).$$

The stochastic process Z is called a bridge because its values at 0 and 1 are the constant vectors $\mu(\varnothing) = 0$ and $\mu(I)$, respectively. The increments of the process $Z(t)$ are exchangeable: if $h > 0$ and $0 \le t_1 < t_2 < \ldots < t_k \le 1 - h$ with $t_{i+1} - t_i \ge h$, then the finite sequence of the r.v.s $Z(t_i + h) - Z(t_i)$, $1 \le i \le k$, is exchangeable; namely, for every permutation σ: $\{1, \ldots, k\} \to \{1, \ldots, k\}$ the distribution of the vector $(Z(t_1 + h) - Z(t_1), \ldots, Z(t_i + h) - Z(t_i), \ldots, Z(t_k + h) - Z(t_k))$ and the distribution of the vector $(Z(t_{\sigma(1)} + h) - Z(t_{\sigma(1)}), \ldots, Z(t_{\sigma(i)} + h) - Z(t_{\sigma(i)}), \ldots, Z(t_{\sigma(k)} + h) - Z(t_{\sigma(k)})$ coincide. Therefore, the stochastic process Z is a Poisson bridge. It is composed as a sum of a linear (deterministic) drift, $t\mu^{NA}(I)$, and a pure jump Poisson bridge $\sum_{i=1}^{\infty} \mu(a_i)\mathbb{1}(X_i \le t)$.

As the function $t \mapsto Z(t)$ is monotonic everywhere, the limit, $\lim_{h \to 0+} Z(t - h)$, exists everywhere, and is denoted $Z(t-)$. We say that the Poisson bridge Z *hits* the point $y \in \mathbb{R}^k$ (respectively, the set $A \subset \mathbb{R}^k$) if $\exists\, 0 \le t \le 1$ s.t. $Z(t) = y$ or $Z(t-) = y$ (respectively, $\exists\, 0 \le t \le 1$ s.t. $Z(t) \in A$ or $Z(t-) \in A$). The hitting probability of the point $y \in \mathbb{R}^k$ (respectively, of the set A) by the process Z is $\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } Z(t) = y \text{ or } Z(t-) = y)$ (respectively, $\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } Z(t) \in A \text{ or } Z(t-) \in A)$).

Our result on the existence of the asymptotic value of a two-house w.m.g. $[(q_1, q_2); \mu]$ relies on the fact that the probability that the auxiliary Poisson bridge Z^{μ} hits the quota (q_1, q_2) is zero. In fact, our proof shows that if μ is a two-dimensional vector measure and $(0, 0) < q = (q_1, q_2) < \mu(I)$ is a quota s.t. the hitting probability of the point q is 0, then for every smooth function f: $\mathbb{R}^2 \to \mathbb{R}$ the game v defined by $v(S) = f(\mu(S))$ if $\mu(S) > q$ and $= 0$ otherwise has an asymptotic value.

It is, therefore, most important for the study of the asymptotic value of two-house w.m.g.s in particular, and of vector measure games in general, to establish conditions on a vector measure μ so that for every point $0 < q < \mu(I)$ the probability that Z^{μ} hits the point q is zero.

Lemma 1. *Assume that μ is an n-dimensional vector measure. Then, the set of points $y \in \mathbb{R}^n$ such that Z hits y with positive probability has measure zero.*

Proof. For every $\alpha \in \mathbb{R}$ let A_{α} be the subset of points $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} y_i = \alpha$. Note that the Poisson bridge is monotonic ($s \le t \Rightarrow Z(s) \le Z(t)$) everywhere. Therefore, everywhere, for any two distinct points $y, y' \in A_{\alpha}$ the path $(t \mapsto Z(t))$ does not hit both y and y'; namely, if either $Z(t-) = y$ or $Z(t) = y$, then for every $0 \le s \le 1$ we have $Z(s-) \ne y'$ and $Z(s) \ne y'$. Therefore, the set of all points $y \in A_{\alpha}$ with $P(\exists\, 0 \le t \le 1 \text{ s.t. } Z(t) = y \text{ or } Z(t-) = y) > 1/k$ has less than k points, and thus, there are at most countably many points $y \in A_{\alpha}$ with $P(\exists\, 0 \le t \le 1 \text{ s.t. } Z(t) = y \text{ or } Z(t-) = y) > 0$. A subset of $\mathbb{R}^n$ that intersects each set A_{α} in at most countably many points has measure zero. □

A useful tool in analyzing the stochastic process is the *reflection principle*. In its simplest form it states that the reversed Poisson bridge Z^*, defined by $Z^*(t) = t\mu^{NA}(I) + \sum_i \mathbb{1}(X_i \ge 1 - t)$, has the same distribution as Z. An alternative definition of Z^* is obtained by setting the r.v.s $Y_i = 1 - X_i$ and defining $Z^*(t) = t\mu^{NA}(I) + \sum_i \mathbb{1}(Y_i \le t)$. The reflection following a stopping time is especially useful. Let $\mathcal{F}_t$ be the σ-algebra of events generated by the r.v.s $X_i\mathbb{1}(X_i \le t)$. A stopping time is a $[0, 1]$-valued measurable function T defined on the probability space on which the r.v.s X_i are defined (and thus, on which the Poisson bridge Z is defined), such that the event $T \le t$ is in $\mathcal{F}_t$. Given a stopping time T we can define the Poisson bridge Z^{T*} obtained by reflection following time T. Namely, set $Y_i = X_i$ if $X_i \le T$ and $Y_i = 1 - X_i + T$ if $X_i > T$, and define $Z^{T*}(t) = t\mu^{NA}(I) + \sum_i \mu(a_i)\mathbb{1}(Y_i \le t)$. As the r.v.s Y_i are i.i.d. uniformly distributed on $[0, 1]$, we deduce that Z^{T*} has the same distribution as Z.

Another helpful transformation is as follows. Assume that X_i, $i \ge 1$, and Y_i, $i \ge 1$, are i.i.d. uniformly distributed on $(0, 1)$. Assume w.l.o.g. that for $i < j$, $X_i \ne X_j$ and $Y_i \ne Y_j$, and for all i, j, $X_i \ne Y_j$. Let T be a stopping time w.r.t. $\mathcal{F}_t$. Define the stochastic bridge $Z^{T, Y}$ by

$$Z^{T, Y}(t) = t\mu^{NA}(I) + \sum_i \mu(a_i)\mathbb{1}(X_i^{T, Y} \le t),$$

where $X_i^{T,Y} = X_i$ if $X_i \le T$ and $X_i^{T,Y} = T + Y_i(1-T)$ if $X_i > t$. The important observation is that the distributions of Z and of $Z^{T,Y}$ coincide.

An important concept, which is used implicitly in our proof, is that of a two-sided stopping time. A *two-sided stopping time* is a $[0,1]$-valued measurable function T s.t. T is a stopping time w.r.t. the increasing family of σ-algebras $(\mathcal{F}_t)_{0\le t\le 1}$ and $1-T$ is a stopping time w.r.t. the increasing family of σ-algebras $(\mathcal{G}_t)_{0\le t\le 1}$, where $\mathcal{G}_t$ is the σ-algebra generated by the r.v.s $X_i \mathbb{1}(X_i \ge 1-t)$.

Let $A \subset \mathbb{R}^k$ s.t. $x \in A$ and $y \ge x$ implies that $y \in A$. Then, the A-entry time of the Poisson bridge Z^μ (where μ is a k-dimensional vector measure), namely, $\inf\{t\colon Z^\mu(t) \in A\}$, is a two-sided stopping time.

A useful transformation that builds on a two-sided stopping time T is as follows. Assume that X_i, $i \ge 1$, and Y_i^j, $i \ge 1$ and $j = 1, 2$, are i.i.d. uniformly distributed on $(0,1)$. Assume w.l.o.g. that for $i < \ell$, $X_i \ne X_\ell$ and $Y_i^j \ne Y_\ell^j$, and for i, ℓ, $X_i \ne Y_\ell$. Let T be a two-sided stopping time w.r.t. $(\mathcal{F}_t)_t$. Define the stochastic bridge

$$Z^{T,Y}(t) = t\mu^{NA}(I) + \sum_i \mu(a_i)\mathbb{1}(X_i^{T,Y} \le t),$$

where $X_i^{T,Y} = TY_i^1$ if $X_i < T$, $X_i^{T,Y} = X_i$ if $X_i = T$, and $X_i^{T,Y} = T + Y_i^2(1-T)$ if $X_i > t$. The important observation is that the distributions of Z and of $Z^{T,Y}$ coincide.

In fact, we will implicitly use a simple extension of the above-mentioned transformation.

3.2. The one-dimensional Poisson bridge. In this section, we fix a finite positive measure μ on $(I, \mathcal{C})$ with infinitely many atoms $\{a_i\}_{i=1}^\infty$. Let $\alpha = \mu(I) - \sum_{i=1}^\infty \mu(a_i)$. Let $\{X_i\}_{i=1}^\infty$ be i.i.d. r.v.s uniformly distributed on $(0,1)$ and, as before, let $Z(t)(\omega) = \sum_{i=1}^\infty \mu(a_i) \cdot \mathbb{1}(X_i(\omega) \le t) + \alpha \cdot t$. Let $\mathrm{Im}(Z) = \{Z(t) \mid t \in [0,1]\}$ be a random subset of $[0,1]$, which is the image of the process $Z(t)$. Note that surely $0, 1 \in \mathrm{Im}(Z)$.

The first lemma is a direct application of Chebyshev's inequality:

Lemma 2 (Neyman 1988, Lemma 11). *Assume that* $\max_{i\ge1}\mu(a_i) > 0$. *For all* $0 \le t < t' \le 1$ *and for all* $c > 0$,

$$\Pr(|\, Z(t') - Z(t) - (t'-t)\cdot\mu(I)\,| > c\cdot\mu(I)) < \frac{(t'-t)\max_{i\ge1}\mu(a_i)}{c^2\cdot\mu(I)}.$$

Lemma 11 in Neyman (1988) is formally stated for the case of a measure with finitely many atoms, but the proof applies to the case of countably many atoms as well.

The second lemma is an application of the first lemma which uses the monotonicity of the Poisson bridge.

Lemma 3 (Neyman 1988, Lemma 14). *For all* $c > 0$,

$$\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } |Z(t) - t\mu(I)| > c\mu(I)) \le \frac{8\max_{i\ge1}\mu(a_i)}{c^3\mu(I)}.$$

3.2.1. The purely atomic case. We start by stating the essential result of Berbee (1981).

Proposition 1 (Berbee 1981, Theorem 3). *Assume that μ is a purely atomic positive scalar measure with infinitely many atoms a_i s.t. $\mu(a_i) > 0$ and $0 < q < \mu(I)$. Then,*

$$\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } Z(t) = q \text{ or } Z(t-) = q) = 0.$$

Corollary 1. *If μ is purely atomic, then for every $\varepsilon > 0$ and $0 < q < \mu(I)$ there exists $\delta > 0$ such that*

$$\Pr(\exists\, t \in [0,1] \quad Z(t) \in [q-\delta, q+\delta]) < \varepsilon.$$

PROOF. Fix a decreasing sequence $0 < \delta_i \downarrow 0$. Set $A_i = \{\omega\colon \exists 0 \le t \le 1 \text{ s.t. } q - \delta_i \le Z(t) \le q + \delta_i\}$. If $\omega \in \bigcap_i A_i$, then for every i there is $t_i = t_i(\omega)$ such that $q - \delta_i \le Z(t_i)(\omega) \le q + \delta_i$. The sequence $(t_i)_i$ has a monotonic subsequence $(t_{i_j})_j$, such that either t_{i_j} is monotonic nonincreasing or strictly increasing. If t_{i_j} is monotonic nonincreasing, then, as Z is right continuous everywhere, we deduce that $Z(t) = q$ where $t = \lim_j t_{i_j}$, and if t_{i_j} is strictly monotonic increasing, then $Z(t-) = q$. Therefore, using Proposition 1, we deduce that $P(\bigcap_i A_i) = 0$. The sequence of events A_i is decreasing and therefore $0 = P(\bigcap_i A_i) = \lim_{i\to\infty} P(A_i)$, implying that for every $\varepsilon > 0$, there is i such that $P(A_i) < \varepsilon$. □

3.2.2. The mixed case with countably many atoms. Obviously, the conclusion of Corollary 1 does not extend to a process with a nonatomic part. However, in that case, we shall show that if the measure μ has infinitely many atoms and one restricts attention to a sufficiently small interval of time, then the hitting probability of the interval $[q-\delta, q+\delta]$ is bounded by ε.

LEMMA 4. *Let μ be a positive scalar measure with infinitely many atoms and let $0 < q < \mu(I)$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $s \in [0, 1)$,*

$$\Pr(\exists\, t \in [s, s+\delta] \text{ s.t. } Z(t) \in [q-\delta, q+\delta]) < \varepsilon.$$

PROOF. $Z(t)$ converges in probability to 0 as $t \to 0+$ and to $\mu(I)$ as $t \to 1-$. Therefore, there is $\delta_1 > 0$ sufficiently small such that

$$P(Z(2\delta_1) \ge q - \delta_1) + P(Z(1-\delta_1) \le q + \delta_1) < \varepsilon.$$

In particular, for every $0 \le s \le \delta_1$ and $\delta < \delta_1$, we have $P(\exists\, t \in [s, s+\delta]\ Z(t) \in [q-\delta, q+\delta]) \le P(Z(2\delta_1) \ge q - \delta_1) < \varepsilon$, and for every $1 - \delta_1 \le s \le 1$ and $\delta < \delta_1$, we have $P(\exists\, t \in [s, s+\delta]\ Z(t) \in [q-\delta, q+\delta]) \le P(Z(1-\delta_1) \le q+\delta_1) < \varepsilon$.

Denote $A(t) = \sum_{i=0}^{\infty} \mu(a_i) \cdot \mathbb{1}(X_i \le t)$ and recall that $\alpha = \mu^{NA}(I)$. Recall that we assumed w.l.o.g. that for every ω and $i \ne j$ we have $X_i(\omega) \ne X_j(\omega)$. Therefore, the function $t \mapsto Z(t)(\omega)$ is right continuous everywhere and it is left continuous at all points $0 < s \le 1$, with $s \notin \{X_i(\omega)\colon i \ge 1\}$. For every $0 < s < 1$, the probability that there is $i \ge 1$ s.t. $X_i = s$ equals 0. Therefore, for every fixed $s \in (0,1)$, the function $t \mapsto Z(t)(\omega)$ is w.p. 1 continuous at s. As μ has infinitely many atoms the distribution of $A(s)$ (and thus, also of $Z(s)$) is nonatomic (e.g., by using Berbee's (1981) result) and thus, $P(Z(s) = q) = P(A(s) = q - s\alpha) = 0$. Therefore, for every $\varepsilon > 0$ and $s \in (0,1)$ there is $\delta = \delta(s, \varepsilon) > 0$ such that $P(\exists\, t \in [s, s+\delta] \text{ s.t. } Z(t) \in [q-\delta, q+\delta]) < \varepsilon$. The left-hand side probability is, for each fixed $\delta > 0$, continuous in s. Therefore, for each fixed $\delta > 0$ the set of s such that $P(\exists\, t \in [s, s+\delta] \text{ s.t. } Z(t) \in [q-\delta, q+\delta]) < \varepsilon$ is open in $[\delta_1, 1-\delta_1]$. Using compactness of $[\delta_1, 1-\delta_1]$, there exists $\delta_1 > \delta > 0$ such that for every $s \in [\delta_1, 1-\delta_1]$, we have $P(\exists\, t \in [s, s+\delta] \text{ s.t. } Z(t) \in [q-\delta, q+\delta]) < \varepsilon$. This last inequality also holds (as shown above) for every $s < \delta_1$ and for every $s > 1 - \delta_1$. □

3.2.3. The mixed case with a nonatomic part. The following lemma provides stochastic information about the Poisson bridge induced by a scalar measure with a nonatomic part.

LEMMA 5. *Let μ be a positive scalar measure with $\mu^{NA}(I) := \alpha > 0$ and $0 < q < \mu(I)$. Then, for every K and every $\delta > 0$,*

$$\begin{aligned} &P\big(\exists 0 \le t \le 1 \text{ s.t. } Z(t) - Z(t-) \ge \delta \text{ and } q - K\delta < Z(t) < q + K\delta\big) \\ &\quad \le 2m(\delta)\delta K/\alpha \to_{\delta\to 0+} 0, \end{aligned}$$

where $m(\delta)$ is the number of different atoms a of μ with mass $\mu(a) \ge \delta$.

PROOF. Let $a_1, a_2, \ldots$ be the sequence of atoms of μ and assume w.l.o.g. that $\mu(a_i) \geq \mu(a_{i+1})$. Observe that if for some value of t we have $Z(t) - Z(t-) \geq \delta$, then there is $i \leq m(\delta)$ such that $t = X_i$. Let A_i denote the event $q - K\delta < Z(X_i) < q + K\delta$. The event $\exists\, 0 \leq t \leq 1$ s.t. $Z(t) - Z(t-) \geq \delta$ and $q - K\delta < Z(t) < q + K\delta$ equals $\bigcup_{i \leq m(\delta)} A_i$. Therefore, we have to prove that $P(\bigcup_{i \leq m(\delta)} A_i) \leq 2m(\delta)\delta K/\alpha$ and that $2m(\delta)\delta K/\alpha \to 0$ as $\delta \to 0+$.

