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THE PARTITION VALUE*§

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The partition value is a new approach to the value concept. It links together the asymptotic and the axiomatic approach. Using this approach we prove the existence of a continuous value on each of the following spaces: bv'NA, A, $A * bv'$NA, $A * bv'$NA $* bv'$NA and the space W spanned by those spaces and ASYMP.

1. Introduction. In their book, "*Values of Non Atomic Games*," R. J. Aumann and L. S. Shapley extended the concept of value to certain classes of nonatomic games, i.e., infinite person games in which no individual player has significance.

A natural interest arises in values which are limits, in an appropriate sense, of Shapley values of finite games that approximate nonatomic games. Indeed, the asymptotic value stems from such an approach. Roughly speaking, the asymptotic value is defined on each game v for which all the sequences of Shapley values, corresponding to sequences of finite games that approximate v, have the same limit.

The asymptotic value has desirable properties like continuity, diagonality and uniqueness; unfortunately, however, the asymptotic value does not exist for many economically important games. The space of all games possessing an asymptotic value is denoted ASYMP.

We suggest a different approach, which yields a new kind of value: the partition value. Like the asymptotic value, the partition value is defined by means of a limiting process of finite games. But in contrast to the asymptotic value, the sort of limit used for the partition value is so weak that any other conceivable constructive approach to the value concept, by means of limits of finite games, will necessarily yield a value which is also a partition value.

The partition value, like the asymptotic value, is continuous and diagonal. Also, on every subspace of ASYMP there exists a unique partition value, the asymptotic value. Moreover, every partition value on a space Q can be extended to a partition value on the space generated by $Q \cup$ ASYMP.

In this paper, we shall define the partition value, prove its basic properties, and prove the existence of a partition value on each of the following important spaces: ASYMP, bv'NA, the algebra A spanned by games of the form $f \circ \mu$ (where f is a continuous function in bv' and μ is a probability measure on NA), and the spaces $A * bv'$NA, bv'NA $* bv'$NA and $A * bv'$NA $* bv'$NA which are the symmetric spaces spanned by $A \cdot bv'$NA, bv'NA $\cdot bv'$NA and $A \cdot bv'$NA $\cdot bv'$NA, respectively. Finally, we shall prove the existence of a partition value on the space W generated by all the above spaces. Moreover, on each of the spaces mentioned, there exists a unique partition value. The above mentioned spaces are important in the analysis of models that have both economic and political aspects.

The partition value also turns out to be a useful tool in proving the existence of values. In this paper, for example, the partition value was used to demonstrate the fact (not previously known) that on the spaces A, pNA $* bv'$NA (or even $A * bv'$NA), bv'NA $* bv'$NA, $A * bv'$NA $* bv'$NA and W values do exist.

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We mention that the existence of a value on most[1] of these spaces also follows from the recent work of J. F. Mertens [5] which however uses entirely different methods.

A recent development is related to the results presented in this paper, namely the proof by A. Neyman [9] that $bv'\mathrm{NA} \subset \mathrm{ASYMP}$. This result, together with the arguments of Propositions 5.5 and 5.10 below, yield that A and $A * bv'\mathrm{NA}$ are contained in ASYMP (and so a fortiori have a partition value). However, $A * bv'\mathrm{NA} * bv'\mathrm{NA}$ is not in ASYMP (Proposition 5.42 below) though it does have a partition value. Moreover, it is worth mentioning that the proof of the existence of a partition value on $bv'\mathrm{NA}$, A and $A * bv'\mathrm{NA}$ is much shorter and simpler than the proof that they are included in ASYMP.

2. Preliminaries. Most of the definitions and notations are according to [1].

Let $(I, \mathcal{C})$ be a measurable space (i.e., I is a set and $\mathcal{C}$ is a σ field of subsets of I). We will assume ([1], assumption (2.1)) that $(I, \mathcal{C})$ is isomorphic to $([0, 1], \mathfrak{B})$ where $\mathfrak{B}$ is the σ field of Borel sets on $[0, 1]$ (i.e., there is a one-one mapping from I onto $[0, 1]$ that is measurable in both directions). A *set function* is a real valued function v on $\mathcal{C}$ such that $v(\emptyset) = 0$. The members of I are called *players*, the members of $\mathcal{C}$ *coalitions*, and the set functions *games*. A game v is *monotonic* if for each $S, T \in \mathcal{C}$, $S \subset T$ implies $v(S) \leqslant v(T)$. If Q is a set of games, Q^+ denotes the subset of monotonic games in Q. A game v is of *bounded variation* if it is the difference between two monotonic games. The space of all games of bounded variation is called $\underline{\mathrm{BV}}$. Let Q be any subset of BV. A mapping of Q into BV is positive if it maps Q^+ into BV^+.

A nondecreasing sequence of coalitions of the form

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = I$$

will be called a chain. The variation of v over a chain

$$\Omega : S_0 \subset S_1 \subset \cdots \subset S_n = I$$

is defined by

$$\|v\|_\Omega = \sum_{i=0}^{n-1} |v(S_{i+1}) - v(S_i)|.$$

For each game v in BV the variation norm is defined by

$$\|v\| = \sup \|v\|_\Omega$$

where the sup is taken over all chains Ω. We shall use the same definition for the variation norm of finite game.

All topological notations appearing in this paper refer to the topology generated by the variation norm.

Let $\mathcal{G}$ be the group of all the automorphisms of $(I, \mathcal{C})$ (i.e., isomorphisms of that space onto itself), each $\theta \in \mathcal{G}$ induces a linear mapping θ^* of BV onto itself defined by

$$(\theta^* v)(S) = v(\theta S) \quad \text{for each } S \in \mathcal{C}.$$

A subset Q of BV is called *symmetric* if $\theta^* Q = Q$ for each $\theta \in \mathcal{G}$. The subspace of BV consisting of all bounded, finitely additive set functions is denoted $\underline{\mathrm{FA}}$.

Let Q be a symmetric subspace of BV. A *value* on Q is a positive linear mapping φ

[1] Merten's work is not based on limits of finite games and therefore is not geared to handle ASYMP and related spaces (such as W); specifically, there are games in ASYMP that are not in the space DIFF defined in [5]. Whether or not there exists a partition value on DIFF is an open question at this writing.

from Q into FA that satisfies:

$$\varphi \text{ is a symmetric, i.e., } \varphi(\theta^* v) = \theta^*(\varphi v) \text{ for each } v \in Q \text{ and } \theta \in \mathcal{G}. \tag{2.1}$$

$$\varphi \text{ is efficient, i.e., } (\varphi v)(I) = v(I) \text{ for all } v \in Q. \tag{2.2}$$

Let Q be a subset of BV and $\varphi : Q \to \mathrm{BV}$ an operator. The norm of φ is defined by

$$\|\varphi\| = \sup \frac{\|\varphi v\|}{\|v\|}$$

where the supremum is taken over all nonzero $v \in Q$.

The subspace of BV consisting of all nonatomic measures on $(I, \mathcal{C})$ is denoted by NA.

NA^1 is the set of all measures μ in NA^+ satisfying $\mu(I) = 1$.

Define DIAG to be the set of all $v \in \mathrm{BV}$ satisfying: *There exists a positive integer k, a k-dimensional vector ξ of* NA^1 *measures, and a neighborhood U in E^k of the diagonal* $[0, \xi(I)]$ *such that if $\xi(S) \in U$ then $v(S) = 0$.* Let Q be the symmetric subspace of BV, φ a value on Q. φ is a *diagonal value* if $\varphi v = 0$ for all $v \in Q \cap \mathrm{DIAG}$.