For every fixed i, let Z^{-i} be the Poisson bridge $Z^{-i}(t) = t\alpha + \sum_{j \neq i} \mathbb{1}(X_j \leq t)$. Note that for $t' > t$, we have $Z^{-i}(t') - Z^{-i}(t) \geq (t' - t)\alpha$. Therefore, given the values of X_j, $1 \leq j \neq i$, the conditional probability of $q - K\delta < Z(X_i) < q + K\delta$, namely, $P(q - K\delta < Z(X_i) < q + K\delta \mid X_j, j \neq i)$ is $\leq 2K\delta/\alpha$. Therefore, $P(A_i) \leq 2K\delta/\alpha$. Therefore, $P(\bigcup_{i \leq m(\delta)} A_i) \leq 2m(\delta)K\delta/\alpha$. As $\sum_{i=1}^{\infty} \mu(a_i) < \infty$, we deduce that $m(\delta)\delta \to 0$ as $\delta \to 0+$ and, therefore, $2m(\delta)K\delta/\alpha \to 0$ as $\delta \to 0+$. □

3.3. The two-dimensional Poisson bridge. In this section, $Z = (Z^1, Z^2)$ is the Poisson bridge associated with the two-dimensional vector measure $\nu = (\nu_1, \nu_2)$. Given $q \in \mathbb{R}^2$ and $\delta > 0$, we denote by $B_\delta(q)$ the set of all points $x \in \mathbb{R}^2$ such that $\|q - x\|_\infty \leq \delta$.

LEMMA 6. *Let ν_1 have infinitely many atoms and assume that $A(\nu_1) \cap A(\nu_2) = \varnothing$. Furthermore, assume that either ν_2 has infinitely many atoms or a positive nonatomic part. Let $q = (q_1, q_2) \in \mathbb{R}^2$ with $\nu(\varnothing) < q < \nu(I)$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\Pr(\exists\, t \in [0, 1] \text{ s.t. } Z(t) \in B_\delta(q)) < \varepsilon.$$

PROOF. Note that in the case that one of the measures is purely atomic we have finished by Corollary 1. Assume that both measures have a nonatomic part. Let t denote the stopping time defined by

$$t(\omega) = \inf\{t\colon Z^2(t) \geq q_2 - \delta\}.$$

As $A(\nu_1) \cap A(\nu_2) = \varnothing$, the stopping time t is independent of the Poisson bridge Z^1. Obviously, on $t(\omega) < 1 - 3\delta/\beta$, where $\beta = \mu_2^{NA}(I)$, we have $Z^2(t(\omega) + 3\delta/\beta) \geq q_2 + 2\delta > q_2 + \delta$. Therefore,

$$\begin{aligned} &\Pr(\exists\, t \in [0, 1] \text{ s.t. } Z(t) \in B_\delta(q) \text{ and } t(\omega) < 1 - 3\delta/\beta) \\ &\quad \leq \Pr(\exists\, t \in [t(\omega), t(\omega) + 3\delta/\beta] \text{ s.t. } |Z^1(t) - q_1| < \delta) \end{aligned} \tag{4}$$

which, by Lemma 4, is less than $\varepsilon/2$ for sufficiently small δ. For sufficiently small $\delta > 0$, we have $\Pr(t(\omega) > 1 - 3\delta/\beta) \leq \varepsilon/2$. Therefore, $\Pr(\exists\, t \in [0, 1] \text{ s.t. } Z(t) \in B_\delta(q)) < \varepsilon$. □

3.4. The k-dimensional Poisson bridge. Let μ be an $\mathbb{R}^k$-valued vector measure and let $Z = (Z(t))_{0 \leq t \leq 1}$ be the associated Poisson bridge. Let $A(\mu) = \{a_i \in I \mid i \geq 0\}$ be a countable set containing all atoms of μ and assume that $a_i \neq a_j$ for $i \neq j$.

A sequence of lists of vectors in $\mathbb{R}^k_+$, $w_1^n, \ldots, w_{\ell_n}^n, w_{\ell_n+1}^n, \ldots, w_{m_n}^n$ (together with the implicit sequence of positive integers ℓ_n) is called *μ-admissible* if

$$\ell_n \to \infty,$$

$$\sum_{i=1}^{\ell_n} \|w_i^n - \mu(a_i)\| \to_{n \to \infty} 0,$$

$$\max_{\ell_n < i \leq m_n} \|w_i^n\| \to_{n \to \infty} 0,$$

$$\sum_{i=\ell_n+1}^{m_n} w_i^n \to \mu^{NA}(I).$$

Assume that $w_1^n, \ldots, w_{\ell_n}^n, w_{\ell_n+1}^n, \ldots, w_{m_n}^n$ is a μ-admissible sequence.

Let Z_n be the Poisson bridge defined by

$$Z_n(t) = \sum_{i=1}^{\ell_n} w_i^n \mathbb{1}(X_i \le t) + \sum_{j=\ell_n+1}^{m_n} w_j^n \mathbb{1}(Y_j \le t),$$

where X_i is the sequence of i.i.d. $[0,1]$-valued uniformly distributed r.v.s that define the stochastic process Z and Y_j, $j \ge 1$, is another sequence of i.i.d. $[0,1]$-valued uniformly distributed r.v.s and the sequence (X_i) is independent of the sequence (Y_j).

3.4.1. Distance between $Z_n(t)$ and $Z(t)$.

PROPOSITION 2. $\forall \varepsilon, \delta > 0\ \exists N$ *s.t.* $\forall n > N$,

$$\Pr(\| Z_n(t) - Z(t) \| < \delta\ \forall t \in [0,1]) > 1 - \varepsilon \quad \textit{and}$$
$$\Pr(\| Z_n(t) - Z(t) \| < \delta\ \forall t \in [0,1] \mid (X_i)_{i\ge 1}) > 1 - \varepsilon \quad \textit{everywhere.}$$

PROOF. It is sufficient to prove the proposition for $k = 1$. Indeed, if for some n we have $\Pr(|Z_n^j(t) - Z^j(t)| < \delta\ \ \forall t \in [0,1]) > 1 - \varepsilon/k$ for all $1 \le j \le k$, then $\Pr(\| Z_n(t) - Z(t) \|_\infty < \delta\ \forall t \in [0,1]) > 1 - \varepsilon$. Assume that $k = 1$ and set $\alpha := \mu^{NA}(I)$ and $\alpha_n := \sum_{\ell_n < i \le m_n} w_i^n$. Let $\delta > 0$ be sufficiently small, e.g., $\delta < 1$. Taking N_1 sufficiently large so that $\forall n > N_1$, we have $\sum_{i=1}^{\ell_n} |w_i^n - \mu(a_i)| + \sum_{i > \ell_n} \mu(a_i) < \delta/6$ and $\alpha - \delta/6 < \alpha_n < \alpha + \delta/6$. Then, for $n > N_1$,

$$\left| \sum_{i=1}^{\ell_n} w_i^n \mathbb{1}(X_i \le t) + t\alpha_n - \sum_{i=1}^{\infty} \mu(a_i) \mathbb{1}(X_i \le t) - t\alpha \right| < \delta/3. \tag{5}$$

Assume w.l.o.g. that $\alpha \le 1$ and set $Z_n^c(t) = \sum_{i=\ell_n+1}^{m_n} w_i^n \mathbb{1}(Y_i^n \le t)$. As $\max_{\ell_n < i \le m_n} w_i^n \to_{n\to\infty} 0$, it follows from Lemma 2 that for every $\varepsilon > 0$, there is N_2 sufficiently large such that for all $n > N_2$, we have $\alpha_n < 4/3$ and

$$\Pr(|\alpha_n t - Z_n^c(t)| > \delta/3) < \varepsilon\delta/5. \tag{6}$$

So far we have shown that $\forall \varepsilon, \delta > 0$, $\exists N_2$ (sufficiently large) s.t. $\forall t \in [0,1]$ and $\forall n > N_2$ inequality (6) holds.

Let M be a positive integer with $M < 5/\delta$ and let $\{t_j\}_{j=0}^M$ be an increasing sequence in $[0,1]$ with $t_0 = 0$, $t_M = 1$, and $t_{j+1} - t_j < \delta/4$.

Inequality (6) implies, in particular, that $\forall n > N_2$ and $\forall j,\ 1 \le j \le M$, we have $\Pr(|Z_n^c(t_j) - \alpha_n t_j| \ge \delta/3) < \varepsilon\delta/5$. Therefore,

$$\Pr(\exists\, 1 \le j \le M \text{ s.t. } |Z_n^c(t_j) - \alpha_n t_j| \ge \delta/3) < M\varepsilon\delta/5 < \varepsilon.$$

If $t_j \le t \le t_{j+1}$ and $|Z_n^c(t) - \alpha_n t| \ge 2\delta/3$, then either $Z_n^c(t) \ge \alpha_n t + 2\delta/3$, implying that $Z_n(t_{j+1}) \ge \alpha_n t_{j+1} + \delta/3$, or $Z_n^c(t) \le \alpha_n t - 2\delta/3$, implying that $Z_n^c(t_j) \le \alpha_n t_j - \delta/3$. Therefore, the event $\{\exists\, 0 \le t \le 1 \text{ s.t. } |Z_n^c(t) - \alpha_n t| \ge 2\delta/3\}$ is a subset of the event $\{\exists\, 1 \le j \le M$ s.t. $|Z_n^c(t_j) - \alpha_n t_j| \ge \delta/3\}$. Therefore,

$$\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } |Z_n^c(t) - \alpha_n t| \ge 2\delta/3) < \varepsilon,$$

which together with inequality (5) implies that for all $n > N := \max(N_1, N_2)$, we have

$$\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } |Z_n(t) - Z(t)| \ge \delta \mid (X_i)_{i\ge 1}) < \varepsilon. \quad \square \tag{7}$$

3.5. Hitting affine sets of codimension 2. The result of this section is used for the proof of the result regarding the asymptotic value of m-dimensional vector measure games. Therefore, it may be omitted in a first reading.

LEMMA 7. *Assume that each one of the probability measures ν_i, $i = 1, \ldots, m-1$, has infinitely many atoms, that the probability measure ν_m has infinitely many atoms or a positive nonatomic part, and that $A(\nu_i) \cap A(\nu_j) = \varnothing$ for all $i \neq j$. Let $\mu = (\mu_1, \mu_2)$ be a vector of two distinct convex combinations of $\nu_1, \ldots, \nu_m$. Then, for all $q = (q_1, q_2) \in \mathbb{R}^2$ with $\mu(\varnothing) < q < \mu(I)$ and $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\Pr(\exists\, t \in [0,1] \text{ s.t. } Z^\mu(t) \in B_\delta(q)) < \varepsilon.$$

PROOF. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\theta = (\theta_1, \ldots, \theta_m)$ be two distinct probability vectors, namely, $0 \le \lambda_j, \theta_j$ and $\sum_{j=1}^m \lambda_j = 1 = \sum_{j=1}^m \theta_j$ such that $\mu_1 = \sum_j \lambda_j \nu_j$ and $\mu_2 = \sum_j \theta_j \nu_j$. Let $J = \{j : 1 \le j \le m \text{ s.t. } \lambda_j \theta_i \ge \lambda_i \theta_j \ \forall\, 1 \le i \le m\}$. Note that J is nonempty and that for every $j \in J$, we have $\lambda_j > 0$ and $\theta_j/\lambda_j < 1$. Assume that there is $j \in J$ with $j < m$. (Otherwise, we replace the set J defined above with the set $J = \{i : 1 \le i \le m \text{ s.t. } \theta_i \lambda_j \ge \theta_j \lambda_i \ \forall\, 1 \le j \le m\}$.) Let μ^J be the restriction of the vector measure μ to the set $\bigcup_{j\in J} A(\nu_j)$ and set $\mu^c = \mu - \mu^J$. The vector measure μ^J is purely atomic (with a countable set of atoms $A(\mu^J) = \bigcup_{j \in J} A(\nu_j)$) and its range is one-dimensional; there is a vector $0 \neq x \in \mathbb{R}^2_+$ such that for every $z \in \mathcal{R}(\mu^J)$ there is $\eta \ge 0$ such that $z = \eta x$. Let $y \in \mathbb{R}^2$ be orthogonal to x with $\langle y, \mu(I)\rangle = \langle y, \mu^c(I)\rangle > 0$. W.l.o.g. $\|x\|_2 = \|y\|_2 = 1$. Set $\bar{q}_1 = \langle (q_1, q_2), x\rangle$ and $\bar{q}_2 = \langle (q_1, q_2), y\rangle$.

If $\bar{q}_2 < 0$, then there is $\delta > 0$ sufficiently small such that $B_\delta(q) \cap \mathcal{R}(\mu) = \varnothing$ and, therefore, $\Pr(\exists\, t \in [0,1] \text{ s.t. } Z^\mu(t) \in B_\delta(q)) = 0 < \varepsilon$.

If $\bar{q}_2 = 0$, then there is $\eta > 0$ sufficiently small such that $\Pr(\langle Z^\mu(\eta'), x\rangle \le \bar{q}_1/2 \ \forall\, \eta' \le \eta) > 1 - \varepsilon/2$, and for a sufficiently small $\delta > 0$, we have $\Pr(\langle Z^\mu(\eta), y\rangle \ge \delta) > 1 - \varepsilon/2$ and thus, $\Pr(\langle Z^\mu(\eta'), y\rangle \ge \delta \ \forall\, \eta' \ge \eta) > 1 - \varepsilon/2$. There is $\delta' > 0$ sufficiently small such that if $\langle Z^\mu(\eta'), y\rangle \ge \delta \ \forall\, \eta' \ge \eta$ and $\forall\, \eta' \le \eta \ Z^\mu(\eta')x \le \bar{q}_1 x/2$, then $Z^\mu(t) \notin B_{\delta'}(q)$ $\forall\, 0 \le t \le 1$. Therefore, $\Pr(Z^\mu(t) \notin B_{\delta'}(q) \ \forall\, 0 \le t \le 1) < \varepsilon$.

Assume that $\bar{q}_2 > 0$. Set $\alpha = \mu^{NA}(I)$. If $\langle \alpha, y\rangle = 0$, then the positive scalar measure $\mu^c y$ is purely atomic with infinitely many atoms and, therefore, there is $\delta > 0$ sufficiently small such that $\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } |\langle Z^{\mu^c}(t), y\rangle - \bar{q}_2| < \delta) < \varepsilon$ and, therefore, for sufficiently small $\delta' > 0$, $\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } Z^\mu(t) \in B_{\delta'}(q)) < \varepsilon$. Assume thus that $\langle \alpha, y\rangle > 0$. Note that $Z^\mu(t)(\omega) = q$ only if $\langle Z^{\mu^c}(t)(\omega), y\rangle = \bar{q}_2$. The stochastic process $t \mapsto \langle Z^{\mu^c}(t), y\rangle$ is strictly increasing and, therefore, there is at most one value of $t = t(\omega)$ such that $\langle Z^{\mu^c}(t)(\omega), y\rangle = \bar{q}_2$.

Define the stopping time $\omega \mapsto s(\omega)$ by

$$s(\omega) := \inf\{0 \le t \le 1 : \langle Z^\mu(t), y\rangle \ge \bar{q}_2\} = \inf\{0 \le t \le : \langle Z^{\mu^c}(t), y\rangle \ge \bar{q}_2\}.$$

The stopping time s is independent of the Poisson bridge Z^{μ^J}. The distribution of $Z^{\mu^J}(s)$ is nonatomic and, therefore,

$$\begin{aligned} 0 &= \Pr(\langle Z^{\mu^J}(s), x\rangle = \bar{q}_1 - \langle Z^{\mu^c}(s), x\rangle) \\ &= \Pr(\langle Z^\mu(s), x\rangle = \bar{q}_1) \\ &\ge \Pr(\exists\, 0 \le t \le 1 \text{ s.t. } Z^\mu(t) = q). \end{aligned}$$

By the reflection principle, we have $\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } Z^\mu(t-) = q) = 0$ and, thus, $\Pr(\exists\, 0 \le t \le 1 \text{ s.t. } q \in Z^\mu(t) \cup Z^\mu(t-)) = 0$. Therefore, for every $\varepsilon > 0$, there is $\delta > 0$ s.t.

$$\Pr(\exists\, t \in [0,1] \text{ s.t. } Z^\mu(t) \in B_\delta(q)) < \varepsilon. \quad \square$$

4. Preliminary results. Our proof will make use of previously known results in value theory, which for completeness will now be stated.

The first result states that the Shapley value of a finite w.m.g. with sufficiently small weights is well approximated by the proportion of the weights.

PROPOSITION 3 (NEYMAN 1981). *For every $\varepsilon > 0$ there exists $K(\varepsilon)$ such that if $v = [q; w_1, \dots, w_m]$ is a w.m.g. such that*

$$K(\varepsilon)\max_j w_j \le q \le \sum_j w_j - K(\varepsilon)\max_j w_j, \tag{8}$$

then

$$\sum_j \left| \psi_j v - w_j \Big/ \sum_\ell w_\ell \right| < \varepsilon. \tag{9}$$

The next result states that all w.m.g.s have an asymptotic value. Given a scalar measure μ and a quota $0 < q < \mu(I)$, we denote by $[q; \mu]$ the w.m.g. v defined by

$$v(S) = \begin{cases} 1 & \text{if } \mu(S) \ge q, \\ 0 & \text{otherwise,} \end{cases}$$

and by $[q+; \mu]$ the w.m.g. v defined by

$$v(S) = \begin{cases} 1 & \text{if } \mu(S) > q, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 4 (NEYMAN 1988). *The asymptotic value of the w.m.g. $v = [q; \mu]$, where $0 < q < \mu(I)$, exists.*

We now comment on several simple corollaries of the above-mentioned propositions. Let $K(\varepsilon)$ be given by Proposition 3. Then, for every quota q and weights $w_1, \dots, w_m$ satisfying inequalities (8), the Shapley value of the w.m.g. $v = [q+; w_1, \dots, w_m]$ satisfies inequality (9) and the Shapley value of the simple game u defined by $u(S) = 1$ if $\sum_{i \in S} w_i = q$ and $u(S) = 0$ otherwise obeys

$$\sum_j |\psi_j u| < 2\varepsilon. \tag{10}$$

Indeed, the w.m.g. $v = [q+; w_1, \dots, w_m]$ is the dual of the w.m.g. $w := [\sum_i w_i - q; w_1, \dots, w_m]$, namely, for every coalition $S \subset I := \{1, \dots, m\}$,

$$v(I) - v(S^c) = w(S)$$

(where S^c is the complement $I \setminus S$ of S in I) and, therefore, the Shapley values of v and w coincide. Therefore, the Shapley value of the game v obeys inequality (9). The game u equals $[q; w_1, \dots, w_m] - v$ and, therefore, $|\psi_j u| = |\psi_j [q; w_1, \dots, w_m] - \psi_j v|$, which by the triangle inequality is $\le |\psi_j [q; w_1, \dots, w_m] - w_j / \sum_\ell w_\ell| + |\psi_j v - w_j / \sum_\ell w_\ell|$. Summing over j we deduce that (10) holds.

Also, if μ is a positive measure with atoms $\{a_i\}_i$ and q is a quota with $K(\varepsilon)\max_i \mu(a_i) < q < \mu(I) - K(\varepsilon)\max_i \mu(a_i)$, then the (asymptotic) value φv of the game $v = [q; \mu]$ satisfies

$$|\varphi v(S) - \mu(S)/\mu(I)| \le \varepsilon/2 \quad \text{for every coalition } S \tag{11}$$

and the (asymptotic) value φu of the game u, $u(S) = \mathbb{1}(\mu(S) = q)$, satisfies

$$|\varphi v(S) - \mu(S)/\mu(I)| \le \varepsilon \quad \text{for every coalition } S. \tag{12}$$

A direct corollary of the above-mentioned results asserts that if μ is a (mixed) scalar measure on $(I, \mathscr{C})$ and q is a quota such that for a given $\varepsilon > 0$ we have

$$K(\varepsilon) \max_{a\in A(\mu)} \mu(a) \le q \le \mu(I) - K(\varepsilon) \max_{a\in A(\mu)} \mu(a),$$

then the asymptotic values of the games $v = [q;\mu]$, $v^+ = [q+;\mu]$, and $v^= = [q=;\mu]$, where $v^=(S) = 1$ if $\mu(S) = q$ and $= 0$ otherwise, have asymptotic values φv, φv^+, and $\varphi v^=$, respectively, and for every coalition S, we have

$$|\varphi v(S) - \mu(S)/\mu(I)| \le \varepsilon/2, \quad |\varphi v^+(S) - \mu(S)/\mu(I)| \le \varepsilon/2, \quad \text{and} \quad |\varphi v^=(S)| \le \varepsilon.$$

5. Values of smooth vector measure games. The main result of this section, Theorem 4, establishes an asymptotic property of values of smooth vector measure games with finitely many players. This asymptotic property is used later in the proof of our main result. In addition, we derive Theorem 5 as a corollary of Theorem 4.

Fix an $\mathbb{R}^k$-valued vector measure μ. Recall that a sequence of lists of vectors in $\mathbb{R}^k_+$, $w_1^n, \dots, w_{\ell_n}^n, w_{\ell_n+1}^n, \dots, w_{m_n}^n$, is called μ-*admissible* if

$$\ell_n \to \infty, \qquad \sum_{i=1}^{\ell_n} \|w_i^n - \mu(a_i)\| \to_{n\to\infty} 0,$$

$$\max_{\ell_n < i \le m_n} \|w_i^n\| \to_{n\to\infty} 0, \qquad \text{and} \qquad \sum_{i=\ell_n+1}^{m_n} w_i^n \to \mu^{NA}(I).$$

The limiting results of this section assume a fixed $\mathbb{R}^k$-valued vector measure μ and a μ-admissible sequence $w_1^n, \dots, w_{\ell_n}^n, w_{\ell_n+1}^n, \dots, w_{m_n}^n$, and a continuously differentiable function f defined on a convex subset $R \supset \mathscr{R}(\mu)$ of $\mathbb{R}^k$ such that for every n and $S \subset \{1, \dots, m_n\}$, the vector $\sum_{i\in S} w_i^n$ is in R. It follows that all the games $v_n := [f; w_1^n, \dots, w_{m_n}^n]$, defined by $v_n(S) = f(\sum_{i\in S} w_i^n)$, as well as the game $f \circ \mu$ are well defined.

The Shapley value of player i in the game v_n is denoted $\Psi_{i,n}$. Z is the Poisson bridge associated with the vector measure μ. Recall that $Z(t) = t\mu^{NA}(I) + \sum_{i=1}^\infty \mathbb{1}(X_i \le t)\mu(a_i)$, where $X_1, X_2, \dots$ is a sequence of i.i.d. r.v.s that are uniformly distributed on $[0,1]$.

The proofs will make use of additional Poisson bridges. For every i, set $Z^{-i}(t) = Z(t) - \mathbb{1}(X_i \le t)\mu(a_i)$. The Poisson bridges Z_n are defined by means of an enlarged sequence $X_1, Y_1, X_2, Y_2, \dots$ of i.i.d. r.v.s that are uniformly distributed on $[0,1]$. For every n and every $i \le \ell_n$, set $Z_n(t) = \sum_{i=1}^{\ell_n} \mathbb{1}(X_i \le t) w_i^n + \sum_{j=\ell_n+1}^{m_n} \mathbb{1}(Y_j \le t) w_j^n$ and $Z_n^{-i}(t) = Z_n(t) - \mathbb{1}(X_i \le t) w_i^n$ (namely, $Z_n^{-i}(t) = \sum_{1\le j\le \ell_n, j\ne i} \mathbb{1}(X_j \le t) w_j^n + \sum_{j=\ell_n+1, j\ne i}^{m_n} \mathbb{1}(Y_j \le t) w_j^n$).

The marginal contribution of player $i \le \ell_n$ in the game $[f; w_1^n, \dots, w_{m_n}^n]$ and in the order induced by the values of $X_1(\omega), \dots, X_{\ell_n}(\omega)$, $Y_{\ell_n+1}(\omega), \dots, Y_{m_n}(\omega)$ equals $f(Z_n(X_i(\omega))(\omega)) - f(Z_n(X_i(\omega)-)(\omega))$ (where $Z_n(X_i(\omega)-)$ denotes the limit $\lim_{t\uparrow X_i(\omega)} Z_n(t)$). Therefore,

$$\Psi_{i,n} = E(f(Z_n(X_i)) - f(Z_n(X_i-))). \tag{13}$$

As $Z_n(X_i) = Z_n^{-i}(X_i) + w_i^n$ and $Z_n(X_i-) = Z_n^{-i}(X_i)$, we have

$$\Psi_{i,n} = E(f(Z_n^{-i}(X_i) + w_i^n) - f(Z_n^{-i}(X_i))). \tag{14}$$

THEOREM 4. *Assume that f is continuously differentiable on a neighborhood of R. Then, there exists a vector $p \in \mathbb{R}^k$ and a series Ψ_j with $\sum_{j=1}^\infty |\Psi_j| < \infty$ such that*

$$\sum_{j=\ell_n+1}^{m_n} |\Psi_{j,n} - p \cdot w_j^n| \to_{n\to\infty} 0,$$

where $\cdot$ stands for the inner product and

$$\lim_{n\to\infty} \Psi_{j,n} = \Psi_j.$$

The vector p is given by

$$p = \int_0^1 E(\nabla f(Z(t)))\, dt.$$

Proof. Recall that $\Psi_{j,n} = E(f(Z_n(Y_j)) - f(Z_n(Y_j-)))$ for $\ell_n < j \le m_n$. By the mean value theorem, for every $\ell_n < j \le m_n$ and every ω, there is $0 \le \theta \le 1$ such that

$$f(Z_n(Y_j)) - f(Z_n(Y_j-)) = \nabla f(Z_n(Y_j-) + \theta w_j^n)\cdot w_j^n.$$

$Z_n(Y_j) - (Z_n(Y_j-) + \theta w_j^n) = (1-\theta)w_j^n$. Therefore, as f is continuously differentiable, we deduce that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\|w_j^n\| < \delta$, then

$$\|\nabla f(Z_n(Y_j)) - \nabla f(Z_n(Y_j-) + \theta w_j^n)\| \le \varepsilon$$

and, therefore,

$$|f(Z_n(Y_j)) - f(Z_n(Y_j-)) - \nabla f(Z_n(Y_j))\cdot w_j^n| \le \varepsilon \|w_j^n\|.$$

For n sufficiently large, for every $\ell_n < j \le m_n$ we have $\|w_j^n\| < \delta$ and, therefore.

$$|\Psi_{j,n} - E(\nabla f(Z_n(Y_j)))\cdot w_j^n| \le \varepsilon\|w_j^n\| \quad \forall\, \ell_n < j \le m_n.$$

The r.v. $\sup_{0\le t\le 1}\|Z_n(t) - Z(t)\|$ converges in probability to 0 as $n\to\infty$. Therefore, $\max_{\ell_n<j\le m_n}\|\nabla f(Z_n(Y_j)) - \nabla f(Z(Y_j))\| \to_{n\to\infty} 0$ in probability. As ∇f is bounded in a neighborhood of $R \supset \mathcal{R}(\mu)$, we deduce that $\max_{\ell_n<j\le m_n}\|E(\nabla f(Z_n(Y_j))) - E(\nabla f(Z(Y_j)))\| \to_{n\to\infty} 0$ and, therefore, that for every $\varepsilon > 0$ there is $n(\varepsilon)$ such that for every $n \ge n(\varepsilon)$ and every $\ell_n < j \le m_n$, we have

$$\begin{aligned}\left|\Psi_{j,n} - \int_0^1 E(\nabla f(Z(t)))\,dt\cdot w_j^n\right| &= |\Psi_{j,n} - E(E(\nabla f(Z(Y_j)) \mid Y_j))\cdot w_j^n| \\ &= |\Psi_{j,n} - E(\nabla f(Z(Y_j)))\cdot w_j^n| \\ &\le 2\varepsilon\|w_j^n\|\end{aligned}$$

and, thus, $\sum_{j=\ell_n+1}^{m_n}|\Psi_{j,n} - p\cdot w_j^n| \le 2\varepsilon\sum_{\ell_n<j\le m_n}\|w_i^n\|$, implying that the sum $\sum_{j=\ell_n+1}^{m_n}|\Psi_{j,n} - p\cdot w_j^n|$ converges to 0 as $n\to\infty$.

By Equation (14) and the convergence in probability of $\sup_{0\le t\le 1}\|Z_n(t) - Z(t)\|$ to 0 as $n\to\infty$, we have

$$\lim_{n\to\infty}\Psi_{i,n} = E(f(Z(X_i)) - f(Z(X_i-))) := \Psi_i. \quad \square$$

A corollary of the above theorem is

Theorem 5. *Let μ be a vector measure and assume that f is a smooth function defined on the range of μ. Then, $f\circ\mu$ has an asymptotic value $\varphi(f\circ\mu)$. For every coalition $S\subset I$, the asymptotic value $\varphi(f\circ\mu)(S)$ is given by*

$$\begin{aligned}\varphi(f\circ\mu)(S) &= \sum_{i:\, a_i\in S}\Psi_i + p\cdot\mu^{NA}(S) \\ &= \sum_{i:\, a_i\in S}\Psi_i + E\left(\int_0^1 f_{\mu^{NA}(S)}(Z(t))\,dt\right),\end{aligned} \tag{15}$$

where $p = E(\int_0^1 \nabla f(Z(t))dt)$.

As mentioned earlier, the existence part of Theorem 5 follows from the known inclusion $pM \subset ASYMP$. A game of the form $f \circ \mu$ where f is smooth on the range of the vector measure μ is in pM, the closed space generated by polynomials of scalar measure games.

Step 1. Approximate $f \circ \mu$ with $g \circ \mu$ with g a polynomial.

Step 2. If g is a polynomial, then $g \circ \mu$ is a linear combination of polynomials of scalar measures and thus is in pM.

Formula (15) of the asymptotic value generalizes the classical diagonal formula for the value of smooth nonatomic vector measure games. Indeed, if μ is a nonatomic vector measure, then $Z(t) = t\mu(I)$. Therefore, the formula reduces to the classical diagonal formula

$$\varphi(f \circ \mu)(S) = \int_0^1 \nabla f(t\mu(I))\, dt \cdot \mu(S) = \int_0^1 f_{\mu(S)}(t\mu(I))\, dt.$$

The Poisson bridge $Z^\mu(t)$ is thus termed the *diagonal of the vector measure* μ.

6. Proof of the main result. Let $\nu = (\nu_1, \nu_2)$ satisfy the conditions of Theorem 2. We begin the proof by illustrating that any game in $LPS(\nu)$ is a linear combination of games or duals of games in the following two classes of games:

1. $v = (f \circ \nu)\mathbb{1}(\lambda\nu_1 + (1-\lambda)\nu_2 = \theta)$, where $0 \le \lambda \le 1$, $0 < \theta < 1$, and f bounded and piecewise smooth on L.

2. $v = (f \circ \nu)\mathbb{1}(\mu_1 > q_1$ and $\mu_2 > q_2)$, where $\mu_i = \lambda_i\nu_1 + (1-\lambda_i)\nu_2$, $0 \le \lambda_i \le 1$, $0 < q_i < \mu_i(I)$ for $i = 1, 2$, and f continuously differentiable on $\{x \in \mathbb{R}^2 \mid 0 \le x \le \nu(I)\}$.

Denote the first class of games by $\mathcal{U}_1$ and the second class of games by $\mathcal{U}_2$. Note that the first class of games, $\mathcal{U}_1$, also includes all games of the form

3. $f(x) = \mathbb{1}(x = (q_1, q_2))$ for some $0 < (q_1, q_2) < \nu(I)$.

For any smooth function, f, defined on the rectangle $[0, \nu_1(I)] \times [0, \nu_2(I)]$ and any $R \subset [0, \nu_1(I)] \times [0, \nu_2(I)]$, we can define the game v_R as follows:

$$v_R(S) = \begin{cases} f(\nu(S)) & \text{if } \nu(S) \in R \\ 0 & \text{otherwise.} \end{cases}$$

Next, we illustrate classes of sets R so that the games v_R defined above are linear combinations of games of the form 1 or 2 above.

To prove that every game in $LPS(\nu)$ is a linear combination of games in $\mathcal{U}_1 \cup \mathcal{U}_2$, we define three classes of games. The first class, $\mathcal{R}_1$, is the class of all such games, where R is a nonincreasing line segment with end points A and B, where $0 \le A = (x_1, y_1) \le \nu(I)$ and $0 \le B = (x_2, y_2) \le \nu(I)$ with $(x_1 - x_2)(y_1 - y_2) \le 0$ and if $A = B$, then $A \notin \{0, \mu(I)\}$. Then, any game in $\mathcal{R}_1$ is obviously a linear combination of games in $\mathcal{U}_1$.

Next, consider the class $\mathcal{R}_2$, where the set R is defined by means of two vectors $0 \le A = (x_1, y_1) \le \nu(I)$ and $0 \le B = (x_2, y_2) \le \nu(I)$ with $x_1 > x_2$ and $y_1 < y_2$ as follows: $z \in R$ if and only if z is in the convex hull of the points A, B, and $C := (x_1, y_2)$. See Figure 1.

We will show that $v := v_R$ is a linear combination of games in $\mathcal{U}_1 \cup \mathcal{U}_2$.