The space of all real valued functions f of bounded variation on $[0, 1]$ that obey $f(0) = 0$ and are continuous at 0 and 1 is denoted bv'. The closed symmetric subspace of bv spanned by the set function of the form $f \circ \mu$ where $f \in bv'$ and $\mu \in \mathrm{NA}^1$ is called $bv'\mathrm{NA}$. $p\mathrm{NA}$ is the closed subspace of $bv'\mathrm{NA}$ spanned by all powers of NA^1 measures. A is the closed algebra generated by all set functions of the form $f \circ \mu$ where f is a continuous function in bv', and μ is in NA^1.

If Q_1 and Q_2 are subsets of BV then $Q_1 * Q_2$ is the closed linear symmetric subspace spanned by all games of the form $v_1 \cdot v_2$ where $v_1 \in Q_1$ and $v_2 \in Q_2$. If B is a set of games then Q^B will denote symmetric subspace spanned by B, and $\bar{B}$ the closure of B (in the variation norm).

The Shapley value on finite games will be denoted by ψ. Let v be a finite game, N the set of players. For each player a and order $\mathcal{R}$ on N, $P_a^{\mathcal{R}}$ is the set of all players preceding a in $\mathcal{R}$. The Shapley value ψv is given by

$$(\psi v)(a) = \frac{1}{|N|!} \sum_{\mathcal{R}} \left[v\left(P_a^{\mathcal{R}} \cup \{a\}\right) - v\left(P_a^{\mathcal{R}}\right) \right].$$

$(\psi v)(a)$ can be regarded as the expectation of the contributions of player a, where each order $\mathcal{R}$ has the same probability, namely $1/|N|!$. We will write $v(P_a^{\mathcal{R}} \cup a)$ instead of $v(P_a^{\mathcal{R}} \cup \{a\})$.

A partition π of the underlying space $(I, \mathcal{C})$ is a finite family of disjoint measurable subsets whose union is I. A partition π_2 is a *refinement* of another partition π_1 if each member of π_1 is a union of members of π_2. In such a case we denote it by $\pi_2 \succ \pi_1$. For any two partitions π_1 and π_2, $\pi_1 * \pi_2$ is the partition such that for each partition π if $\pi \succ \pi_1$ and if $\pi \succ \pi_2$ then $\pi \succ \pi_1 * \pi_2$. In other words $\pi_1 * \pi_2$ is the smallest partition (with respect to cardinality) which is a refinement of both π_1 and π_2. A sequence $(\pi_m)_{m=1}^{\infty}$ of partitions is called *admissible* if it satisfies:

(1) it is *decreasing*, i.e., $\pi_{m+1} \succ \pi_m$ for each m;

(2) it is *separating*, i.e., for each $s, t \in I$ with $s \neq t$, there is an m such that s and t are in different members of π_m.

For each partition π and set function v, let v_π be the finite game whose players are the members of π, i.e.,

$$v_\pi(A) = v\left(\bigcup_{a \in A} a\right) \quad \text{for all } A \subset \pi.$$

Let $T \in \mathfrak{B}$ and let $(\pi_m)_{m=1}^{\infty}$ be an admissible sequence whose first term is π_1

$= \{T, I \setminus T\}$. For each m let $\overline{T}_m = \{S \in \pi_m \mid S \subset T\}$, i.e., $\overline{T}_m$ or simply $\overline{T}$ is the coalition corresponding to T in the finite game v_{π_m}. $(\psi v_{\pi_m})(\overline{T})$ is the Shapley value of the coalition T_m. If the numbers $(\psi v_{\pi_m})(\overline{T})$ approach a limit as $m \to \infty$ and this limit is independent of the particular sequence $(\pi_m)_{m=1}^{\infty}$ then we will denote it by $(\psi v)(T)$. If $(\psi v)(T)$ exists for all $T \in \mathfrak{B}$, then the set function ψv is called the *asymptotic value* of v. It is clear that if ψv exists it is necessarily unique and finitely additive.

The set of all games $v \in \mathrm{BV}$ having an asymptotic value is denoted ASYMP. The closed linear subspace of BV spanned by all games v where $v \in \mathrm{ASYMP} \cup (A * bv'\mathrm{NA} * bv'\mathrm{NA})$, is denoted by W. It is worth mentioning that W contains each of the following important spaces: $bv'\mathrm{NA}$, A, $A * bv'\mathrm{NA}$, $p\mathrm{NA} * bv'\mathrm{NA}$ and $bv'\mathrm{NA} * bv'\mathrm{NA}$.

3. The partition value and its basic properties. Throughout the rest of this paper we use the terminology introduced in §2 (preliminaries).

DEFINITION 3.1. Let Q be a symmetric subspace of BV. A value φ on Q is a *partition value* if for each v in Q and S in $\mathfrak{B}$ there exists an admissible sequence of partitions $(\pi_n)_{n=1}^{\infty}$ such that

$$\pi_1 \succ \{S, I \setminus S\}, \text{ i.e., } \pi_1 \text{ is a refinement of } \{S, I \setminus S\}, \quad \text{and} \tag{3.2.1}$$

$$\lim_{n \to \infty} (\psi v_{\pi_n})(\overline{S}) = (\varphi v)(S) \tag{3.2.2}$$

(where ψ is the Shapley value for finite games).

By definition, the partition value is a value. However, to guarantee that an operator φ satisfying (3.2.1) and (3.2.2) is a value, fewer conditions are sufficient.

PROPOSITION 3.3. *Let Q be a symmetric subspace of* BV, *and let φ be a linear symmetric operator from Q into* FA *obeying* (3.2). *Then φ is a partition value.*

PROOF. The efficiency and the positivity of φ follow from the efficiency and positivity of the Shapley value for finite games.

PROPOSITION 3.4. *Every partition value φ is continuous in the variation norm and* $\|\varphi\| \leqslant 1$.

PROOF. If u is a finite game with n players and ψ is the Shapley value for finite games, then

$$(\psi u)(\{i\}) = \frac{1}{n!} \sum_{\mathfrak{R}} \left[u(P_i^{\mathfrak{R}} \cup \{i\}) - u(P_i^{\mathfrak{R}}) \right],$$

$$\|\psi u\| = \sum_{i=1}^{n} |(\psi u)(\{i\})| \leqslant \frac{1}{n!} \sum_{\mathfrak{R}} \sum_{i=1}^{n} |u(P_i^{\mathfrak{R}} \cup \{i\}) - u(P_i^{\mathfrak{R}})|$$

$$\leqslant \max_{\mathfrak{R}} \sum_{i=1} |u(P_i^{\mathfrak{R}} \cup \{i\}) - u(P_i^{\mathfrak{R}})| \leqslant \|u\|.$$

Let Q be a symmetric subspace of BV, and let φ be a partition value on it. We shall show that for every v in Q

$$\|\varphi v\| \leqslant \|v\|$$

since $\varphi v \in \mathrm{FA}$, $\|\varphi v\| = \sup_{S \in \mathfrak{B}} [(\varphi v)(S) - (\varphi v)(I \setminus S)]$. For $S \in \mathfrak{B}$, there exists an admissible sequence $(\pi_k)_{k=1}^{\infty}$ with $\pi_k \succ \{S, I \setminus S\}$, such that

$$\lim_{n \to \infty} (\psi v_{\pi_n})(\overline{S}) = (\varphi v)(S).$$

Thus by the efficiency we also have

$$\lim_{n \to \infty} (\psi v_{\pi_n})(\overline{I \setminus S}) = (\varphi v)(I \setminus S).$$

Using the inequality

$$\|\psi v_{\pi_n}\| \leqslant \|v_{\pi_n}\| \leqslant \|v\|$$

we deduce that $\|\varphi v\| \leqslant \|v\|$, which completes the proof of proposition 3.4.

PROPOSITION 3.5. *Every partition value is a diagonal value.*

PROOF. The partition value is continuous (proposition 3.4). And [8] asserts that every continuous value is a diagonal value.

It is worth mentioning that not every value is a diagonal one. For counter examples refer to [7] and [11].

PROPOSITION 3.6. *Let Q be a symmetric subspace of* ASYMP. *Then the operator which associates to each v in Q its asymptotic value, is the unique partition value on Q. Furthermore, if φ is a partition value on a symmetric subspace Q of* BV, *then φ has a unique extension to a partition value φ on the space generated by $Q \cup$* ASYMP.

PROOF. Follows immediately from the definitions of the asymptotic value and the partition value, and from proposition 3.3.

4. General preparation for proofs of existence theorems. Let $B \subset$ BV be a set of games. An operator $\gamma : B \to$ BV is *strongly positive* iff for every $v_i \in B$, $\theta_i \in \mathcal{G}$, $a_i \in E^1$ $(1 \leqslant i \leqslant n)$, monotonicity of $\sum_{i=1}^n a_i\theta_i^* v_i$ implies that $\sum_{i=1}^n a_i\theta_i^*(\gamma v_i)$ is a monotinic game. γ is *efficient* if $(\gamma v)(I) = v(I)$ for each v in B.

Denote by Q^B the linear symmetric subspace of BV spanned by B. Thus $v \in Q^B$ iff $v = \sum_{i=1}^k a_i\theta_i^* v_i$ for $v_i \in B$, $\theta_i \in \mathcal{G}$, $a_i \in E^1$ $(1 \leqslant i \leqslant k)$.

PROPOSITION 4.1. *Let $B \subset$ BV be a set of games. If $\gamma : B \to$ FA is a strongly positive efficient operator, then there exists a unique extension of γ to a value φ on Q^B, such that:*

$$\varphi\left(\sum_{i=1}^k a_i\theta_i^* v_i\right) = \sum_{i=1}^k a_i\theta_i^*(\gamma v_i).$$

PROOF. Extend γ to $\varphi : Q^B \to$ FA by $\varphi(\sum a_i\theta_i^* v_i) = \sum a_i\theta_i^*(\gamma v_i)$. We have to show first that φ is well defined, i.e., that if

$$\sum a_i\theta_i^* v_i = \sum b_j\hat{\theta}_j^* \hat{v}_j \quad \text{then} \quad \sum a_i\theta_i^*(\gamma v_i) = \sum b_j\hat{\theta}_j^*(\gamma\hat{v}_j).$$

Obviously, it is sufficient to show that if $v_i \in B$ and $\sum a_i\theta_i^* v_i$ vanishes identically then so does $\sum a_i\theta_i^*(\gamma v_i)$. Indeed, if $v \equiv 0$, then in particular v and $-v$ are monotonic set functions. Since γ is strongly positive, both φv and $-\varphi v$ are nonnegative, which implies that $\varphi v \equiv 0$.

Thus φ is a (well-defined) operator. Also, according to its definition, φ is linear and symmetric. The efficiency and the positivity of φ follow from the efficiency and strong positivity of γ. Therefore φ is the extension of γ to a value on Q^B.

LEMMA 4.2. *Let $B \subset$ BV be a symmetric set of games and $\gamma : B \to$ FA operator. Assume that the following condition holds:*

For each measurable partition π, coalition $S \in \mathfrak{B}$ and finite set of games $v^1, \ldots, v^n$ in B, there exists a sequence of partitions $(\pi_k)_{k=1}^\infty$ such that:

$$\begin{aligned} &\pi_k \succ \pi * \{S, I \setminus S\}, \\ &\lim_{k\to\infty} (\psi v^i_{\pi_k})(\bar{S}) = (\gamma v^i)(S) \quad \textit{for each } 1 \leqslant i \leqslant n. \end{aligned} \tag{4.3}$$

Then γ has a unique extension to a partition value on $\bar{Q}^B$ (the closure of Q^B in the variation norm).

This lemma is very useful in order to prove the existence of a partition value on a given closed symmetric space. We shall make use of it, in all the proofs of the existence theorems.

For the proof of lemma 4.2 we need the following:

PROPOSITION 4.4. *Let Q be a symmetric subspace of* BV. *If φ is a value on Q with $\|\varphi\| \leqslant 1$ then, φ can be extended uniquely to a continuous value on $\bar{Q}$.*

PROOF. Let $w \in \bar{Q}$ then $w = \lim_{n\to\infty} w_n$ where $w_n \in Q$. Since φ is a continuous value on Q the equation $\varphi w = \lim_{n\to\infty} \varphi w_n$ extends φ to a well-defined operator on $\bar{Q}$. Obviously φ is now a linear symmetric and efficient operator on $\bar{Q}$. Since $\|\varphi w_n\| \leqslant \|w_n\|$ for each n then, $\|\varphi w\| \leqslant \|w\|$ and therefore $\|\varphi\| \leqslant 1$. Hence, proposition 4.6 in [1] asserts that φ is positive, which yields that φ is a continuous value on $\bar{Q}$.

PROOF OF LEMMA 4.2. Since the Shapley value ψ is a positive and efficient operator, the second condition in (4.3) guarantees that γ is a strongly positive and efficient operator. Thus, proposition 4.1 implies that the equation

$$\varphi\left(\sum_{i=1}^{n} a_i\theta_i^* v_i\right) = \sum_{i=1}^{n} a_i\theta_i^*(\gamma v_i)$$

defines a value φ on Q^B. As the proof of proposition 3.4 does not depend on the fact that $(\pi_k)_{k=1}^{\infty}$ is admissible, φ is continuous with $\|\varphi\| \leqslant 1$. By proposition 4.4, φ can be extended uniquely to a continuous value on $\bar{Q}^B$. To complete the proof we have to show that φ is a partition value on $\bar{Q}^B$, i.e., that for every v in $\bar{Q}^B$ and S in $\mathfrak{B}$ there exists an *admissible* sequence of partitions $(\pi^n)_{n=1}^{\infty}$ such that

$$\pi^1 \succ \{S, I\backslash S\},$$

$$\lim_{n\to\infty} (\psi v_{\pi^n})(\bar{S}) = (\varphi v)(S). \tag{4.5}$$

Let $(\Delta_n)_{n=1}^{\infty}$ be an arbitrary admissible sequence, let v be any game in $\bar{Q}^B$, and let S in $\mathfrak{B}$. Then there are $v^n \in Q^B$ $(n = 1, 2, \ldots)$ such that $v = \lim_{n\to\infty} v^n$ and hence

$$(\varphi v)(S) = \lim_{n\to\infty} (\varphi v^n)(S). \tag{4.6}$$

We are now ready to construct, by induction, an admissible sequence $(\pi^n)_{n=1}^{\infty}$ satisfying the condition (4.5).

From the definition of φ there exists a sequence of measurable partitions $(\pi_m^1)_{m=1}^{\infty}$ satisfying

$$\pi_m^1 \succ \{S, I\backslash S\},$$

$$\lim_{n\to\infty} \left(\psi v_{\pi_m^1}^1\right)(\bar{S}) = (\varphi v^1)(S).$$

Let us choose one partition from $(\pi_m^1)_{m=1}^{\infty}$, which will be denoted by π^1, such that

$$\pi^1 \succ \{S, I\backslash S\}$$

and

$$|(\psi v_{\pi^1}^1)(\bar{S}) - (\varphi v^1)(S)| < 1.$$

Assume that π^t is a measurable partition satisfying

$$\pi^t \succ \{S\colon I\backslash S\}$$

and

$$|(\psi v_{\pi^t}^t)(\bar{S}) - (\varphi v^t)(S)| < 1/t$$

then define $\tilde{\pi}^t = \Delta^t * \pi^t$.