W.l.o.g. we assume that f is a smooth function defined on the rectangle $[0, \nu_1(I)] \times [0, \nu_2(I)]$. Set $\lambda = (y_2 - y_1)/(y_2 - y_1 + x_1 - x_2)$, $\mu = \lambda\nu_1 + (1-\lambda)\nu_2$, and $q = \lambda x_1 + (1-\lambda)y_1 = \lambda x_2 + (1-\lambda)y_2$. Define the following games:

$$v_0(S) = \begin{cases} f(\nu(S)) & \text{if } \mu(S) > q, \\ 0 & \text{otherwise,} \end{cases}$$

$$v_1(S) = \begin{cases} f(\nu(S)) & \text{if } \nu_1(S) > x_1 \text{ and } \mu(S) > q, \\ 0 & \text{otherwise,} \end{cases}$$

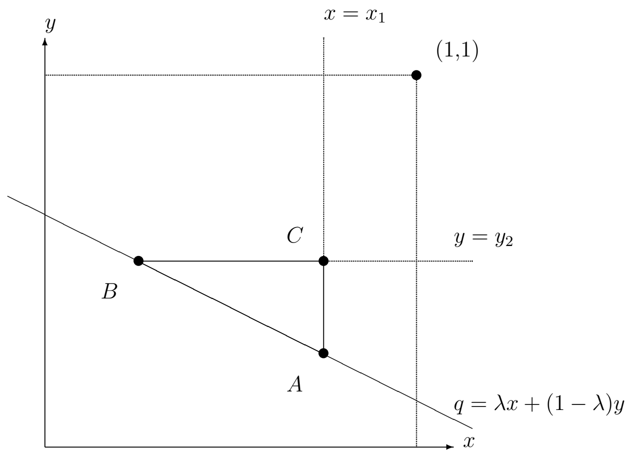

FIGURE 1. The region R.

$$v_2(S) = \begin{cases} f(\nu(S)) & \text{if } \nu_2(S) > y_2 \text{ and } \mu(S) > q, \\ 0 & \text{otherwise,} \end{cases}$$

$$v_3(S) = \begin{cases} f(\nu(S)) & \text{if } \nu_1(S) > x_1 \text{ and } \nu_2(S) > y_2, \\ 0 & \text{otherwise.} \end{cases}$$

The game $v - v_0 + v_1 + v_2 - v_3$ is a linear combination of games in $\mathcal{U}_1$, and, therefore, any game $v \in \mathcal{R}_2$ is a linear combination of games in $\mathcal{U}_1 \cup \mathcal{U}_2$.

The class $\mathcal{R}_3$ is obtained by setting $D = (x_2, y_1)$ and R is the convex hull of A, B, and D. Then, a similar decomposition illustrates that every game $v \in \mathcal{R}_3$ is a linear combination of games in $\mathcal{U}_1 \cup \mathcal{U}_2$.

As the linear combinations of the games in $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ include all games in $LPS(\nu)$, it is sufficient to show that every game in $\mathcal{U}_1 \cup \mathcal{U}_2$ has an asymptotic value to conclude the validity of Theorem 2.

6.1. Setting the stage. Recall that w.l.o.g. we assume that $\mathcal{C}$ is countably generated and separating. In addition, the assumptions on (ν_1, ν_2) imply that $\mathcal{C}$ is an infinite σ-field.

Throughout the proof we fix a coalition $C \in \mathcal{C}$ and a C-admissible sequence $\{\Pi_n\}_{n=1}^{\infty}$. The set of atoms of Π_n is denoted $A(\Pi_n)$.

Let $\nu = (\nu_1, \nu_2)$ satisfy the conditions of the main theorem, $0 \le \lambda_1 < \lambda_2 \le 1$ and $\mu_i = \lambda_i \nu_1 + (1 - \lambda_i)\nu_2$. Let $A(\nu) = \{a_i\}_{i=1}^{\infty}$ and set $\alpha_i = \mu_i(I) - \sum_{j=1}^{\infty} \mu_i(a_j) = \mu_i^{NA}(I)$, $i = 1, 2$.

Let $(c_i)_{i=1}^{\infty}$ be a sequence of distinct elements of I s.t. $\{c_i \mid i \ge 1\} \supset \{a_i \mid i \ge 1\}$, and for every n and every $a \in \Pi_n$, there is $i \ge 1$ s.t. $c_i \in a$. Let $(X_i)_{i=1}^{\infty}$ be a sequence of i.i.d. r.v.s that are uniformly distributed on $[0, 1]$ and w.l.o.g. assume that for all $i \ne j$, $X_i \ne X_j$ everywhere.

For every $a \in I$, we denote by $\pi_n(a)$ the unique atom of Π_n that contains a. For every $a \in A(\Pi_n)$, we denote by $i(a) = \min\{i \mid c_i \in a\}$ and $X_a^{\Pi_n} = X_{i(a)}$.

Let Ω be the sample space on which all these random variables are defined and let P denote the probability measure on Ω. The Poisson bridge associated with a vector measure μ is defined by

$$Z^{\mu}(t) = \sum_{i=1}^{\infty} \mu(c_i) \mathbb{1}(X_i \le t) + t\mu^{NA}(I).$$

The Poisson bridge associated with the finite field Π_n and a vector measure μ is given by

$$Z_n^{\mu}(t) = \sum_{a \in A(\Pi_n)} \mu(a) \mathbb{1}(X_a^{\Pi_n} \le t).$$

Let Z and Z_n stand for Z^ν and Z_n^ν. Set

$$\Omega(n,\delta) := \{\omega \in \Omega : \forall\, 0 \le t \le 1 \quad \|Z_n(t) - Z(t)\| < \delta\},$$

and recall that by Proposition 2,

$$P(\Omega(n,\delta)) \to_{n\to\infty} 1. \tag{16}$$

For every game v, a positive integer n, and an atom a of Π_n, we define the random variable $\Delta v_n(a)$ by

$$\Delta v_n(a) = v\bigg(\sum_{b\in A(\Pi_n)} b\mathbb{1}(X_b^{\Pi_n} \le X_a^{\Pi_n})\bigg) - v\bigg(\sum_{b\in A(\Pi_n)} b\mathbb{1}(X_b^{\Pi_n} < X_a^{\Pi_n})\bigg),$$

where $\sum_{b\in A(\Pi_n)} b\mathbb{1}(X_b^{\Pi_n} \le X_a^{\Pi_n})$ stands for the union of all $b \in A(\Pi_n)$ such that $X_b^{\Pi_n} \le X_a^{\Pi_n}$ and $\sum_{b\in A(\Pi_n)} b\mathbb{1}(X_b^{\Pi_n} < X_a^{\Pi_n})$ stands for the union of all $b \in A(\Pi_n)$ such that $X_b^{\Pi_n} < X_a^{\Pi_n}$.

Let v_n stand for the finite game v_{Π_n}. Recall that for every atom a of Π_n, we have

$$\psi v_n(a) = E(\Delta v_n(a)).$$

Let $\mathcal{F}_t$ denote the σ-field generated by all the random variables $X_j \cdot \mathbb{1}(X_j \le t)$, $j \ge 1$.

6.2. The case $v = \mathbb{1}(\nu = (q_1, q_2))$.

Proposition 5. *Let $0 < (q_1, q_2) < \nu(I)$ and $f(x) = \mathbb{1}(x = (q_1, q_2))$. Then, $f \circ \nu$ has an asymptotic value* $(=0)$.

Proof. By Lemma 6, for every $\varepsilon > 0$ there is $\delta > 0$ sufficiently small so that the Poisson bridge $Z(t)$ associated with the vector measure $\nu = (\nu_1, \nu_2)$ will hit $B_\delta(q) := \{x \in \mathbb{R}^2 : \|x - (q_1, q_2)\| < \delta\}$ with probability $< \varepsilon$.

Fix a sufficiently large $n(\delta)$ such that for every $n \ge n(\delta)$, we have

$$\Pr\Big(\sup_{0\le t\le 1} \|Z_n(t) - Z(t)\| \ge \delta/2\Big) < \varepsilon.$$

On $\|Z_n(t) - Z(t)\| < \delta/2$ and $\inf_{0\le t\le 1} \|q - Z(t)\| \ge \delta$, we have $f(Z_n(t)) - f(Z_n(t-)) = 0$ $\forall\, t$. Therefore, $\sum_{a\in A(\Pi_n)} |f(Z_n(X_a^{\Pi_n}) - f(Z_n(X_a^{\Pi_n}-)| = 0$ on a set of probability $\ge 1 - 2\varepsilon$. As $\sum_{a\in A(\Pi_n)} |f(Z_n(X_a^{\Pi_n}) - f(Z_n(X_a^{\Pi_n}-)| \le \|v\| \le 2$ everywhere, we deduce that for $n \ge n(\delta)$, we have $|\psi v_{\Pi_n}(C)| \le 4\varepsilon$. Therefore, $\limsup_n |\psi v_{\Pi_n}(C)| \le 4\varepsilon$. As this holds for every $\varepsilon > 0$, we deduce that $\lim_{n\to\infty} \psi v_{\Pi_n}(C) = 0$, which completes the proof. □

6.3. $v = (f \circ \nu)\mathbb{1}(\mu = \theta)$.

Proposition 6. *Assume that $\mu = \lambda\nu_1 + (1-\lambda)\nu_2$, $0 \le \lambda \le 1$, $0 < \theta < \mu(I)$, $L = \{q \in \mathbb{R}^2 \mid \lambda q_1 + (1-\lambda)q_2 = \theta, 0 \le q_i \le \nu_i(I)\}$, and f bounded and piecewise smooth on L. Then, v has an asymptotic value φv, and if μ does not have a finite support, then $\varphi v = 0$.*

Proof. If μ has a finite support, then v is a scalar measure game and, thus, has an asymptotic value.

Assume that μ does not have a finite support. Fix $\varepsilon > 0$. If $\mu^{NA}(I) = 0$, then μ is purely atomic with infinitely many atoms. Therefore, by Corollary 1, for sufficiently small $\delta > 0$, the probability that there is $0 \le t \le 1$ such that $|Z^\mu(t) - \theta| \le 2\delta$ is $< \varepsilon$ and, therefore, there is m sufficiently large such that for every $n \ge m$, the probability that there is $0 \le t \le 1$ such that $|Z_n^\mu(t) - \theta| \le \delta$ is $< 2\varepsilon$. Therefore, for sufficiently large n, we have $|\psi v_n(C)| \le 2\varepsilon\|v\|$. Thus, $\limsup_{n\to\infty} |\psi v_n(C)| \le 2\varepsilon\|v\|$. As this inequality holds for every $\varepsilon > 0$, we deduce that $\lim_{n\to\infty} \psi v_n(C) = 0$.

Assume that $\alpha := \mu^{NA}(I) > 0$. It is sufficient to prove that for every $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} |\psi v_n(C)| < \varepsilon.$$

As f is piecewise smooth on L, there is a set D of finitely many points $q = (q_1, q_2)$ such that $L \setminus D$ is a finite union of intervals L_k and f is Lipschitz on each interval L_k.

Fix $\varepsilon > 0$. There is $\delta_1 > 0$ sufficiently small such that for every k,

$$|f(x) - f(y)| < \varepsilon \quad \forall\, x, y \in L_k \text{ s.t. } \|x - y\|_\infty < 6\delta_1$$

and

$$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| < \varepsilon \quad \forall\, x_0 \le x_1 \le \dots \le x_m \in L_k \text{ s.t. } \|x_m - x_0\|_\infty < 6\delta_1.$$

(The last condition is obviously redundant whenever $0 < \lambda < 1$.)

We will fix a sufficiently large constant K and a sufficiently small $\delta > 0$.

6.3.1. Definition of K. Let $K > 3$ be a sufficiently large constant such that for all weights $0 \le w_1, \dots, w_m$ and quota θ with

$$K \max_i w_i/3 \le \theta \le \sum_i w_i - K \max_i w_i/3, \tag{17}$$

the value of the w.m.g. $u = [\theta; w_1, \dots, w_m]$ satisfies

$$\sum_{j=1}^{m} \left| \psi_j u - \frac{w_j}{\sum_\ell w_\ell} \right| < \varepsilon. \tag{18}$$

The existence of such a constant K follows from Proposition 3.

As in the closing comments in §4, it follows that if $\theta; w_1, \dots, w_m$ obey the above conditions (17), then the Shapley value of the simple game u^* with player set $\{1, 2, \dots, m\}$ and defined by $u^*(S) = \mathbb{1}(w(S) = \theta)$ satisfies

$$\sum_j |\psi_j u^*| < 2\varepsilon.$$

Indeed, set $u^+ = [\theta+; w_1, \dots, w_m]$ and $\bar{u} = [\sum_{j=1}^n w_j - \theta; w_1, \dots, w_m]$. Then, $u^* = u - u^+$, $\psi u^+ = \psi \bar{u}$, and the w.m.g.s u and $\bar{u}$ obey conditions (17).

6.3.2. Definition of δ. Let r be the stopping time

$$r(\omega) := \inf\{0 \le t \le 1 \mid Z^\mu(t) \ge \theta\}.$$

As $Z^\mu(t) \to_{t\to 1-} \mu(I)$ and $Z^\mu(t) \to_{t\to 0+} \mu(\varnothing) = 0$ in probability, we deduce that there exists $\xi > 0$ such that

$$P(2\xi < r < 1 - 2\xi) > 1 - \varepsilon. \tag{19}$$

Set $m(\delta) = |\{j \ge 1 : \|\nu(c_j)\|_1 \ge \delta\}|$, where for a set S we denote by $|S|$ the number of elements in S. Note that $m(\delta)\delta \to_{\delta\to 0+} 0$. Let $D_{K\delta} = \bigcup_{q\in D} B_{K\delta}(q)$ (where, as in §3.3, $B_{K\delta}(q) = \{x \in \mathbb{R}^2 : \|x - q\|_\infty \le K\delta\}$). Recall that Z and Z_n stand for Z^ν and Z_n^ν, respectively. It follows from Lemma 6 that

$$P(\exists\, 0 \le t \le 1 \text{ s.t. } Z(t) \in D_{K\delta}) \to_{\delta\to 0+} 0.$$

Obviously,

$$\sum_{j:\,\|\nu(c_j)\|_1\le\delta}\|\nu(c_j)\|_1\to_{\delta\to 0+}0.$$

Let $\delta>0$ be sufficiently small so that

$$2K\delta+2K\delta\,\frac{\max_{i=1,2}\nu_i^{NA}(I)}{\alpha}+\sum_{j:\,\|\nu(c_j)\|_1\le\delta}\|\nu(c_j)\|_1<\delta_1, \tag{20}$$

$$\frac{m(\delta)\delta K}{\alpha\xi}<\varepsilon, \tag{21}$$

and

$$P(\exists\, 0\le t\le 1 \text{ s.t. } Z(t)\in D_{K\delta})<\varepsilon. \tag{22}$$

6.3.3. The stopping time $s=s_\delta$. Define the $(\mathcal{F}_t)_t$-stopping time s_δ, or s for short, by

$$s(\omega)=\inf\{t\ge 0\mid Z^\mu(t)\ge\theta-K\delta\}.$$

Note that s is a two-sided stopping time. Define the $(\mathcal{F}_t)_t$-stopping time s_1 by

$$s_1(\omega)=\begin{cases} s(\omega)+2K\delta/\alpha & \text{if } s(\omega)<1-2K\delta/\alpha \text{ and } Z^\mu(s(\omega))<\theta,\\ 1 & \text{if } s(\omega)\ge 1-2K\delta/\alpha \text{ and } Z^\mu(s(\omega))<\theta,\\ s & \text{otherwise.}\end{cases}$$

We now call attention to an implicit (but essential) property of the stopping times s and s_1, which is used implicitly in the proof. For every sequence of independent r.v.s X_i, Y_i^j, $j=0,2,3$ and $i\ge 1$ that are uniformly distributed on $(0,1)$, the $[0,1]$-valued random variables X_i^*, defined by

$$X_i^*(\omega)=\begin{cases} Y_i^0(\omega)s(\omega) & \text{if } X_i(\omega)<s(\omega),\\ s(\omega)=X_i(\omega) & \text{if } X_i(\omega)=s(\omega),\\ s(\omega)+(s_1(\omega)-s(\omega))Y_i^2 & \text{if } s(\omega)<X_i(\omega)<s_1(\omega),\\ s_1(\omega)+(1-s_1(\omega))Y_i^3 & \text{if } s_1(\omega)<X_i(\omega),\end{cases}$$

are i.i.d. uniformly distributed on $(0,1)$.

6.3.4. Partitioning the probability space. Recall that Z and Z_n stand for Z^ν and Z_n^ν, respectively.

Let $\Omega(m)$ be the event of all ω such that $Z(s(\omega))\notin D_{K\delta}$, $Z^\mu(s(\omega))\le\theta-(K-1)\delta$, and $s(\omega)<1-2\xi$. Let $\Omega(+)$ be the event of all ω such that $Z^\mu(s(\omega))\ge\theta+K\delta$.

LEMMA 8.

$$P(\Omega(+)\cup\Omega(m))\ge 1-3\varepsilon.$$

PROOF. Recall that $\mu^{NA}(I)=\alpha>0$ by assumption. If $\omega\notin\Omega(+)\cup\Omega(m)$, then either there is $0\le t\le 1$ such that $Z(t)\in D_{K\delta}$, or there is t with $\theta-K\delta<Z^\mu(t)<\theta+K\delta$ and $Z^\mu(t)-Z^\mu(t-)\ge\delta$, or $s(\omega)\ge 1-2\xi$. Using Lemma 5 with inequality (21) and inequalities (22) and (19) completes the proof. □

We continue by deriving a useful inequality about the distance between $Z(s)$ and $Z(s_1)$. For every $j\ge 1$, we have

$$P(X_j\in(s,s_1]\mid\mathcal{F}_s)\le\frac{s_1-s}{1-s}.$$

Therefore, on $s(\omega) < 1-2\xi < 1-2K\delta/\alpha$, and thus, in particular, on $\Omega(m)$, we have for every $j \ge 1$ that

$$P(X_j \in (s, s_1] \mid \mathcal{F}_s) \le \frac{K\delta}{\alpha\xi}.$$

Therefore, using inequality (21), on $s(\omega) < 1-2\xi < 1-2K\delta/\alpha$, we have

$$P(\exists\, j \text{ s.t. } \|\nu(c_j)\|_1 \ge \delta \text{ and } X_j \in (s, s_1] \mid \mathcal{F}_s) < \varepsilon, \tag{23}$$

and thus, by using inequality (20),

$$P(\|Z(s_1) - Z(s)\|_\infty \ge \delta_1 \mid \mathcal{F}_s) < \varepsilon \quad \text{on } s < 1-2\xi. \tag{24}$$

It follows, in particular, that for $\omega \in \Omega(m)$ there is k such that whenever $x, y \in L$ satisfy $Z(s) - (\delta_1, \delta_1) \le x, y \le Z(s) + 2(\delta_1, \delta_1)$, then $x, y \in L_k$ and thus, in particular, $|f(x) - f(y)| < \varepsilon$, and whenever $x_0 < x_1 \le \ldots \le x_m \in L$ (which is possible only when $\lambda(1-\lambda) = 0$) and $Z(s) - (\delta_1, \delta_1) \le x_0, x_m \le Z(s) + 2(\delta_1, \delta_1)$, then $x_0, x_m \in L_k$ and thus, in particular, $\sum_{i=1}^m |f(x_i) - f(x_{i-1})| < \varepsilon$.