From the assumptions it follows that for the partition $\tilde{\pi}^t$, the game v^{t+1} and the coalition S there exists a sequence of measurable partitions $(\pi_m^{t+1})_{m=1}^{\infty}$ satisfying

$$\pi_m^{t+1} \succ \tilde{\pi}^t \quad \text{for each } m$$

and

$$\lim_{m\to\infty} (\psi v_{\pi_m^{t+1}}^{t+1})(\bar{S}) = (\varphi v^{t+1})(S).$$

Let us choose one partition from $(\pi_m^{t+1})_{m=1}^{\infty}$ which will be denoted by π^{t+1}, such that

$$|(\psi v_{\pi^{t+1}}^{t+1})(\bar{S}) - (\varphi v^{t+1})(S)| < 1/(t+1).$$

By this process we shall get a sequence of measurable partitions $(\pi^k)_{k=1}^{\infty}$ satisfying:
1. π^{k+1} is a refinement of π^k for each k.
2. $(\pi^k)_{k=1}^{\infty}$ is separating because π^{k+1} is a refinement of Δ^k for each $k \geqslant 1$ and $(\Delta^k)_{k=1}^{\infty}$ is separating.

Hence $(\pi^k)_{k=1}^{\infty}$ is an admissible sequence. Moreover,

$$\begin{gathered} \pi^1 \succ \{S, I\setminus S\}, \\ |(\psi v_{\pi^k}^k)(\bar{S}) - (\varphi v^k)(S)| < 1/k \quad \text{for each } k. \end{gathered} \tag{4.7}$$

To complete the proof, we have to show that $(\pi^k)_{k=1}^{\infty}$ satisfies the second condition of (4.5).

For every measurable partition π

$$\begin{aligned} |\psi v_\pi(\bar{S}) - \psi v_\pi^k(\bar{S})| &\leqslant \|\psi v_\pi - \psi v_\pi^k\| \\ &\leqslant \|v_\pi - v_\pi^k\| \leqslant \|v - v^k\| \underset{t\to\infty}{\to} 0. \end{aligned} \tag{4.8}$$

(4.6), (4.7) and (4.8) imply that each term in the right side of the following inequality

$$\begin{aligned} |\psi v_{\pi^k}(\bar{S}) - (\varphi v)(S)| &\leqslant |\psi v_{\pi^k}(\bar{S}) - \psi v_{\pi^k}^k(\bar{S})| \\ &\quad + |\psi v_{\pi^k}^k(\bar{S}) - (\varphi v^k)(S)| + |\varphi v^k(S) - \varphi v(S)| \end{aligned}$$

tends to zero as k tends to infinity. Thus

$$\lim_{k\to\infty} (\psi v_{\pi^k})(\bar{S}) = (\varphi v)(S)$$

and the second condition of (4.5) is fulfilled.

The proofs of the existence theorems which will be introduced in the next section rely on the following results of A. Neyman [6].

LEMMA 4.9. *Let $(\pi_k)_{k=1}^{\infty}$ be a sequence of partitions satisfying:*
I. *There exists L such that $|\{t \mid t = \lambda(a),\ a \in \pi_k\}| \leqslant L$ for each k; and*
II. $\max_{a\in\pi_k} \lambda(a) \to_{k\to\infty} 0$.
Then for every $f \in bv'$

$$\operatorname*{Sup}_{A_k \subset \pi_k} |\psi(f \circ \lambda)_{\pi_k}(A_k) - \lambda(A_k)\cdot f(1)| \underset{k\to\infty}{\to} 0.$$

5. Existence of a partition value on bv'NA, A, $A * bv'$NA, bv'NA $* bv'$NA and W.

In this section, we shall prove the existence of a partition value on the spaces bv'NA, A, $A * bv'$NA, bv'NA $* bv'$NA, $A * bv'$NA $* bv'$NA and W. This, of course, establishes the existence of a value on all the above spaces; moreover, this value may be viewed as a limit of Shapley values of finite games.

The existence of a (unique) value on bv'NA was first proved by Aumann and

Shapley [1, chapter I, §8]. However, on the other spaces the existence of a value was previously unknown. It is worth mentioning that the existence of a partition value on the closed space W generated by ASYMP and all the other spaces mentioned above gives us an important wider space which properly contains ASYMP (theorem 5.42 below) and upon which a value exists.

In this section the following notation is used: Let π be an arbitrary measurable partition and μ a vector of NA^1 measures. $\pi(k, \mu)$ denotes the set of all measurable partitions $\tilde{\pi}$ satisfying:

I. $\tilde{\pi} \succ \pi$,

II. for each $a \in \tilde{\pi}$ and $b \in \pi$ if $a \subset b$, then $\mu(a) = (1/k)\mu(b)$.

(Note that Lyapunov's theorem [4] asserts that for each vector ν of NA^1 measures, partition π and positive integer k, $\pi(k, \nu) \neq \emptyset$.) Denote the partition $\pi * \{S, I \backslash S\}$ by π_S and define $\pi(k, \mu, S)$ to be $\pi_S(k, \mu)$. For brevity, from now on, we shall use the word partition instead of measurable partition.

THEOREM 5.1. *There exists a partition value on* $bv'\mathrm{NA}$.

REMARK. As already mentioned, the existence of a (unique) value on $bv'\mathrm{NA}$ was proved by Aumann and Shapley [1, chapter I, §8], who further proved that this (unique) value φ on $bv'\mathrm{NA}$ satisfies

$$\varphi(f \circ \mu) = f(1) \cdot \mu \quad \text{for each } f \in bv' \text{ and } \mu \in \mathrm{NA}^1.$$

However the proof of theorem 5.1 does not depend on Aumann and Shapley's result and thus it may serve also as an alternative proof for their existence theorem.

The proof of theorem 5.1 is based upon

PROPOSITION 5.2. *For each partition* π, $S \in \mathfrak{B}$ *and the finite number of games* $f_1 \circ \nu_1, \ldots, f_n \circ \nu_n$ *where* $f_i \in bv'$, $\nu_i \in \mathrm{NA}^1$, $1 \leqslant i \leqslant n$, *there exists a sequence* $(\pi_k)_{k=1}^{\infty}$ *of partitions such that*

$$\pi_k \succ \pi * \{S, I \backslash S\},$$

$$\lim_{k \to \infty} \psi(f_i \circ \nu_i)_{\pi_k}(\bar{S}) = \nu_i(S) \cdot f_i(1).$$

PROOF OF PROPOSITION 5.2. Since for each positive integer k, coalition S, and $1 \leqslant i \leqslant n$, $\pi(k, \nu, S) \subset \pi(k, \nu_i, S)$ where $\nu = (\nu_1, \ldots, \nu_n)$, it is sufficient to show that for each partition π, $S \in \mathfrak{B}$, $f \circ \mu$ where $f \in bv'$, $\mu \in \mathrm{NA}^1$ and for each sequence $(\pi_k)_{k=1}^{\infty}$ where $\pi_k \in \pi(k, \mu, S)$

$$\lim_{k \to \infty} \psi(f \circ \mu)_{\pi_k}(\bar{S}) = f(1) \cdot \mu(S).$$

Since $\mu(\pi_k) \to_{k \to \infty} 0$ and since $|\{t \mid t = \mu(a), a \in \pi_k\}|$ is bounded, the last equation follows directly from lemma 4.9.

PROOF OF THEOREM 5.1. Let B be the set of all games having the form $f \circ \mu$ where $f \in bv'$ and $\mu \in \mathrm{NA}^1$. Obviously $\overline{Q}^B = bv'\mathrm{NA}$. Define $\gamma : B \to \mathrm{FA}$ by

$$\gamma(f \circ \mu) = f(1) \cdot \mu.$$

We have to show first that γ is a well-defined operator. Proposition 5.2 asserts that if $f \circ \mu$ and $g \circ \nu$ are games in B, then there exists a sequence of partitions $(\pi_k)^{\infty}$ such that

$$\lim_{k \to \infty} \psi(f \circ \mu)_{\pi_k}(\bar{S}) = f(1) \cdot \mu(S)$$

and

$$\lim_{k \to \infty} \psi(g \circ \nu)_{\pi_k}(\bar{S}) = g(1) \cdot \nu(S).$$

Therefore if $f \circ \mu = g \circ \nu$ then $f(1) \cdot \mu = g(1) \cdot \nu$ and thus γ is well defined. The rest of the proof is an immediate consequence of proposition 5.2 and lemma 4.2.