6.3.5. The random decomposition of v_n. We will define auxiliary finite games $v_n^0(\omega)$, $v_n^1(\omega)$, $v_n^2(\omega)$, $v_n^3(\omega)$, and $u_n^1(\omega)$.

It will follow from the definition of the auxiliary random games that

$$\|v_n^0(\omega)\| + \|v_n^1(\omega)\| + \|v_n^2(\omega)\| + \|v_n^3(\omega)\| \le \|v\| \quad \forall\, n, \tag{25}$$

$$\psi v_n(C) = E(\psi v_n^0(C) + \psi v_n^1(C) + \psi v_n^2(C) + \psi v_n^3(C)), \tag{26}$$

and for sufficiently large n,

$$P(|\psi v_n^i(C)| \ge \eta_1) < \varepsilon_1 \quad \forall\, 0 \le i \le 3. \tag{27}$$

It follows that for sufficiently large n,

$$P(|\psi v_n^0(C) + \psi v_n^1(C) + \psi v_n^2(C) + \psi v_n^3(C)| \ge 4\eta_1) < 4\varepsilon_1, \tag{28}$$

and thus,

$$|\psi v_n(C)| \le 4\eta_1 + 4\varepsilon_1 \|v\|. \tag{29}$$

The set of players of $v_n^i(\omega)$ is the set of atoms of Π_n and it is partitioned into a set of null players $D_n^i(\omega)$ and a set of essential players $S_n^i(\omega)$. Note that some essential players may be null players as well. This partition is, however, useful as it enables us to define the game $v_n^i(\omega)$ by specifying $v_n^i(\omega)(S)$ for subsets S of the set of essential players $S_n^i(\omega)$.

The sets $S_n^i(\omega)$ are defined by means of the two $(\mathcal{F}_t)_t$-stopping times, the two-sided stopping time $s = s_\delta$ and s_1. The two stopping times s and s_1 partition the interval $[0, 1]$ into four random intervals: $J_0(\omega) = [0, s(\omega))$, $J_1(\omega) = [s(\omega), s(\omega)]$, $J_2(\omega) = (s(\omega), s_1(\omega)]$, and $J_3(\omega) = (s_1(\omega), 1]$. Note that $J_2(\omega)$ may be empty.

The set $S_n^i(\omega)$ of essential players in $v_n^i(\omega)$ is the set of all atoms a of Π_n such that $X_a^{\Pi_n}(\omega) \in J_i(\omega)$.

For a coalition $S \subset S_n^i(\omega)$, we define

$$v_n^i(S) = v_n\left(\bigcup_{j<i} S_n^j(\omega) \cup S\right) - v_n\left(\bigcup_{j<i} S_n^j(\omega)\right).$$

Note that by convention a union over an empty class of sets is the empty set, and thus, $v_n^0(S) = v_n(S)$ for $S \subset S_n^0(\omega)$.

Assume w.l.o.g. that $\|f\|_\infty, \|v\| \le 1$.

Observe that for every $\omega \in \Omega(+)$ and n with $\sup_{0\le t\le 1} |Z_n^\mu(t) - Z^\mu(t)| < \delta$ the games $v_n^0(\omega)$, $v_n^1(\omega)$, $v_n^2(\omega)$, and $v_n^3(\omega)$ vanish, and therefore,

$$\left\{\omega \in \Omega(+) : \sum_{i=0}^{3} \psi v_n^i(\omega)(C) \neq 0\right\} \subset \Omega \setminus \Omega(n, \delta).$$

Therefore, for sufficiently large n, we have

$$P\left(\left\{\omega \in \Omega(+) : \sum_{i=0}^{3} \psi v_n^i(\omega)(C) \neq 0\right\}\right) < \varepsilon. \tag{30}$$

Similarly, for $\omega \in \Omega(m) \cap \Omega(n, \delta)$, the games $v_n^0(\omega)$, $v_n^1(\omega)$, and $v_n^3(\omega)$ vanish. Therefore, for sufficiently large n, we have

$$P(\{\omega \in \Omega(m) : \psi v_n^0(\omega)(C) + \psi v_n^1(\omega)(C) + \psi v_n^3(\omega)(C) \neq 0\}) < \varepsilon. \tag{31}$$

On $\Omega(m)$ we approximate the game $v_n^2(\omega)$ by a game $u_n(\omega)$. This is a crucial step in the proof.

We provide an alternative definition of $v_n^2(\omega)$ for $\omega \in \Omega(m)$ with $Z_n^\mu(s(\omega)) < \theta$. Denote by $\nu^n(\omega)$ the vector measure on Π_n that is given by

$$\nu^n(\omega)(S) = \sum_{a \in A(\Pi_n): a \subset S} \nu(a) \mathbb{1}(s(\omega) < X_a^{\Pi_n} \le s_1(\omega))$$

and by $\mu^n(\omega)$ the scalar measure on Π_n that is given by

$$\mu^n(\omega)(S) = \sum_{a \in A(\Pi_n): a \subset S} \mu(a) \mathbb{1}(s(\omega) < X_a^{\Pi_n} \le s_1(\omega)).$$

For $\omega \in \Omega(m)$ with $Z_n^\mu(s(\omega)) < \theta$, we have for every coalition $S \in \Pi_n$,

$$v_n^2(\omega)(S) = \begin{cases} f(Z_n^\nu(s(\omega)) + \nu^n(\omega)(S)) & \text{if } Z_n^\mu(s(\omega)) + \mu^n(S) = \theta, \\ 0 & \text{otherwise.} \end{cases}$$

For $\omega \in \Omega(m)$ with $Z_n^\mu(s(\omega)) < \theta$, the game $v_n^2(\omega)$ is approximated by the game $u_n(\omega)$ that is given by

$$u_n(\omega)(S) = \begin{cases} f(x(Z(s(\omega)))) & \text{if } Z_n^\mu(s(\omega)) + \mu^n(S) = \theta, \\ 0 & \text{otherwise,} \end{cases}$$

where for $0 \le y \le \nu(I)$ we denote by $x(y)$ the unique point in L of the form $(1-\beta)y + \beta\nu(I)$. It follows that for $\omega \in \Omega(m) \cap \Omega(n, \delta_1)$ with $Z(s_1) \le Z(s) + (\delta_1, \delta_1)$, we have (assuming w.l.o.g. that $\|v\| \le 1$ and that $\delta > 0$ is sufficiently small so that on $\Omega(m)$ we have $x(Z(s)) \le Z(s) + (\delta_1, \delta_1)$)

$$\|v_n^2(\omega) - u_n(\omega)\| < 3\varepsilon.$$

Using inequality (24) the probability that $\omega \in \Omega(m)$ and $Z(s_1)$ is not $\le Z(s) + (\delta_1, \delta_1)$ is $\le \varepsilon$. Therefore, we deduce that for sufficiently large n,

$$P(\{\omega \in \Omega(m) : \|v_n^2(\omega) - u_n(\omega)\| > 3\varepsilon\}) < 3\varepsilon$$

and, therefore,

$$P(\{\omega \in \Omega(m) : |\psi v_n^2(\omega)(C) - \psi u_n(\omega)(C)| > 3\varepsilon\}) < 3\varepsilon. \tag{32}$$

Note that for $\omega \in \Omega(m)$ with $Z_n^\mu(s(\omega)) < \theta$, the game $u_n(\omega)$ is a constant $f(x(Z(\sigma(\omega))))$ times a simple game $\mathbb{1}(\mu^n(S) = \theta - Z^\mu(s(\omega)))$.

The probability that $\omega \in \Omega(m)$ and the quota $\theta - Z^\mu(s(\omega))$ and weights $(\mu(a))_{a\in A(\Pi_n)}$ of the simple game $\mathbb{1}(\mu^n(S) = \theta - Z^\mu(s(\omega)))$ violate condition (17) is $< 2\varepsilon$. Indeed, the conditional probability, given $\mathcal{F}_s$, that there is j with $\mu(c_j) \geq \delta$ and $s < X_j < s_1$ is $\leq \varepsilon$ on $s < 1 - 2\xi$, and for sufficiently large n, the probability $P(\sup_t |Z^\mu(t) - Z_n^\mu(t)| \geq \delta)$ is $< \varepsilon$. Therefore (assuming w.l.o.g. that $|f(x)| \leq 1$ for all $x \in L$),

$$P(\{\omega \in \Omega(m) : |\psi u_n(C)| \geq \varepsilon\}) < 2\varepsilon,$$

which together with (32) implies that

$$P(\{\omega \in \Omega(m) : |\psi v_n^2(\omega)(C)| > 4\varepsilon\}) < 5\varepsilon.$$

Combined with (31) we conclude that

$$P\left(\left\{\omega \in \Omega(m) : \left|\sum_{i=0}^{3} \psi v_n^i(\omega)(C)\right| > 4\varepsilon\right\}\right) < 6\varepsilon. \tag{33}$$

Therefore, using Lemma 8, (33), and (30), for sufficiently large n we have

$$P\left(\left\{\omega \in \Omega : \left|\sum_{i=0}^{3} \psi v_n^i(\omega)(C)\right| > 4\varepsilon\right\}\right) < 10\varepsilon.$$

As $|\sum_{i=0}^3 \psi v_n^i(\omega)(C)| \leq \sum_{i=0}^3 \|v_n^i(\omega)\| \leq \|v\| \leq 1$ everywhere, we deduce that for sufficiently large n we have

$$|\psi v_n(C)| \leq E\left(\left|\sum_{i=1}^{3} \psi v_n^i(\omega)(C)\right|\right) < 4\varepsilon + 10\varepsilon.$$

Therefore,

$$\limsup_{n\to\infty} |\psi v_n(C)| \leq 4\varepsilon + 10\varepsilon,$$

and as this holds for every $\varepsilon > 0$, we conclude that $\lim_{n\to\infty} \psi v_n(C) = 0$. □

6.4. $v = (f \circ \mu)\mathbb{1}(\mu_1 > q_1 \text{ and } \mu_2 > q_2)$.

Proposition 7. *Assume that* $\mu_i = \lambda_i \nu_1 + (1-\lambda_i)\nu_2$, $i = 1, 2$, $0 \leq \lambda_2 < \lambda_1 \leq 1$, $0 < q_i < \mu_i(I)$, $L = \{x \in \mathbb{R}^2 \mid \mu_i(I) \geq x_i > q_i\}$, *and* f *continuously differentiable on* $\{x \in \mathbb{R}^2 \mid 0 \leq x \leq \mu(I)\}$. *Then, the game* $v = (f \circ \mu)\mathbb{1}(\mu_1 > q_1$ *and* $\mu_2 > q_2)$, *namely,*

$$v(S) = \begin{cases} f(\mu(S)) & \text{if } \mu(S) \in L, \\ 0 & \text{otherwise,} \end{cases}$$

has an asymptotic value.

Proof. We have to prove that the limit, $\lim_{n\to\infty} \psi v_{\Pi_n}(C)$, exists and is independent of the C-admissible sequence $(\Pi_n)_n$.

We assume w.l.o.g. that $\|v\| \leq 1$, $|f(x)| \leq 1$ $\forall x$, $\nu_i(I) = 1$, and $\|\nabla f(x)\|_\infty \leq 1/2$ $\forall x$. Let $\bar{f}$ be the function that coincides with f on L and equals 0 on the complement of L.

We start by describing the structure of the proof. (Ω, P) is the probability space on which the Poisson bridge $Z(t)$ is defined. $(I, \mathscr{C})$ is the measurable space of players. We define (below) a random measure $\omega \mapsto \tau(\omega)$ on $(I, \mathscr{C})$ and (in §6.4.1) a random coalitional game $\omega \mapsto \tilde{v}(\omega)$ such that $\tilde{v}(\omega)$ is a smooth vector measure game (and thus has an asymptotic value $\varphi\tilde{v}(\omega)$) and the entire proof in §6.4 will show that the asymptotic value of v exists and is given by the formula

$$\varphi v(S) = E(\tau(S) + \varphi\tilde{v}(S)).$$

The random measure τ and the random smooth vector measure game $\tilde{v}$ are defined by means of the $(\mathcal{F}_t)_t$-stopping time r that is defined as the entry time of the Poisson bridge $Z^\mu(t)$ into the set L.

The $(\mathcal{F}_t)_{0\le t\le 1}$-stopping time r is defined by

$$r(\omega) = \inf\Big\{t\colon \min_{i=1,2}(Z^{\mu_i}(t) - q_i) \ge 0\Big\}.$$

The random measure $\tau(\omega)$ is the "jump" of the process at the stopping time r, namely,

$$\tau(\omega)(S) = \begin{cases} f(Z^\mu(r)) & \text{if } \exists\, j \text{ s.t. } c_j \in S \text{ and } X_j = r, \\ f(Z^\mu(r))\dfrac{\mu_i^{NA}(S)}{\mu_i^{NA}(I)} & \text{if } Z^{\mu_i}(r) = q_i \text{ and } Z^{\mu_{3-i}}(r) > q_{3-i}. \end{cases}$$

Note that the probability that there is j such that $X_j = r$ and $Z^\mu(r) \notin L$ is 0, and if $\mu_i^{NA}(I) = 0$, then the probability that $Z^{\mu_i}(r) = q_i$ equals 0. Therefore, τ is well defined almost everywhere.

The random coalitional game $\tilde{v}(\omega)$ is the smooth vector measure game defined by the marginal contribution to the coalition of players appearing up to time r, namely,

$$\tilde{v}(\omega)(S) = f\bigg(Z^\mu(r) + (1-r)\mu^{NA}(S) + \sum_{j\colon c_j\in S}\mu(c_j)\mathbb{1}(X_j > r(\omega))\bigg) - f(Z^\mu(r)).$$

Given $\varepsilon > 0$, we approximate the stopping time r by a stopping time $s \le r$ and a stopping time $s_1 \ge r$; see §§6.4.3 and 6.4.4, respectively.

The two stopping times s and s_1 partition the interval $[0,1]$ into four random intervals: $J_0(\omega) = [0, s(\omega))$, $J_1(\omega) = [s(\omega), s(\omega)]$, $J_2(\omega) = (s(\omega), s_1(\omega)]$, and $J_3(\omega) = (s_1(\omega), 1]$. The random partition of the interval $[0,1]$ induces a random partition of the players $A(\Pi_n)$ (of v_n) according to the values of $X_a^{\Pi_n}$: $S_n^k(\omega)$ is the set of atoms $a \in A(\Pi_n)$ such that $X_a^{\Pi_n}(\omega) \in J_k(\omega)$. For a coalition $S \in \Pi_n$, we define

$$v_n^k(\omega)(S) = v_n\bigg(\bigcup_{j<k} S_n^j(\omega) \cup (S \cap S_n^k(\omega))\bigg) - v_n\bigg(\bigcup_{j<k} S_n^j(\omega)\bigg).$$

It follows from the definition of s and s_1 that s is a (two-sided) stopping time and s_1 is measurable with respect to $\mathcal{F}_s$. Therefore,

$$\psi v_n(C) = E\bigg(\sum_{k=0}^{3}\psi v_n^k(C)\bigg).$$

We will show that

$$\psi v_n^3(C) \quad \text{is approximately } \varphi\tilde{v}(C),$$
$$\psi v_n^0(C) \quad \text{is approximately } 0, \text{ and}$$
$$\psi v_n^1(C) + \psi v_n^2(C) \quad \text{is approximately } \tau(C).$$

Therefore, for sufficiently large n,

$$\sum_{k=0}^{3}\psi v_n^k(C)) \quad \text{is approximately } \tau(C) + \varphi\tilde{v}(C)$$

and, therefore,

$$\psi v_n(C) = E\left(\sum_{k=0}^{3} \psi v_n^k(C)\right) \quad \text{is approximately } \varphi v = E(\tau(C) + \varphi\tilde{v}(C)).$$

By conditioning on the values of $Z^\mu(s)$ and s, in §6.4.5 we partition the probability space Ω into three parts: $\Omega(+)$, $\Omega(m) = \Omega_1(m) \dot\cup \Omega_2(m))$, and $\Omega(c) := \Omega \setminus (\Omega(+) \cup \Omega(m))$. It will turn out that $P(\Omega(c))$ is sufficiently small; see Lemma 9. Therefore, it suffices to establish the approximations on $\Omega(+)$ and $\Omega(m)$.

The approximations (for sufficiently large n) of ψv_n^k, $k = 0, 1, 2, 3$, on $\Omega(+)$ is essentially straightforward; see §6.4.7: for sufficiently large n we show that on $\Omega(+)$, $\psi v_n^0(C) = 0$ with high probability, $\psi v_n^1(C)$ is approximately $\tau(C)$, and $\psi v_n^3(C)$ is approximately $\varphi\tilde{v}(C)$. Similarly, the approximation of ψv_n^0, ψv_n^1, and ψv_n^3 on $\Omega(m)$ is essentially straightforward (see (58), (60), and §6.4.7): for sufficiently large n we show that on $\Omega(m)$, $\psi v_n^0(C) = 0$ with high probability, $\psi v_n^1(C) = 0$, and $\psi v_n^3(C)$ is approximately $\varphi\tilde{v}(C)$. The delicate point is the approximation of ψv_n^2 on $\Omega(m)$.

We approximate $v_n^2(\omega)$ by a scalar measure game $u_n(\omega)$, which is a product of a constant times a w.m.g.: if $\omega \in \Omega_i(m)$, then

$$u_n(\omega)(S) = \bar{f}(Z_n^\mu(s_1))\mathbb{1}\left(\sum_{a \in A(\Pi_n):\, a \subset S,\ s < X_a^{\Pi_n}(\omega) \le s_1} \mu_i(a) > q_i - Z_n^{\mu_i}(s)\right).$$

On $\Omega_i(m)$, the norm distance between $v_n^2(\omega)$ and $u_n(\omega)$ is bounded by $\|Z^\mu(s_1) - Z^\mu(s)\|_1$, which is, by our construction, small with high probability. Therefore, ψv_n^2 is approximated by ψu_n, and we prove that $\psi u_n(C)$ approximates $\tau(C)$.

We continue by introducing some notation that will enable us to illustrate and derive our probabilistic approximations. For two random variables C and D, an event $\Omega' \subset \Omega$, and a scalar $\varepsilon \ge 0$, we write

$$C \sim_\varepsilon D \quad \text{on } \Omega'$$

whenever the event $\{\omega \in \Omega' : |C(\omega) - D(\omega)| > \varepsilon\}$ has probability $\le \varepsilon$, and for a vector $\varepsilon = (\varepsilon_1, \varepsilon_2)$, we write $C \sim_\varepsilon D$ on Ω' whenever the event $\{\omega \in \Omega' : |C(\omega) - D(\omega)| > \varepsilon_1\}$ has probability $\le \varepsilon_2$.

Several straightforward properties of the relation $\sim_\varepsilon$ follow. If $C \sim_\varepsilon D$ on Ω' and $F \sim_\eta G$ on Ω', then $C + F \sim_{\varepsilon+\eta} D + G$ on Ω', and if $C \sim_{(\varepsilon_1, \varepsilon_2)} D$ on Ω', then $C \sim_{\max_i \varepsilon_i} D$ on Ω'. Also, if $|C|, |D| \le 1$ and $C \sim_\varepsilon D$ on Ω, then $|E(C) - E(D)| \le 3\varepsilon$, and if $C \sim_\theta D$ on Ω_1 and $C \sim_\eta D$ on Ω_2, then $C \sim_{\theta+\eta} D$ on $\Omega_1 \cup \Omega_2$.

This symbolism enables us to summarize the previously mentioned approximations by the relations

$$\begin{aligned}
0 &\sim_{\varepsilon_1} \psi v_n^0(C) && \text{on } \Omega(+) \cup \Omega(m),\\
\tau(C) &\sim_{\varepsilon_2} \psi v_n^1(C) && \text{on } \Omega(+),\\
0 &= \psi v_n^2(C) && \text{on } \Omega(+),\\
\tau(C) &\sim_{\varepsilon_3} \psi v_n^2(C) && \text{on } \Omega(m),\\
0 &\sim_{\varepsilon_4} \psi v_n^1(C) && \text{on } \Omega(m),\\
\varphi\tilde{v}(C) &\sim_{\varepsilon_5} \psi v_n^3(C) && \text{on } \Omega(+) \cup \Omega(m),
\end{aligned} \tag{34}$$

which hold for sufficiently large n. In fact, we will prove these approximations with $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon$, $\varepsilon_3 = 14\varepsilon$, and $\varepsilon_5 = 3\varepsilon$.

The approximations in (34) (with $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon$, $\varepsilon_3 = 14\varepsilon$, and $\varepsilon_5 = 3\varepsilon$) imply that for sufficiently large n,

$$\sum_{k=0}^{3} \psi v_n^k(C) \sim_{20\varepsilon} \tau(C) + \varphi\tilde{v}(C) \quad \text{on } \Omega(+) \cup \Omega(m),$$

and therefore, using the inequalities $P(\Omega(+)\cup\Omega(m)) > 1-3\varepsilon$ (see Lemma 9) and $|\sum_{k=0}^{3}\psi v_n^k(C)-(\tau(C)+\varphi\tilde{v}(C))|\le 2$ (as $\|\sum_{k=0}^{3} v_n^k\|\le\|v\|\le 1$ and $\|\tau+\tilde{v}\|\le\|v\|\le 1$), we have

$$\left|E\left(\sum_{k=0}^{3}\psi v_n^k(C)\right)-E(\tau(C)+\varphi\tilde{v}(C))\right|<66\varepsilon.$$

The first part of the proof provides a formula for the candidate φv of the asymptotic value of v. Along the way we make several comments which are not used in the definition of φv but are used later in the proof.

6.4.1. The formula of φv. Recall that f is a smooth function on $\{x: 0\le x\le\mu(I)=[1,1]\}$. In particular, it is defined on $L_1=\{0\le x\le\mu(I)\mid x_1=q_1 \text{ and } x_2>q_2\}$ and on $L_2=\{0\le x\le\mu(I)\mid x_2=q_2 \text{ and } x_1>q_1\}$.

For every $\omega\in\Omega$, we define a smooth vector measure game $\tilde{v}(\omega)=g(\omega)\circ\bar{\mu}(\omega)$, where $g(\omega)$ is a smooth function defined on $\{y-Z^{\mu}(r(\omega)): Z^{\mu}(r(\omega))\le y\le\mu(I)\}$ by

$$g(y-Z^{\mu}(r(\omega)))=f(y)-f(Z^{\mu}(r(\omega))),$$

and $\bar{\mu}(\omega)$ is a vector measure on $(I,\mathscr{C})$ with atomic part $\bar{\mu}^A(\omega)$ given by

$$\bar{\mu}^A(\omega)(S)=\sum_{j:\,c_j\in S}\mu(c_j)\mathbb{1}(X_j>r(\omega))$$

and nonatomic part

$$\bar{\mu}^{NA}(\omega)(S)=(1-r(\omega))\mu^{NA}(S).$$

Observe that $\omega\mapsto\tilde{v}(\omega)$ is $\mathscr{F}_r$-measurable. The vector measure game $\tilde{v}(\omega)$ has, by Theorem 4, an asymptotic value $\varphi\tilde{v}(\omega)$, and

$$\forall\, S\in\mathscr{C},\quad \varphi\tilde{v}(\omega)(S)=p(\omega)\cdot\bar{\mu}^{NA}(S)+\sum_{j:\,c_j\in S}\Psi_j(\omega),$$

where $p(\omega)\in\mathbb{R}^2$. The maps $\omega\mapsto p(\omega)$, $\omega\mapsto\Psi_j(\omega)$, and $\omega\mapsto\varphi\tilde{v}(\omega)$ are $\mathscr{F}_r$-measurable.

For every coalition $S\in\mathscr{C}$, $\tau(S)$ and $\varphi\tilde{v}(S)$ are real-valued random variables, $\omega\mapsto\tau(\omega)(S)$ and $\omega\mapsto\varphi v^2(\omega)(S)$, respectively. The candidate φv of the asymptotic value is the measure φv on $(I,\mathscr{C})$ defined by

$$\varphi v(S)=E(\tau(S)+\varphi\tilde{v}(S)).$$

We will prove that $\forall\,\varepsilon>0$, we have

(35) $$\limsup_{n\to\infty}|\psi v_{\Pi_n}(C)-\varphi v(C)|<\varepsilon.$$

As inequality (35) holds for every $\varepsilon>0$, we deduce that $\lim_{n\to\infty}\psi v_{\Pi_n}(C)=\varphi v(C)$, which finishes the proof.

Fix $\varepsilon>0$ and assume w.l.o.g. that $\varepsilon<1$. As f is smooth, there is $\delta_1>0$ sufficiently small such that

(36) $$|f(x)-f(y)|+\|\nabla f(x)-\nabla f(y)\|_1<\varepsilon\quad\forall\, 0\le x,y\le\mu(I)\text{ s.t. }\|x-y\|_\infty<\delta_1.$$

6.4.2. Definition of K. Let $K>3$ be a sufficiently large constant such that for every w.m.g. $u=[\theta; w_1,\ldots,w_m]$ with

(37) $$K\max_{j=1}^{m} w_j/3\le\theta\le\sum_{j=1}^{m} w_j-K\max_{j=1}^{m} w_j/3,$$

we have

(38) $$\sum_{j=1}^{m}\left|\psi_j u-w_j\Big/\sum_{\ell=1}^{m} w_\ell\right|<\varepsilon.$$

The existence of such a constant K follows from Proposition 3 (e.g., set $K=\max(4,3K(\varepsilon))$, where $K(\varepsilon)$ is given by Proposition 3).

6.4.3. Definition of δ. As $Z^\mu(t) \to_{t\to 1-} \mu(I)$ and $Z^\mu(t) \to_{t\to 0+} \mu(\varnothing) = 0$ in probability, we deduce that there exists $\xi > 0$ such that

$$P(2\xi < r(\omega) < 1 - 2\xi) > 1 - \varepsilon. \tag{39}$$

Set $m(\delta) = |\{j \geq 1 : \|\mu(c_j)\| \geq \delta\}|$. Note that

$$m(\delta)\delta \to_{\delta\to 0+} 0. \tag{40}$$

Therefore, $\alpha_i > 0 \Rightarrow m(\delta)\delta K/(\alpha_i \xi) \to_{\delta\to 0+} 0$.

As $\|\nabla f(x)\|_\infty \leq 1\ \forall\, (0,0) \leq x \leq (1,1) = \mu(I)$ by assumption, for all $(0,0) \leq x, y \leq \mu(I)$, we have $|f(x) - f(y)| \leq \|y - x\|_1$, and for all increasing sequences $(0,0) \leq x = x(0) \leq x(1) \leq \ldots \leq x(m) = y \leq \mu(I)$, we have

$$\sum_{j=1}^{m} |f(x(j)) - f(x(j-1))| \leq \|y - x\|_1. \tag{41}$$

Let $B_{K\delta}(q)$ denote the set of all points $x = (x_1, x_2) \in \mathbb{R}^2$ such that $\max_{i=1,2} |x_i - q_i| \leq K\delta$. Note that if $\theta = (\theta_1, \theta_2)$ is the unique point in $\mathbb{R}^2$ such that $\lambda_i\theta_1 + (1-\lambda_i)\theta_2 = q_i$ then there is $\eta > 0$ such that for every $\delta > 0$, we have $\|x - \theta\| \leq \eta K\delta$ whenever $(\lambda_1 x_1 + (1-\lambda_1)x_2, \lambda_2 x_1 + (1-\lambda_2)x_2) \in B_{K\delta}(q)$. Therefore, it follows from Lemma 6 that

$$P(\exists\, 0 \leq t \leq 1 \text{ s.t. } Z^\mu(t) \in B_{K\delta}(q)) \to_{\delta\to 0+} 0.$$

Let $\delta > 0$ be sufficiently small so that

$$P(\exists\, 0 \leq t \leq 1 \text{ s.t. } Z^\mu(t) \in B_{K\delta}(q)) < \varepsilon, \tag{42}$$

and for $i = 1, 2$, we have

$$\alpha_i > 0 \Longrightarrow \sum_{j:\, \|\mu(c_j)\| \leq \delta} \|\mu(c_j)\| < \varepsilon^2 \xi \alpha_i, \tag{43}$$

$$\alpha_i > 0 \Longrightarrow 2K\delta \frac{\alpha_{3-i}}{\alpha_i} + \sum_{j:\, \|\mu(c_j)\| \leq \delta} \|\mu(c_j)\| < \delta_1/2, \tag{44}$$

$$\alpha_i > 0 \Longrightarrow \frac{2m(\delta)\delta K}{\alpha_i \xi} < \varepsilon, \tag{45}$$

$$\alpha_i > 0 \Longrightarrow \|\mu^{NA}(I)\|_1 \frac{2\delta K}{\alpha_i} < \varepsilon/2, \tag{46}$$

and

$$\alpha_i = 0 \Longrightarrow P(\exists\, 0 \leq t \leq 1 \text{ s.t. } Z^{\mu_i}(t) \in (q_i - K\delta, q_i + K\delta)) < \varepsilon/2. \tag{47}$$

6.4.4. The stopping time $s = s_\delta$. Define the $(\mathcal{F}_t)_{0\leq t\leq 1}$-stopping time s_δ, or s for short, by

$$s_\delta(\omega) = \inf\Big\{t : \max_i (q_i - Z^{\mu_i}(t)(\omega)) \leq K\delta\Big\}. \tag{48}$$

Note that $s = s_\delta$ is a two-sided stopping time.

6.4.5. Partitioning the probability space. Define the events $\Omega_i(m)$, $i = 1, 2$, and $\Omega(+)$ by

(49) $$\Omega_i(m) = \{\omega\colon Z^{\mu_i}(s(\omega)) < q_i - (K-1)\delta \text{ and } Z^{\mu_{3-i}}(s(\omega)) \ge q_{3-i} + K\delta \text{ and } s(\omega) < 1 - 2\xi\}$$

and

(50) $$\Omega(+) = \Big\{\omega\colon \min_i (Z^{\mu_i}(s(\omega)) - q_i) \ge K\delta\Big\}.$$

LEMMA 9.
$$P(\Omega(+) \cup \Omega_1(m) \cup \Omega_2(m)) \ge 1 - 3\varepsilon.$$

PROOF. If $\omega \notin \Omega(+) \cup \Omega_1(m) \cup \Omega_2(m)$, then either

$$\omega \in \Omega^{\xi} := \{\omega\colon s(\omega) \ge 1 - 2\xi\},$$

or

$$\omega \in \Omega^{q} := \{\omega\colon \exists\, 0 \le t \le 1 \text{ s.t. } Z^{\mu}(t) \in B_{K\delta}(q)\},$$

or $\omega \in \Omega_1^* \cup \Omega_2^*$, where

$$\Omega_i^* := \{\omega\colon \exists\, 0 \le t \le 1 \text{ s.t. } q_i - K\delta < Z^{\mu_i}(t) < q_i + K\delta \text{ and } Z^{\mu_i}(t) - Z^{\mu_i}(t-) \ge \delta\}.$$

As $s \le r$, inequality (39) implies that $P(\Omega^{\xi}) < \varepsilon$. Inequality (42) implies that $P(\Omega^q) < \varepsilon$. If $\alpha_i > 0$, then (45) implies that $P(\Omega_i^*) < \varepsilon/2$. If $\alpha_i = 0$, then inequality (47) implies that $P(\Omega_i^*) < \varepsilon/2$. Altogether, $P(\Omega^{\xi} \cup \Omega^q \cup \Omega_1^* \cup \Omega_i^*) < 3\varepsilon$ and, therefore, $P(\Omega(+) \cup \Omega_1(m) \cup \Omega_2(m)) \ge 1 - 3\varepsilon$. □

We define the $(\mathcal{F}_t)_t$-stopping time s_1 as follows: $s_1(\omega) = s(\omega) + 2K\delta/\alpha_i$ if $\alpha_i > 0$ and $\omega \in \Omega_i(m)$, and $s_1(\omega) = s(\omega)$ otherwise.

We continue with deriving a useful inequality about the distance between $Z(r)$ and $Z(s_1)$ on $\Omega_i(m)$ in the case that $\alpha_i > 0$. On $\Omega_i(m)$, $s(\omega) < 1 - 2\xi$ and $\alpha_i > 0 \Rightarrow 1 - 2\xi < 1 - 2K\delta/\alpha_i$ by (45). Therefore, for every $j \ge 1$, we have

$$P(X_j \in (s, s_1] \mid \mathcal{F}_s) \le \frac{s_1 - s}{1 - s} \le \frac{K\delta}{\alpha_i \xi} \quad \text{on } \Omega_i(m).$$

Therefore, using inequality (45), we have

(51) $$P(\exists\, j \text{ s.t. } \|\mu_i(c_j)\| \ge \delta \text{ and } X_j \in (s, s_1] \mid \mathcal{F}_s) < \varepsilon/2 \quad \text{on } \Omega_i(m),$$

and thus, by using inequality (44),

(52) $$P(\|Z(s_1) - Z(s)\| \ge \delta_1 \mid \mathcal{F}_s) < \varepsilon/2 \quad \text{on } \Omega_i(m).$$

As $Z(s) \le Z(r) \le Z(s_1)$ on $\Omega_i(m)$, it follows that

(53) $$P(\|Z(r) - Z(s_1)\| \ge \delta_1 \mid \mathcal{F}_s) < \varepsilon/2 \quad \text{on } \Omega_i(m),$$

and therefore, also

(54) $$P(|f(Z(r)) - f(Z(s_1))| \ge \varepsilon \mid \mathcal{F}_s) < \varepsilon/2 \quad \text{on } \Omega_i(m).$$

We have $Z^{\mu}(s_1) - Z^{\mu}(s) = (s_1 - s)\mu^{NA}(I) + \sum_j \mu(c_j)\mathbb{1}(s < X_j \le s_1)$. In what follows, we bound the conditional probability, given $\mathcal{F}_s$, that $\sum_j \|\mu(c_j)\|\mathbb{1}(s < X_j \le s_1) \ge 2K\delta\varepsilon$.