5. a. *The partition value on A.* For the next theorems we shall need the following: Let S be any coalition in $\mathfrak{B}$ and μ a vector of NA^1 measures.

Let $(\pi_k)_{k=1}^{\infty}$ be a sequence of partitions with $\lim_{k\to\infty} \mu(a_k) = 0$ for each $a_k \in \pi_k$. Denote by Ω_k the set of all the orders on π_k and assume that each of them has the same probability (i.e., $1/n(k)$ where $n(k) = |\pi_k|$). For each neighborhood U of the diagonal $[0, \mu(I)]$ of the vector measure μ, define

$$\Omega_k^U = \left\{ \mathfrak{R} \in \Omega_k \mid \mu\left(P_a^{\mathfrak{R}} \cup a\right) \in U \text{ for each } a \in \pi_k \right\}.$$

Let X_k be a random variable on Ω_k. Denote by $X_k|_{\Omega_k^U}$ the restriction of the random variable X_k to Ω_k^U, where each order in Ω_k^U has the same probability: $1/|\Omega_k^U|$.

For each positive integer k, let X_k be a random variable defined on Ω_k. $(X_k)_{k=1}^{\infty}$ is bounded if there is a number M such that for each k and for each $\mathfrak{R} \in \Omega_k$, $|X_k(\mathfrak{R})| \leqslant M$.

PROPOSITION 5.3. *Let $(X_k)_{k=1}^{\infty}$ be a bounded sequence of random variables where each X_k is defined on Ω_k. Then, for each neighborhood U of the diagonal of the vector measure μ*

$$\lim_{k\to\infty} \left[E(X_k) - E\left(X_{k\,|\,\Omega_k^U}\right) \right] = 0 \quad \textit{where } E \textit{ is the expectation operator.}$$

PROOF. Since $(X_k)_{k=1}^{\infty}$ is a bounded sequence, the proof is an immediate consequence of Aumann-Shapley's corollary 18.10 [1, chapter III] which asserts that

$$\lim_{k\to\infty} \operatorname{Prob}\left(\Omega_k^U\right) = 1.$$

(Note that the condition of corollary 18.10 is that $(\pi_k)_{k=1}^{\infty}$ is an admissible sequence, but the proof of lemma 18.7 there, implies that this condition can be replaced by the condition that $\lim_{k\to\infty} \mu(a_k) = 0$ for each $a_k \in \pi_k$.)

THEOREM 5.4. *There exists a partition value φ on A satisfying*

$$\varphi\left(\prod_{i=1}^{n} (f_i \circ \mu_i)\right) = \sum_{j=1}^{n} \mu_j \cdot \int_0^1 \left(\prod_{i\neq j} f_i\right) df_j$$

where for each $1 \leqslant i \leqslant n$, f_i is a continuous function in bv' and $\mu_i \in \mathrm{NA}^1$. (If $n = 1$, $\prod_{i\neq j} f_i$ should be replaced by 1.)

PROOF. Let cbv be the subspace of bv consisting of all continuous functions of bv. $(cbv)^+$ is the set of all nondecreasing functions in cbv. Let B be the subset of A consisting of all games of the form $\prod_{i=1}^{k} f_i \circ \mu_i$ where $f_i \in (cbv)^+$ and $\mu_i \in \mathrm{NA}^1$ $(1 \leqslant i \leqslant k)$. Obviously $A = \overline{Q}^B$. With the above notations:

PROPOSITION 5.5. *For each partition π, $S \in \mathfrak{B}$, $v \in B$ $(v = \prod_{i=1}^{n} (f_i \circ \mu_i)$, $f_i \in (cbv)^+$, $\mu_i \in \mathrm{NA}^1)$ and for each $\pi_k \in \pi(k, \mu, S)$ where $\mu = (\mu_1, \ldots, \mu_n)$ we have that*

$$\lim_{k\to\infty} (\psi v_{\pi_k})(\bar{S}) = \sum_{j=1}^{n} \mu_j \cdot \int_0^1 \prod_{i\neq j} df_j.$$

Assuming proposition 5.5 (which we will prove later), the proof of theorem 5.4 is completed as follows: Define $\gamma : B \to \mathrm{FA}$ by

$$\gamma\left(\prod_{i=1}^{n} f_i \circ \mu_i\right) = \sum_{j=1}^{n} \mu_j \cdot \int_0^1 \left(\prod_{i\neq j} f_i\right) df_j.$$

We will prove first that γ is a well-defined operator. Let $\prod_{i=1}^{n} g_j \circ \mu_i$, $\prod_{j=1}^{m} g_i \circ \nu_j$ be two games in B. Denote $\xi = (\mu_1, \mu_2, \ldots, \mu_n, \nu_1, \nu_2, \ldots, \nu_m)$ and let $\pi_k \in \pi(k, \xi, S)$. Since $\pi(k, \xi, S) \subset \pi(k, \nu, S) \cap \pi(k, \mu, S)$ where $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_m)$, it follows from proposition 5.5 that

$$\lim_{k\to\infty} \psi\left(\prod_{i=1}^{n} f_i \circ \mu_i\right)_{\pi_k} (\bar{S}) = \sum_{j=1}^{n} \mu_j(S) \int_0^1 \left(\prod_{i\neq j} f_i\right) df_j$$

and

$$\lim_{k\to\infty} \psi\left(\prod_{i=1} g_i \circ \nu_i\right)_{\pi_k} (\bar{S}) = \sum_{j=1}^{m} \nu_j(S) \int_0^1 \left(\prod_{i\neq j} g_i\right) dg_j$$

which imply that if $\prod_{i=1}^{n} f_i \circ \mu_i = \prod_{j=1}^{m} g_j \circ \nu_j$ then

$$\sum_{j=1}^{n} \mu_j \int_0^1 \left(\prod_{i\neq j} f_i\right) df_j = \sum_{j=1}^{m} \nu_j \int_0^1 \left(\prod_{i\neq j} g_i\right) dg_j.$$

Thus γ is well defined.

Let π be a partition, $S \in \mathfrak{B}$, n a positive integer, and $v^1, v^2, \ldots, v_n$ in B, where $v^t = \prod_{i=1}^{m_t} f_i^t \circ \mu_i^t$, $1 \leqslant t \leqslant n$. Let $\mu = (\mu_1^1, \ldots, \mu_{m_1}^1, \ldots, \mu_1^n, \ldots, \mu_{m_n}^n)$, μ consists of all measures involved in v^t for $1 \leqslant t \leqslant n$. For each $1 \leqslant t \leqslant n$, denote $\mu^t = (\mu_1^t, \ldots, \mu_{n_t}^t)$. Since $\pi(k, \mu, S) \subset \pi(k, \mu^t, S)$ for each $S \in \mathfrak{B}$ and $1 \leqslant t \leqslant n$, proposition 5.5 implies that if $\pi_k \in \pi(k, \mu, S)$ then

$$\lim_{k\to\infty} (\psi v_{\pi_k}^t)(\bar{S}) = (\gamma v)(S).$$

Theorem 5.4 now follows directly from lemma 4.2.

Proof of Proposition 5.5. Let π be any partition, $S \in \mathfrak{B}$, and let $v \in B$ be given by $v = \prod_{i=1}^{n} (f_i \circ \mu_i)$ $(f_i \in cbv^+,\ \mu_i \in \mathrm{NA}^1)$, and $\pi_k \in \pi(k, \mu, S)$ where $\mu = (\mu_1, \ldots, \mu_n)$. Define

$$W_1(a, \mathfrak{R}) = \prod_{i=2}^{n} (f_i \circ \mu_i)(P_a^{\mathfrak{R}})$$

for $2 \leqslant i \leqslant n-1$ define

$$W_i(a, \mathfrak{R}) = (f_1 \circ \mu_1)(P_a^{\mathfrak{R}} \cup a) \cdots (f_{i-1} \circ \mu_{i-1})(P_a^{\mathfrak{R}} \cup a) \cdot (f_{i+1} \circ \mu_{i+1})(P_a^{\mathfrak{R}}) \cdots (f_n \circ \mu_n)(P_a^{\mathfrak{R}}),$$

$$W_n(a, \mathfrak{R}) = \prod_{i=1}^{n-1} \left[(f_i \circ \mu_i)(P_a^{\mathfrak{R}} \cup a)\right].$$

It is easy to verify that

$$v(P_a^{\mathfrak{R}} \cup a) - v(P_a^{\mathfrak{R}}) = \sum_{i=1}^{n} W_i(a, \mathfrak{R})\left[(f_i \circ \mu_i)(P_a^{\mathfrak{R}} \cup a) - (f_i \circ \mu_i)(P_a^{\mathfrak{R}})\right].$$

It is sufficient to prove that, for each i, $1 \leqslant i \leqslant n$,

$$\lim_{k\to\infty} \sum_{\substack{a\in\pi_k \\ a\in S}} E\left\{W_i(a, \mathfrak{R})\left[(f_i \circ \mu_i)(P_a^{\mathfrak{R}} \cup a) - (f_i \circ \mu_i)(P_a^{\mathfrak{R}})\right]\right\} = \mu_i(S) \int_0^1 \left(\prod_{j\neq i} f_j\right) df_i.$$

Let $g_i = \prod_{k\neq i} f_k$, g_i and f_i are monotonic continuous functions. Therefore

$$\int_0^1 g_i df_i = \lim_{L\to\infty} \sum_{j=1}^{L} g_i\left(\frac{j-1}{L}\right)\left[f_i\left(\frac{j}{L}\right) - f_i\left(\frac{j-1}{L}\right)\right].$$

Let ϵ be a positive number. There is an integer, L, such that

$$\begin{aligned} &\text{I} \quad \left|\sum_{j=1}^{L} g_i\left(\frac{j-1}{L}\right)\left[f_i\left(\frac{j}{L}\right)-f_i\left(\frac{j-1}{L}\right)\right]-\int_0^1 g_i df_i\right| < \epsilon, \\ &\text{II} \quad y-x<\frac{2}{L} \Rightarrow g_i(y)-g_i(x)<\epsilon \quad \text{where } 0 \leqslant x \leqslant y \leqslant 1. \end{aligned} \tag{5.6}$$

For each $1 \leqslant j \leqslant L$, let us define a function $f_i^j(x)$ by

$$f_i^j(x)=\begin{cases} 0, & x \leqslant \dfrac{j-1}{L}, \\ f_i(x)-f_i\left(\dfrac{j-1}{L}\right), & \dfrac{j-1}{L} \leqslant x \leqslant \dfrac{j}{L}, \\ f_i\left(\dfrac{j}{L}\right)-f_i\left(\dfrac{j-1}{L}\right), & \dfrac{j}{L} \leqslant x \leqslant 1, \end{cases}$$

$f_i=\sum_{j=1}^{L} f_i^j$. Hence

$$\begin{aligned} &\sum_{\substack{a\in\pi_k\\ a\in S}} E\left\{W_i(a,\mathcal{R})\left[(f_i\circ\mu_i)(P_a^{\mathcal{R}}\cup a)-(f_i\circ\mu_i)(P_a^{\mathcal{R}})\right]\right\} \\ &\quad=\sum_{\substack{a\in\pi_k\\ a\in S}} E\left\{\sum_{j=1}^{L} W_i(a,\mathcal{R})\left[(f_i^j\circ\mu_i)(P_a^{\mathcal{R}}\cup a)-(f_i^j\circ\mu_i)(P_a^{\mathcal{R}})\right]\right\} \\ &\quad=\sum_{j=1}\sum_{\substack{a\in\pi_k\\ a\in S}} E\left\{W_i(a,\mathcal{R})\left[(f_i^j\circ\mu_i)(P_a^{\mathcal{R}}\cup a)-(f_i^j\circ\mu_i)(P_a^{\mathcal{R}})\right]\right\}. \end{aligned}$$

Let

$$T_k=\left|\sum_{j=1}^{L}\sum_{\substack{a\in\pi_k\\ a\in S}} E\left\{W_i(a,\mathcal{R})\left[(f_i^j\circ\mu_i)(P_a^{\mathcal{R}}\cup a)-(f_i^j\circ\mu_i)(P_a^{\mathcal{R}})\right]\right\}-\mu_i(S)\int_0^1 g_i df_i\right|.$$

To complete the proof of the proposition, we have to show that $\lim_{k\to\infty} T_k=0$.

$$\begin{aligned} T_k \leqslant &\left|\sum_{j=1}^{L}\sum_{\substack{a\in\pi_k\\ a\in S}} E\left\{W_i(a,\mathcal{R})\left[(f_i^j\circ\mu_i)(P_a^{\mathcal{R}}\cup a)-(f_i^j\circ\mu_i)(P_a^{\mathcal{R}})\right]\right\}\right. \\ &\qquad\left.-\mu_i(S)\sum g_i\left(\frac{j-1}{L}\right)\left[f_i\left(\frac{j}{L}\right)-f_i\left(\frac{j-1}{L}\right)\right]\right|+\epsilon\cdot\mu_i(S) \\ \leqslant &\left|\sum_{j=1}^{L}\sum_{\substack{a\in\pi_k\\ a\in S}} E\left\{\left[W_i(a,\mathcal{R})-g_i\left(\frac{j-1}{L}\right)\right]\left[(f_i^j\circ\mu_i)(P_a^{\mathcal{R}}\cup a)-(f_i^j\circ\mu_i)(P_a^{\mathcal{R}})\right]\right\}\right| \\ &+\left|\sum_{j=1}^{L} g_i\left(\frac{j-1}{L}\right)\cdot\left\{\sum_{\substack{a\in\pi_k\\ a\in S}} E\left[(f_i^j\circ\mu_i)(P_a^{\mathcal{R}}\cup a)-(f_i^j\circ\mu_i)(P_a^{\mathcal{R}})\right]\right.\right. \\ &\qquad\left.\left.-\mu_i(S)\left[f_i\left(\frac{j}{L}\right)-f_i\left(\frac{j-1}{L}\right)\right]\right\}\right|+\epsilon\cdot\mu_i(S). \end{aligned}$$

Lemma 4.9 implies that

$$\lim_{k\to\infty} \sum_{\substack{a\in\pi_k\\ a\in S}} E\left[(f_i^j \circ \mu_i)(P_a^{\mathcal{R}} \cup a) - f_i^j(P_a^{\mathcal{R}})\right] = \mu_i(S)\cdot f_i^j(1).$$

Therefore, for large k,

$$\left| \sum_{\substack{a\in\pi_k\\ a\in S}} E\left[(f_i^j \circ \mu_i)(P_a^{\mathcal{R}} \cup a) - (f_i^j \circ \mu_i)(P_a^{\mathcal{R}})\right] - \mu_i(S)\left[f_i\left(\frac{j}{L}\right) - f\left(\frac{j-1}{L}\right)\right]\right| < \frac{\epsilon}{L}.$$

Hence, for large k,

$$T_k \leqslant \left| \sum_{j=1}^{L} \sum_{\substack{a\in\pi_k\\ a\in S}} E\left\{\left[W_i(a, \mathcal{R}) - g\left(\frac{j-1}{L}\right)\right]\left[(f_i^j \circ \mu_i)(P_a^{\mathcal{R}} \cup a) - (f_i^j \circ \mu_i)(P_a^{\mathcal{R}})\right]\right\}\right| + \left| \sum_{j=1}^{L} g_i\left(\frac{j-1}{L}\right)\cdot \frac{\epsilon}{L}\right| + \epsilon\cdot\mu_i(S).$$