For every j, $E(\mathbb{1}(s < X_j \le s_1) \mid \mathcal{F}_s) = (s_1 - s)/(1-s)$. Therefore,

$$E\left(\sum_{j:\, \|\mu(c_j)\| \le \delta} \|\mu(c_j)\| \mathbb{1}(s < X_j \le s_1) \,\middle|\, \mathcal{F}_s\right) = \frac{s_1 - s}{1-s} \sum_{j:\, \|\mu(c_j)\| \le \delta} \|\mu(c_j)\|.$$

On $\Omega_i(m)$, $s < 1 - 2\xi$ and $s_1 - s = 2K\delta/\alpha_i$ and, therefore, $(s_1 - s)/(1-s) \le 2K\delta/2\alpha_i \xi$. By (43), $\sum_{j:\, \|\mu(c_j)\| \le \delta} \|\mu(c_j)\| < \varepsilon^2 \xi \alpha_i$. Therefore,

$$E\left(\sum_{j:\, \|\mu(c_j)\| \le \delta} \|\mu(c_j)\| \mathbb{1}(s < X_j \le s_1) \,\middle|\, \mathcal{F}_s\right) \le K\delta\varepsilon^2 \quad \text{on } \Omega_i(m).$$

Using Markov's inequality, we deduce that

$$P\left(\sum_{j:\, \|\mu(c_j)\| \le \delta} \|\mu(c_j)\| \mathbb{1}(s < X_j \le s_1) > 2K\delta\varepsilon \,\middle|\, \mathcal{F}_s\right) < \varepsilon/2 \quad \text{on } \Omega_i(m).$$

Therefore, using (51), we have

$$P\left(\sum_{j} \|\mu(c_j)\| \mathbb{1}(s < X_j \le s_1) > 2K\delta\varepsilon \,\middle|\, \mathcal{F}_s\right) < \varepsilon \quad \text{on } \Omega_i(m),$$

implying, in particular, that

$$P\left(\left\{\omega \in \Omega_i(m) : \sum_j \|\mu(c_j)\| \mathbb{1}(s < X_j \le s_1) > 2K\delta\varepsilon\right\}\right) < \varepsilon. \tag{55}$$

6.4.6. The random decomposition of v_{Π_n}. Let v_n stand for the finite game v_{Π_n}. We will define auxiliary finite games $v_n^0(\omega)$, $v_n^1(\omega)$, $v_n^2(\omega)$, and $v_n^3(\omega)$, and an auxiliary smooth vector measure game $v^3(\omega)$. In addition, we define an auxiliary finite game $u_n(\omega)$, which is a product of a constant and a w.m.g.

It will follow from the definition of the auxiliary random games that

$$\|v_n^0(\omega)\| + \|v_n^1(\omega)\| + \|v_n^2(\omega)\| + \|v_n^3(\omega)\| \le \|v\| \quad \forall\, n \tag{56}$$

and

$$\psi v_n = E(\psi v_n^0 + \psi v_n^1 + \psi v_n^2 + \psi v_n^3). \tag{57}$$

The set of players of $v_n^i(\omega)$ is the set of atoms of Π_n, and it is partitioned into a set of null players $D_n^i(\omega)$ and a set of essential players $S_n^i(\omega)$. Note that some essential players may be null players as well. This partition is, however, useful as it enables us to define the game $v_n^i(\omega)$ by specifying $v_n^i(\omega)(S)$ for subsets S of the set of essential players.

The sets $S_n^i(\omega)$ of essential players in $v_n^i(\omega)$ are defined by means of the two $(\mathcal{F}_t)_t$-stopping times s and s_1. The two stopping times s and s_1 partition the interval $[0,1]$ into four random intervals: $J_0(\omega) = [0, s(\omega))$, $J_1(\omega) = [s(\omega), s(\omega)]$, $J_2(\omega) = (s(\omega), s_1(\omega)]$, and $J_3(\omega) = (s_1(\omega), 1]$.

The set $S_n^i(\omega)$ of essential players in $v_n^i(\omega)$ is the set of atoms $a \in A(\Pi_n)$ such that $X_n^{\Pi_n}(\omega) \in J_i(\omega)$. For a coalition $S \subset S_n^i(\omega)$, we define

$$v_n^i(\omega)(S) = v_n\left(\bigcup_{j<i} S_n^j(\omega) \cup S\right) - v_n\left(\bigcup_{j<i} S_n^j(\omega)\right).$$

Note that on $\Omega(+) \cap \Omega(n,\delta)$ and on $\Omega_i(m) \cap \Omega(n,\delta)$, the game $v_n^0(\omega)$ vanishes. Also, $s_1 = s$ on $\Omega(+)$, and therefore, the game v_n^2 vanishes on $\Omega(+)$, and on $\Omega_i(m) \cap \Omega(n,\delta)$, the game $v_n^1(\omega)$ vanishes. As $P(\Omega(n,\delta)) \to_{n\to\infty} 1$, we deduce that for sufficiently large n, we have

(58) $$\psi v_n^0(C) \sim_\varepsilon 0 \quad \text{on } \Omega(+) \cup \Omega(m),$$

(59) $$\psi v_n^2(C) \sim_\varepsilon 0 \quad \text{on } \Omega(+),$$

(60) $$\psi v_n^1(C) \sim_\varepsilon 0 \quad \text{on } \Omega(m).$$

For every $\omega \in \Omega(+)$, there exists a unique j such that $X_j = s$ and $\tau(\omega)(C) = f(Z^\mu(s(\omega)))$ if $c_j \in C$ and $=0$ otherwise. The continuity of the function f and the convergence in probability of $Z_n^\mu(s)$ and $Z^\mu(s-)_n$ to $Z^\mu(s)$ and $Z^\mu(s-)$, respectively, imply that for sufficiently large n we have $\bar{f}(Z_n^\mu(s-)) \sim_{\varepsilon/2} 0$ and $\bar{f}(Z_n^\mu(s)) \sim_{\varepsilon/2} \bar{f}(Z^\mu(s))$ on $\Omega(+)$, and therefore,

(61) $$\psi v_n^1(C) \sim_\varepsilon \tau(C) \quad \text{on } \Omega(+).$$

6.4.7. $\psi v_n^3(C) \sim_{3\varepsilon} \varphi\tilde{v}(C)$ on $\Omega(+) \cup \Omega(m)$. To every $\omega \in \Omega$ and n we associate the finite game $\tilde{v}_n(\omega)$ on the set of players $A(\Pi_n)$, defined as follows: $S_n^-(\omega) = \{a \in A(\Pi_n) : X_a^{\Pi_n} \le r\}$ and $S_n^+(\omega) = \{a \in A(\Pi_n) : X_a^{\Pi_n} > r\}$, and for $S \subset A(\Pi_n)$, we define

$$\tilde{v}_n(\omega)(S) = v(S \cup S_n^-(\omega)) - v(S_n^-(\omega)).$$

It follows from Theorem 4 that

$$\psi\tilde{v}_n(C) \to_{n\to\infty} \varphi\tilde{v}(C).$$

The next lemma asserts that for sufficiently large n with high probability, the bounded variation norm distance between $\tilde{v}_n$ and v_n^3 is small.

LEMMA 10. *Let* $B = \max\{\|\nabla f(x)\|_\infty : 0 \le x \le \mu(I)\}$ *and* $B(\omega) = \max\{\|\nabla f(x) - \nabla f(y)\|_\infty : 0 \le x, y \le \mu(I)$ *with* $\|x-y\| \le \|Z^\mu(s_1) - Z^\mu(s)\|\}$. *Then,*

$$\|\tilde{v}_n(\omega) - v_n^3(\omega)\| \le B\|Z_n^\mu(s_1) - Z_n^\mu(s)\|_1 + 2B(\omega).$$

PROOF. Given an order $\mathcal{R}$ of the players $A(\Pi_n)$ and an atom $a \in A(\Pi_n)$, we set

$$\Delta(\tilde{v}_n - v_n^3, \mathcal{R}, a) := \tilde{v}_n(\omega)(P_a^{\mathcal{R}} \cup a) - v_n^3(\omega)(P_a^{\mathcal{R}} \cup a) - (\tilde{v}_n(\omega)(P_a^{\mathcal{R}}) - v_n^3(\omega)(P_a^{\mathcal{R}})).$$

We have to prove that for every order $\mathcal{R}$, we have

$$\sum_{a \in A(\Pi_n)} |\Delta(\tilde{v}_n - v_n^3, \mathcal{R}, a)| \le B\|Z^\mu(s_1) - Z^\mu(s)\|_i + 2B(\omega).$$

If $a \in S_n^-(\omega)$, then $\tilde{v}_n(\omega)(P_a^{\mathcal{R}} \cup a) = \tilde{v}_n(\omega)(P_a^{\mathcal{R}})$ and $v_n^3(\omega)(P_a^{\mathcal{R}} \cup a) = v_n^3(\omega)(P_a^{\mathcal{R}})$, and therefore, $\Delta(\tilde{v}_n(\omega) - v_n^3(\omega), \mathcal{R}, a) = 0$. If $a \in S_n^+(\omega) \setminus S_n^3(\omega)$, then $v_n^3(\omega)(P_a^{\mathcal{R}} \cup a) - v_n^3(\omega)(P_a^{\mathcal{R}}) = 0$ and $|\tilde{v}_n(\omega)(P_a^{\mathcal{R}} \cup a) - \tilde{v}_n(\omega)(P_a^{\mathcal{R}}) = 0| \le B\|\mu(a)\|_1$. Therefore,

$$\sum_{a \in S_n^+(\omega) \setminus S_n^3(\omega)} |\Delta(\tilde{v}_n - v_n^3, \mathcal{R}, a)| \le B\|\mu(S_n^+(\omega) \setminus S_n^3(\omega))\|_1 \le B\|Z_n^\mu(s_1) - Z_n^\mu(s)\|_1.$$

If $a \in S_n^3(\omega)$, then

$$\tilde{v}_n(\omega)(P_a^{\mathcal{R}} \cup a) - \tilde{v}_n(\omega)(P_a^{\mathcal{R}}) = \int_0^1 \langle \nabla f(x + t\mu(a)), \mu(a) \rangle \, dt,$$

where $x = \mu(S_n^- \cup P_a^{\mathcal{R}})$ and

$$v_n^3(\omega)(P_a^{\mathcal{R}} \cup a) - v_n^3(\omega)(P_a^{\mathcal{R}}) = \int_0^1 \langle \nabla f(y + t\mu(a)), \mu(a) \rangle \, dt,$$

where $y = \mu(P_a^{\mathcal{R}} \cup (A(\Pi_n) \setminus S_n^3(\omega)))$. Note that $\|x - y\|_1 \le \|Z^\mu(s_1) - Z^\mu(s)\|$, and therefore, $|v_n^3(\omega)(P_a^{\mathcal{R}} \cup a) - v_n^3(\omega)(P_a^{\mathcal{R}}) - (\tilde{v}_n(\omega)(P_a^{\mathcal{R}} \cup a) - \tilde{v}_n(\omega)(P_a^{\mathcal{R}}))| \le B(\omega)\|\mu(a)\|_1$. Therefore, as $\sum_{a \in S_n^3} \|\mu(a)\|_1 = \|\mu(S_n^3)\|_1 \le 2$, we deduce that

$$\sum_{a \in S_n^3(\omega)} |\Delta(\tilde{v}_n - v_n^3, \mathcal{R}, a)| \le 2B(\omega).$$

Altogether,

$$\sum_{a \in A(\Pi_n)} |\Delta(\tilde{v}_n - v_n^3, \mathcal{R}, a)| \le B\|Z^\mu(s_1) - Z^\mu(s)\|_i + 2B(\omega). \quad \square$$

Lemma 11. *$\psi v_n^3(C) \sim_{3\varepsilon} \varphi \tilde{v}_n(C)$ on $\Omega(+) \cup \Omega(m)$.*

Proof. As $Z_n^\mu(r)$ and $Z_n^\mu(s_1)$ converge in probability to $Z^\mu(r)$ and $Z^\mu(s_1)$, respectively, we deduce from (53) that for sufficiently large n,

$$P(\{\omega \in \Omega(m) : \|Z^\mu(r) - Z_n^\mu(s_1)\| \ge 2\delta_1\}) < 2\varepsilon,$$

and therefore, by using (36), we deduce that

$$P(\{\omega \in \Omega(m) : B(\omega) \ge 2\varepsilon\}) < 2\varepsilon.$$

On $\Omega(+)$, we have $v_n^3 = \tilde{v}_n$. As $\delta_1 < \varepsilon$ and $B < 1$, we conclude that

$$\psi v_n^3(C) \sim_{3\varepsilon} \varphi \tilde{v}_n(C) \quad \text{on } \Omega(+) \cup \Omega(m). \quad \square$$

6.4.8. $\psi v_n^2 \sim_{7\varepsilon} \tau(C)$ on $\Omega_i(m)$. For $\omega \in \Omega_i(m)$ and n with $Z_n^{\mu_i}(s(\omega)) < q_i$, we approximate the game $v_n^2(\omega)$ by the auxiliary game $u_n(\omega)$, where $S_n^2(\omega)$ is the set of essential players of $u_n(\omega)$, and for every coalition $S \subset S_n^2(\omega)$,

$$u_n(\omega)(S) = \begin{cases} v_n^2(\omega)(I) & \text{if } Z_n^{\mu_i}(s(\omega)) + \mu_i(S) > q_i, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\Omega(n, \delta) = \{\omega \in \Omega : \sup_{0 \le t \le 1} \|Z(t) - Z_n(t)\| < \delta\}$. Observe that for every n and $\omega \in \Omega(n, \delta) \cap \Omega_i(m)$, the game $u_n(\omega)$ is a product of a constant $v_n^2(\omega)(I)$ and a w.m.g. $\mathbb{1}(\mu_i(S \cap S_n^2(\omega)) \ge q_i - Z_n^{\mu_i}(s(\omega)))$ and the game $v_n^2(\omega)$ obeys $v_n^2(\omega)(S) = v_n(S \cup S_n^0(\omega) \cup S_n^1(\omega))$ for every $S \subset S_n^2(\omega)$. Letting $\mu^n(\omega)$ stand for the vector measure on Π_n that is given by

$$\mu^n(\omega)(S) = \sum_{a \in A(\Pi_n): a \subset S} \mu(a)\mathbb{1}(s(\omega) < X_a^{\Pi_n} \le s_1(\omega)),$$

it follows that on $\Omega_i(m) \cap \Omega(n, \delta)$, we have (using the assumption $\|\nabla f(x)\|_\infty \le 1/2 \ \forall x$)

$$\|v_n^2(\omega) - u_n(\omega)\| \le \|\mu^n(\omega)(I)\|_1. \tag{62}$$

It follows from relation (52) that $P(\{\omega \in \Omega_i(m) : \|Z(s_1) - Z(s)\|_1 \ge \delta_1\}) < \varepsilon$. As $\sup_t\{\|Z_n(t) - Z(t)\|_1\}$ converges in probability to 0, it follows that for every $\delta' > \delta_1$, we have for sufficiently large n that $P(\{\omega \in \Omega_i(m) : \|Z_n(s_1) - Z_n(s)\|_1 \ge \delta'\}) < \varepsilon$ and, thus, $P(\{\omega \in \Omega_i(m) : \|\mu^n(\omega)(I)\|_1 \ge \delta'\}) \le \varepsilon$. Together with inequality (62), we deduce that for sufficiently large n, $P(\{\omega \in \Omega_i(m) : \|v_n^2(\omega) - u_n(\omega)\| > \varepsilon\}) \le \varepsilon$, namely,

$$\|v_n^2(\omega) - u_n(\omega)\| \sim_\varepsilon 0 \quad \text{on } \Omega_i(m). \tag{63}$$

By the weak contraction property of the Shapley value, $|\psi v_n^2(C) - \psi u_n(C)|$ is $\le \|v_n^2(C) - u_n(C)\|$, and thus, for sufficiently large n, we have

$$P(\{\omega \in \Omega_i(m) : |\psi v_n^2(\omega)(C) - \psi u_n(\omega)(C)| > \varepsilon\}) \le \varepsilon,$$

namely,

$$\psi v_n^2(C) \sim_\varepsilon \psi u_n(C) \quad \text{on } \Omega_i(m). \tag{64}$$