Thus

$$T_k \leqslant \left| \sum_{j=1}^{L} \sum_{\substack{a\in\pi_k\\ a\in S}} E\left\{\left[W_i(a, \mathcal{R}) - g_i\left(\frac{j-1}{L}\right)\right]\left[(f_i^j \circ \mu_i)(P_a^{\mathcal{R}} \cup a) - (f_i^j \circ \mu_i)(P_a^{\mathcal{R}})\right]\right\}\right| + (g_i(1) + \mu_i(S))\cdot\epsilon. \tag{5.7}$$

Let $\eta > 0$ and $\Omega_k^\eta = \{\mathcal{R} \mid |(\mu_j - \mu_i)(P_a^{\mathcal{R}} \cup a)| < \eta$ for each $1 \leqslant i, j \leqslant n$ and for each $a \in \pi_k\}$. Denote

$$x_k^{i,j}(\mathcal{R}) = \sum_{\substack{a\in\pi_k\\ a\in S}} \left[W_i(a, \mathcal{R}) - g_i\left(\frac{j-1}{L}\right)\right]\cdot\left[(f_i^j \circ \mu_i)(P_a^{\mathcal{R}} \cup a) - (f_i^j \circ \mu_i)(P_a^{\mathcal{R}})\right].$$

For large k's and for each $\mathcal{R} \in \Omega_k^\eta$

$$|W_i(a, \mathcal{R}) - g_i(P_a^{\mathcal{R}} \cup a)| < \epsilon, \qquad \left|g_i(P_a^{\mathcal{R}} \cup a) - g_i\left(\frac{j-1}{L}\right)\right| < \epsilon.$$

Hence

$$|x_k^{i,j}(\mathcal{R})| \leqslant \sum_{\substack{a\in\pi_k\\ a\in S}} \left[|W_i(a, \mathcal{R}) - g_i(P_a^{\mathcal{R}} \cup a)| + \left|g_i(P_a^{\mathcal{R}} \cup a) - g_i\left(\frac{j-1}{L}\right)\right|\right] \cdot\left[(f_i^j \circ \mu_i)(P_a^{\mathcal{R}} \cup a) - (f_i^j \circ \mu_i)(P_a^{\mathcal{R}})\right] < 2\epsilon\cdot f_i^j(1).$$

Therefore

$$E\left(x_k^{i,j}\Big|_{\Omega_k^\eta}\right) \leqslant 2\epsilon\cdot f_i^j(1).$$

Proposition 5.3 implies that

$$\lim_{k\to\infty}\left[E(x_k^{i,j}) - E\left(x_k^{i,j}\Big|_{\Omega_k^\eta}\right)\right] = 0.$$

Hence for k, large enough

$$\left|\sum_{j=1}^{L} E(x_k^{i,j})\right| \leqslant \sum_{j=1}^{L} |E(x_k^{i,j})| \leqslant \sum_{j=1}^{L} \left[|E(x_k^{i,j}|_{\Omega_k^i})| + \frac{\epsilon}{L} \right]$$

$$\leqslant 2\epsilon \cdot \sum_{j=1}^{L} \left[f_i\left(\frac{j}{L}\right) - f_i\left(\frac{j-1}{L}\right) \right] + \epsilon \leqslant [2f_i(1) + 1] \cdot \epsilon. \qquad (5.8)$$

(5.7) and (5.8) imply that for large k's

$$T_k \leqslant [g_i(1) + \mu_i(S)] \cdot \epsilon + [2f_i(1) + 1] \cdot \epsilon.$$

Hence

$$\lim_{k \to \infty} T_k = 0. \quad \text{Q.E.D.}$$

5. b. *Partition value on $A * bv'$NA.* For the next theorem let us use the following notations. Let φ be the partition value on A which was introduced in the previous theorem. For $f_i \in cbv$ and $\mu_i \in \mathrm{NA}^1$, $1 \leqslant i \leqslant n$, we have that

$$\varphi\left(\prod_{i=1}^{n} (f_i \circ \mu_i)\right) = \sum_{j=1}^{n} \mu_j \cdot \int_0^1 \left(\prod_{i \neq j} f_i\right) df_j.$$

Let φ_t be the operator from B to FA obeying

$$\varphi_t\left(\prod_{i=1}^{n} (f_i \circ \mu_i)\right) = \sum_{j=1}^{n} \mu_j \cdot \int_t^1 \prod_{i \neq j} f_i \, df_j.$$

The fact that φ_t is well defined is an easy consequence of theorem 5.4. But since φ_t is used here only to simplify notations, the proof is omitted.

Let $\varphi_{bv'\mathrm{NA}}$ be the partition value on bv'NA.

For the rest of the paper we shall denote by g_t or by f_t where $0 < t < 1$ a single jump function which has a jump at t, i.e., g_t is of the form

$$g_t(x) = \begin{cases} 0, & 0 \leqslant x < t, \\ \text{Const.}, & t \leqslant x \leqslant 1, \end{cases} \quad \text{or} \quad g_t(x) = \begin{cases} 0, & 0 \leqslant x \leqslant t, \\ \text{Const.}, & t < x \leqslant 1. \end{cases}$$

We are now ready to state the following theorem.

THEOREM 5.9. *There exists a partition value φ on $A * bv'$NA which satisfies for $f_i \in cbv$, μ_i, $\nu \in \mathrm{NA}^1$, $1 \leqslant i \leqslant n$:*

$$\varphi\left[(g_t \circ \nu) \cdot \prod_{i=1}^{n} (f_i \circ \mu_i)\right](S) = g_t(1) \cdot \sum_{j=1}^{n} \mu_j(s) \int_t^1 \left(\prod_{i \neq j} f_i\right) df_j + g_t(1) \cdot \nu(S) \cdot \prod_{i=1}^{n} f_i(t)$$

or with the above notations

$$\varphi\left[(g_t \circ \nu) \cdot \prod_{k=1}^{n} (f_i \circ \mu_i)\right] = g_t(1) \cdot \varphi_t\left(\prod_{i=1}^{n} f_i \circ \mu_i\right) + \varphi_{bv'\mathrm{NA}}(g_t \circ \nu) \cdot \prod_{i=1}^{n} f_i(t).$$

PROOF. Every function in bv' can be approximated by functions of the form $f^c + \sum_{i=1}^{n} f_{t_i}$ where f^c is in cbv. Since $\|u \cdot v\| \leqslant \|u\| \cdot \|v\|$ for each u and v in BV, it is easy to verify that each game in $A * bv'$NA can be approximated by games of the form $\sum_{i=1}^{n} v_i(g_{t_i} \circ \mu_i) + w$, where v_i and w are games in Q^B (B is defined as in the beginning of the proof of theorem 5.4) and μ_i ($1 \leqslant i \leqslant n$), are in NA^1.

Let M be the set of all games of the form $v \cdot (g_t \circ \mu)$ where v is in B, and μ is in NA^1. $A * bv'\mathrm{NA}$ is the closure of the linear subspace generated by $M \cup B$.

The rest of the proof is based upon

PROPOSITION 5.10. *For each partition* π, $S \in \mathfrak{B}$, $v \in M$ *if* $v = (g_t \circ \nu) \cdot \prod_{i=1}^n (f_i \circ \mu_i)$ *then for each* $\pi_k \in \pi(k, \eta, S)$ *where* $\eta = (\mu_1, \ldots, \mu_n, \nu)$

$$\lim_{k\to\infty} (\psi v_{\pi_k})(\bar{S}) = \left(\prod_{i=1}^n f_i\right)(t) \cdot \varphi_{bv}(g_t \circ \nu) + g_t(1) \cdot \varphi_t\left(\prod_{i=1}^n (f_i \circ \mu_i)\right).$$

Assuming proposition 5.10 (which we will prove later), the proof of theorem 5.9 is completed as follows:

Let $\gamma : M \cup B \to \mathrm{FA}$ be the operator which coincides on B with the partition value on A, and is defined on M by

$$\gamma\left(g_t \circ \nu \cdot \prod_{i=1}^n (f_i \circ \mu_i)\right) = \left(\prod_{i=1}^n f_i\right)(t) \cdot \varphi_{bv}(g_t \circ \nu) + g_t(1) \cdot \varphi_t\left(\prod_{i=1}^n (f_i \circ \mu_i)\right).$$

An analogous argument to the ones used in theorems 5.1 and 5.4 leads to the conclusion that γ is a well-defined operator.