As Π_n is admissible, $\max_{a\in A(\Pi_n)} \mu_i^{NA}(a) \to 0$ as $n \to \infty$, and for every $1 \le i < j$, there is a sufficiently large $n(i, j)$ such that for every $n \ge n(i, j)$, the field Π_n separated c_i from c_j, namely, $\pi_n(c_i) \ne \pi_n(c_j)$. Therefore, we have that $\max_{i:\|\mu(c_i)\|<\delta} \|\mu(\pi_n(c_i))\| < \delta$ for sufficiently large n. It follows from (51) that $P(\{\omega \in \Omega_i(m) : \exists\, i \text{ s.t. } \|\mu(c_i)\| \ge \delta$ and $X_i \in (s, s_1]\}) < \varepsilon/2$. Therefore, for sufficiently large n, the probability that $\omega \in \Omega_i(m)$ and the quota and weights of the w.m.g. $\mathbb{1}(\mu_i^n(S) > q_i - Z^i(s(\omega)))$ violate condition (37) is $< \varepsilon$. Therefore (using also the assumption that $\|v\| \le 1$), for sufficiently large values of n, we have

$$P\left(\left\{\omega \in \Omega_i(m) : \left|\psi u_n(\omega)(C) - u_n(\omega)(I)\frac{\mu_i^n(\omega)(C)}{\mu_i^n(\omega)(I)}\right| > \varepsilon\right\}\right) \le \varepsilon, \tag{65}$$

namely, for sufficiently large values of n,

$$\psi u_n(C) \sim_\varepsilon u_n(I)\frac{\mu_i^n(C)}{\mu_i^n(I)} \quad \text{on } \Omega_i(m). \tag{66}$$

Recall that $\sup_t \|Z_n(t) - Z(t)\|$ converges in probability to 0. Therefore, using (55), for sufficiently large n,

$$\mu_i^n(I) \sim_{(3K\delta\varepsilon,\varepsilon)} (s_1 - s)\mu_i^{NA}(I) = 2K\delta \quad \text{on } \Omega_i(m),$$

and similarly,

$$\mu_i^n(C) \sim_{(3K\delta\varepsilon,\varepsilon)} (s_1 - s)\mu_i^{NA}(C) \quad \text{on } \Omega_i(m),$$

and therefore,

$$\frac{\mu_i^n(C)}{\mu_i^n(I)} \sim_{(3\varepsilon,2\varepsilon)} \frac{\mu_i^{NA}(C)}{\mu_i^{NA}(I)} \quad \text{on } \Omega_i(m). \tag{67}$$

By (54), $f(Z(r)) \sim_\varepsilon f(Z(s_1))$ on $\Omega_i(m)$. As $f(Z_n(s_1))$ converges in probability as $n \to \infty$ to $f(Z(s_1))$, we deduce that for sufficiently large n, we have

$$f(Z(r)) \sim_{2\varepsilon} f(Z_n(s_1)) = u_n(I) \quad \text{on } \Omega_i(m),$$

and therefore,

$$\tau(C) = f(Z(r))\frac{\mu_i^{NA}(C)}{\mu_i^{NA}(I)} \sim_{(5\varepsilon,4\varepsilon)} u_n(I)\frac{\mu_i^n(C)}{\mu_i^n(I)} \quad \text{on } \Omega_i(m). \tag{68}$$

The three properties, (64), (66), and (68), prove that for sufficiently large n, we have

$$\psi v_n^2(C) \sim_{(7\varepsilon,6\varepsilon)} \tau(C) \quad \text{on } \Omega_i(m). \quad \square \tag{69}$$

7. Outline of the Proof of Theorem 3. Assume that $\nu = (\nu_1, \ldots, \nu_m)$ is a vector of probability measures that satisfies the conditions of Theorem 3 and let $v \in LPS(\nu)$. There are finitely many distinct pairs (μ_i, q_i), $i = 1, \ldots, k$, where μ_i is a convex combination of $\nu_1, \ldots, \nu_m$ and $0 < q_i < 1$, such that for every list of inequality (or equality) signs $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{<, =, >\}^k$, the restriction of v to the set of coalitions $\mathscr{C}_\epsilon := \{S \in \mathscr{C} : \forall\, i\ \mu_i(S)\ \epsilon_i\ q_i\}$ is a smooth function of ν. Namely, for every list of inequality (or equality) signs $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{<, =, >\}^k$, there is a smooth function f_ϵ: $[0, 1]^m \to \mathbb{R}$ such that

$$v(S) = f_\epsilon(\nu(S)) \quad \text{whenever } S \in \mathscr{C}_\epsilon.$$

We assume w.l.o.g. that $|f(x)| \leq 1$ and $\|\nabla f_\epsilon(x)\|_\infty \leq 1$ for every $x \in [0,1]^m$ and every $\epsilon \in \{<,=,>\}^k$.

Let $f: \mathcal{R}(\nu) \to \mathbb{R}$ with $v(S) = f(\nu(S))$. Note that f need not be continuous. However, for every $x \in \mathcal{R}(\nu)$, the limits $\lim_{\varepsilon\to 0+} f((1-\varepsilon)x + \varepsilon\nu(I))$ and $\lim_{\varepsilon\to 0+} f((1-\varepsilon)x + \varepsilon\nu(\emptyset)) = \lim_{\varepsilon\to 0+} f((1-\varepsilon)x)$ exist and are denoted $f^+(x)$ and $f^-(x)$, respectively.

Throughout the proof we fix a coalition $C \in \mathcal{C}$ and a C-admissible sequence $\{\Pi_n\}_{n=1}^\infty$. The set of atoms of Π_n is denoted $A(\Pi_n)$. Let $(c_i)_{i=1}^\infty$ be a sequence of distinct elements of I s.t. $\{c_i \mid i \geq 1\} \supset \{a_i \mid i \geq 1\}$ and for every n and every $a \in \Pi_n$, there is $i \geq 1$ s.t. $c_i \in a$. Let $(X_i)_{i=1}^\infty$ be a sequence of i.i.d. r.v.s that are uniformly distributed on $[0,1]$ and w.l.o.g. assume that for all $i \neq j$, $X_i \neq X_j$ everywhere. For every $a \in I$, we denote by $\pi_n(a)$ the unique atom of Π_n that contains a. For every $a \in A(\Pi_n)$, we denote by $i(a) = \min\{i \mid c_i \in a\}$ and $X_a^{\Pi_n} = X_{i(a)}$.

Let Ω be the sample space on which all these r.v.s are defined and let P denote the probability measure on Ω. The Poisson bridge associated with a vector measure ν is defined by

$$Z^\nu(t) = \sum_{i=1}^\infty \nu(c_i)\mathbb{1}(X_i \leq t) + t\nu^{NA}(I).$$

We define the stopping times r_i by

$$r_i = \inf\{0 \leq t \leq 1 : Z^{\mu_i}(t) \geq q_i\}.$$

For every $j \geq 1$ and $1 \leq i \leq k$, we have $\Pr(Z^{\mu_i}(X_j) = q_i) = 0$, and by Lemma 7, we have

$$\Pr(Z^{\mu_i}(r_i) = q_i \text{ and } Z^{\mu_{i'}}(r_i) = q_{i'}) = 0 \quad \forall\, 1 \leq i \neq i' \leq k.$$

Therefore, $\Pr(Z^{\mu_i}(r_i) = q_i \text{ and } r_i = r_{i'}) = 0 \quad \forall\, 1 \leq i \neq i' \leq k$, and thus, we can assume w.l.o.g. that

$$\forall\, i \neq i', \quad Z^{\mu_i}(r_i) = q_i \implies r_i \neq r_{i'}.$$

For every ω, there is a permutation π on $\{1,\ldots,k\}$ such that $r_{\pi(1)}(\omega) \leq r_{\pi(2)}(\omega) \leq \cdots \leq r_{\pi(k)}(\omega)$. The above assumption implies that if $r_{\pi(i)}(\omega) = r_{\pi(i+1)}(\omega)$, then $Z^{\mu_{\pi(i)}}(r_{\pi(i)})(\omega) > q_{\pi(i)}$ and $Z^{\mu_{\pi(i+1)}}(r_{\pi(i+1)})(\omega) > q_{\pi(i+1)}$.

First, we define the candidate for the asymptotic value. For every $1 \leq i \leq k$, we define a random nonatomic measure τ_i on $(I, \mathcal{C})$ by

$$\tau_i(\omega)(S) = \begin{cases} (f^+(Z^\nu(r_i)) - f^-(Z^\nu(r_i)))\dfrac{\mu_i^{NA}(S)}{\mu_i^{NA}(I)} & \text{if } Z^{\mu_i}(r_i) = q_i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that on $Z^{\mu_i}(r_i) = q_i$, the point $Z^\nu(r_i)$ may be a point of discontinuity of f.

For every $j \geq 1$, we define a random atomic measure τ^j on $(I, \mathcal{C})$ by

$$\tau^j(\omega)(S) = \begin{cases} f(Z^\nu(X_j)) - f(Z^\nu(X_j-)) & \text{if } c_j \in S \text{ and } \exists\, 1 \leq i \leq k \text{ s.t. } X_j = r_i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that almost everywhere $Z^\nu(X_j)$ and $Z^\nu(X_j-)$ are points of continuity of f.

Set $\tilde{r}_i = r_{\pi(i)}$, $\tilde{r}_0 = 0$, and $\tilde{r}_{k+1} = 1$. Define the random vector measure $\omega \mapsto \nu^i(\omega)$ by

$$\nu^i(S) = (\tilde{r}_i - \tilde{r}_{i-1})\nu^{NA}(S) + \sum_{j:\, c_j \in S} \nu(c_j)\mathbb{1}(\tilde{r}_{i-1} < X_j < \tilde{r}_i).$$

Note that on $\tilde{r}_i = \tilde{r}_{i-1}$, the measure ν^i vanishes.

The random smooth coalitional game $\tilde{v}_i$, $1 \le i \le k+1$, is defined by

$$\tilde{v}_i(S) = f^+(Z^\nu(\tilde{r}_{i-1}) + \nu^i(S)) - f^+(Z^\nu(\tilde{r}_{i-1})).$$

The asymptotic value of v is given by

$$\varphi v(S) = E\left(\sum_{i=1}^{k} \tau_i(S) + \sum_{j \ge 1} \tau^j(S) + \sum_{i=1}^{k+1} \varphi \tilde{v}_i(S)\right).$$

Fix $\varepsilon > 0$. Fix a sufficiently small $\xi > 0$ such that for every $1 \le i \le k$, we have

$$\Pr(2\xi < Z^{\mu_i}(r_i) < 1 - 2\xi) > 1 - \varepsilon/k.$$

Next, we define a sufficiently large positive constant K and a sufficiently small $\delta > 0$. Thereafter, we define for every i two stopping times s_i and $\bar{s}_i$. The stopping time s_i is defined by

$$s_i = \inf\{0 \le t \le 1 : Z^{\mu_i}(t) \ge q_i - K\delta\}.$$

Set $s_0 = 0$ and $s_{k+1} = 1$.

Set

$$\Omega_i(m) = \{\omega\colon Z^{\mu_i}(s_i) < q_i - (K-1)\delta \text{ and } Z^{\mu_i}(s_i) < 1 - 2\xi\},$$

$$\Omega_i(+) = \left\{\omega \notin \bigcup_{j<i} \Omega_j(+) : Z^{\mu_i}(s_i) \ge q_i + K\delta\right\},$$

$$\Omega(+) = \bigcup_i \Omega_i(+) \qquad \text{and} \qquad \Omega(m) = \bigcup_i \Omega_i(m).$$

The stopping time $\bar{s}_i$ is defined by

$$\bar{s}_i(\omega) = \begin{cases} s_i(\omega) + 2K\delta/\mu_i^{NA}(I) & \text{if } \mu_i^{NA}(I) > 0 \text{ and } \omega \in \Omega_i(m), \\ s_i(\omega) & \text{otherwise.} \end{cases}$$

The stopping times s_i and $\bar{s}_i$ enable us to define the random games $v_n^{\ell,i}(\omega)$, $\ell = 0, 1, 2$ and $i = 1, \ldots, k$, as follows. First, let

$$I(\omega) := \{1 \le i \le k : \omega \in \Omega_i(m) \cup \Omega_i(+)\}.$$

If $\omega \in \Omega_i(+)$, then there is j s.t. $s_i(\omega) = X_j(\omega)$ and then $\pi_n(c_j)$ is the unique essential player of $v_n^{1,i}(\omega)$, and

$$v_n^{1,i}(\omega)(\pi_n(c_j)) = f(Z_n^\nu(X_j)) - f(Z_n^\nu(X_j -)).$$

If $\omega \in \Omega_i(m)$, the unique essential player of $v^{1,i}(\omega)$ is the atom $a \in \Pi_n$ with $X_a^{\Pi_n} = s_i$ if there is such an atom, and then

$$v_n^{1,i}(\omega)(a) = f(Z_n^\nu(X_a^{\Pi_n})) - f(Z_n^\nu(X_a^{\Pi_n} -)),$$

and otherwise $v^{1,i}(\omega)$ is identically zero. The set of essential players of $v_n^{0,i}(\omega)$ is the set of all atoms $a \in \Pi_n$ such that $X_a^{\Pi_n} \in J_{0,i}(\omega)$, where $J_{0,i}(\omega) = (\max_{j: s_j < s_i} \bar{s}_j(\omega), s_i)$, and for a coalition S of essential players of $v_n^{0,i}(\omega)$, we have

$$v_n^{0,i}(\omega)(S) = f\left(Z_n^\mu\left(\max_{j: s_j < s_i} \bar{s}_j(\omega)\right) + \nu(S)\right) - f\left(Z_n^\mu\left(\max_{j: s_j < s_i} \bar{s}_j(\omega)\right)\right).$$

The essential players of $v_n^{2,i}(\omega)$ are the set of all atoms $a \in \Pi_n$ such that $X_a^{\Pi_n} \in J_{2,i}(\omega)$, where $J_{2,i}(\omega) = (s_i(\omega), \bar{s}_i(\omega)]$, and for a coalition S of essential players of $v_n^{2,i}(\omega)$, we have

$$v_n^{2,i}(\omega)(S) = f(Z_n^\nu(s_i(\omega)) + \nu(S)) - f(Z_n^\nu(s_i(\omega))).$$

As in the proof of Theorem 2, we have

$$\psi v_n(C) \sim E\left(\sum_{\ell, i} \psi v_n^{\ell, i}(C)\right),$$

and one proves that

$$\sum_i \psi v_n^{1,i}(C) \sim \sum_j \tau^j(C) \quad \text{on } \Omega(+), \tag{70}$$

$$\psi v_n^{1,i}(C) \sim 0 \quad \text{on } \Omega_i(m), \tag{71}$$

$$\psi v_n^{2,i}(C) \sim \tau_i(C) \quad \text{on } \Omega_i(m), \tag{72}$$

$$\psi v_n^{2,i}(C) \sim 0 \quad \text{on } \Omega_i(+), \tag{73}$$

$$\psi v_n^{0,i}(C) \sim \varphi \tilde{v}_{\pi(i)}(C) \quad \text{on } \Omega_i(+) \cup \Omega_i(m). \tag{74}$$

$P(\Omega_i(m) \cap \Omega_j(m))$ is sufficiently small whenever $i \neq j$, and $P(\Omega_i(m) \cap \Omega_j(+))$ is sufficiently small for all i, j. Also, $P(\Omega(m) \cup \Omega(+))$ is sufficiently close to 1. Therefore,

$$\sum_i \psi v_n^{1,i}(C) \sim \sum_j \tau^j(C) \quad \text{on } \Omega(+), \tag{75}$$

$$\sum_i \psi v_n^{1,i}(C) \sim 0 \quad \text{on } \Omega(m), \tag{76}$$

$$\sum_i \psi v_n^{2,i}(C) \sim \sum_i \tau_i(C) \quad \text{on } \Omega(m), \tag{77}$$

$$\sum_i \psi v_n^{2,i}(C) \sim 0 \quad \text{on } \Omega(+), \tag{78}$$

$$\sum_i \psi v_n^{0,i}(C) \sim \sum_i \varphi \tilde{v}_{\pi(i)}(C) \quad \text{on } \Omega(+) \cup \Omega(m), \tag{79}$$

and thus,

$$\left| E\left(\sum_{i,\ell} \psi v_n^{\ell,i}(C)\right) - E\left(\sum_{i=1}^{k} \tau_i(C) + \sum_{j \geq 1} \tau^j(C) + \sum_{i=1}^{k+1} \varphi \tilde{v}_i(C)\right) \right|$$

is sufficiently small, and thus, $|\psi v_n(C) - \varphi v(C)|$ is sufficiently small.

The proofs of (70), (71), and (73) are similar to the proofs of (61), (60), and (59), respectively. The proofs of approximations (72) and (74) are similar to the proofs of §§6.4.8 and 6.4.7, respectively.

8. Necessary and sufficient conditions for the zero-hitting-probability property. The essential property of the vector measure $\nu = (\nu_1, \nu_2)$ that is crucial for the result is that ν has the zero-hitting-probability property.

Therefore, the question of the existence of the asymptotic value boils down to a question of hitting probabilities. We conjecture that the following conditions are necessary and sufficient for the Poisson bridge Z^ν to have the zero-hitting-probability property. Given a finite set A, we denote by ν^{-A} the restriction of ν to the complement of A.

CONJECTURE 1. *The vector measure $\nu = (\nu_1, \nu_2)$ has the zero-hitting-probability property if and only if one of the following conditions is satisfied*:

(a) *There is a finite set A and $0 \le \lambda \le 1$ such that $\lambda\nu_1^{-A} + (1-\lambda)\nu_2^{-A}$ is purely atomic with infinitely many atoms.*

(b) *For every finite set A, the image of the idealized process, $Z^{\nu^{-A}}(t)$, is two-dimensional.*

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