Let π be a partition, $S \in \mathfrak{B}$, n a positive integer and $v^1, \ldots, v^n$ are in $M \cup B$. Let μ be a vector measure consisting of all measures involved in $v^1, \ldots, v^n$. Then for each $\pi_k \in \pi(k, \mu, S)$

$$\lim_{k\to\infty} (\psi v^t_{\pi_k})(\bar{S}) = (\gamma v^t)(S) \quad \text{for each } 1 \leqslant t \leqslant n.$$

(The above equation follows from the proof of theorem 5.5 and from proposition 5.10.) Hence, by lemma 4.2, the proof of theorem 5.9 is completed.

PROOF OF PROPOSITION 5.10. Let π be any measurable partition, $S \in \mathfrak{B}$ and v in M given by $v = (g_t \circ \nu)\prod_{i=1}^n f_i \circ \mu_i$. W.l.o.g. let us assume that for each $1 \leqslant i \leqslant n$ f_i is a monotonic function. Let $\pi_k \in \pi(k, \mu, S)$ where $\mu = (\mu_1, \mu_2, \ldots, \mu_n, \nu)$. Let $\delta > 0$ be given with $2\delta < \min(t, 1-t)$. Define g_t^δ as follows

$$g_t^\delta(x) = \begin{cases} 0, & 0 \leqslant x \leqslant t - \dfrac{\delta}{2}, \\ \dfrac{2g_t(1)}{\delta}\left(x - t - \dfrac{\delta}{2}\right), & t - \dfrac{\delta}{2} \leqslant x \leqslant t, \\ g_t(1), & t \leqslant x \leqslant 1, \end{cases}$$

and denote $u_t = g_t \circ \nu$, $u_t^\delta = g_t^\delta \circ \nu$, $w = u_t - u_t^\delta$ and $u = \prod_{i=1}^n f_i \circ \mu_i$. Note that $w \in bv'\mathrm{NA}$, $u_t^\delta \cdot u \in A$ and $v = u_t^\delta \cdot u + w \cdot u$. Proposition 5.5 asserts that

$$\begin{aligned} \lim_{k\to\infty} \psi(u_t^\delta \cdot u)_{\pi_k}(\bar{S}) &= \sum_{j=1}^n \mu_j(S) \int_0^1 \left[\left(\prod_{i\neq j} f_i\right) g_t^\delta\right] df_j \\ &\quad + \nu(S) \int_0^1 \left(\prod_{i=1}^n f_i\right) dg_t^\delta \underset{\delta\to 0}{\to} \sum \mu_j(S) \int_t^1 \left(\prod_{i\neq j} f_j\right) df_j \\ &\quad + \nu(S) \prod_{i=1}^n f_i(t). \end{aligned} \tag{5.11}$$

Let

$$\Omega_k^\delta = \left\{ \mathfrak{R} : \forall a \in \pi_k \text{ and } \forall 1 \leqslant i \leqslant n \,|(\nu - \mu_i)(P_a^{\mathfrak{R}})| < \delta \right\}.$$

According to Aumann-Shapley's corollary 18.10 [1], we have

$$\lim_{k\to\infty} \operatorname{Prob}(\Omega_k^\delta) = 1. \tag{5.12}$$

Define the random variables x_k, y_k, z_k on Ω_k by

$$x_k = \sum \left[u(P_a^{\mathfrak{R}} \cup a) - u(P_a^{\mathfrak{R}})\right] \cdot w(P_a^{\mathfrak{R}} \cup a), \tag{5.13}$$

$$y_k = \sum \left[w(P_a^{\mathfrak{R}} \cup a) - w(P_a^{\mathfrak{R}})\right] \cdot \left[u(P_a^{\mathfrak{R}}) - \prod_{i=1}^{n} f_i(t)\right], \tag{5.14}$$

$$z_k = \sum \left[w(P_a^{\mathfrak{R}} \cup a) - w(P_a^{\mathfrak{R}})\right]\left(\prod_{i=1}^{n} f_i(t)\right)$$

where each of the summations is taken over all $a \in \pi_k$ with $a \subset S$. Note that $\psi(u \cdot w)_{\pi_k}(\bar{S}) = E(x_k + y_k + z_k)$. As $w \in bv'\mathrm{NA}$ we have by proposition 5.2 that

$$\lim_{k\to\infty} E(z_k) = 0. \tag{5.15}$$

Since $w(T)$ vanishes for each $T \in \mathfrak{B}$ satisfying $\nu(T) \notin [t - \delta/2, t]$, we can assume that for large k's on Ω_k^δ, if $w(P_a^{\mathfrak{R}} \cup a) - w(P_a^{\mathfrak{R}}) \neq 0$ then, the summations appearing in (5.13) and in (5.14) are taken over all $a \in \pi_k$ and $a \subset S$ satisfying

$$t - 2\delta < \mu_i(P_a^{\mathfrak{R}}) < t + \delta$$

for each $1 \leqslant i \leqslant n$. Hence, the monotonicity of u and the inequality $\|w\| \leqslant 2g_t(1)$ imply that for sufficently large k

$$|y_k|_{\Omega_k^\delta}| \leqslant 2g_t(1)\left[\prod_{i=1}^{n} f_i(t+\delta) - \prod_{i=1}^{n} f_i(t-2\delta)\right]$$

and

$$|x_k|_{\Omega_k^\delta}| \leqslant \left[\prod_{i=1}^{n} f_i(t+\delta) - \prod_{i=1}^{n} f_i(t-2\delta)\right] g_t(1).$$

Combining the last two inequalities with (5.12) we obtain by proposition 5.3 that

$$\lim_{k\to\infty} \left[E((x_k + y_k)|_{\Omega_k^\delta}) - E(x_k + y_k)\right] = 0$$

and

$$|E((x_k + y_k)|_{\Omega_k^\delta})| \leqslant 3g_t(1)\left[\prod_{i=1}^{n} f_i(t+\delta) - \prod_{i=1}^{n} f_i(t-2\delta)\right] \underset{\delta\to 0}{\to} 0.$$

These imply that $\lim_{k\to\infty} E(x_k + y_k) = 0$, and together with (5.15) we get

$$\lim_{k\to\infty} E(x_k + y_k + z_k) = 0$$

which completes the proof of the proposition.

5. c. *The partition value on* $bv'\mathrm{NA} * bv'\mathrm{NA}$, $A * bv'\mathrm{NA} * bv'\mathrm{NA}$ *and* W. In this section we will prove the existence of a partition value on each of the spaces $bv'\mathrm{NA} * bv'\mathrm{NA}$ (theorem 5.16 below), $A * bv'\mathrm{NA} * bv'\mathrm{NA}$ (confer theorem 5.43) and on W—the space generated by ASYMP, A, $A * bv'\mathrm{NA}$ and $A * bv'\mathrm{NA} * bv'\mathrm{NA}$ (confer theorem 5.52). We will also prove that $bv'\mathrm{NA} * bv'\mathrm{NA}$ is not contained in ASYMP (proposition 5.42 below).

THEOREM 5.16. *A partition value φ exists on the space $bv'\mathrm{NA} * bv'\mathrm{NA}$ satisfying*

$$\varphi[(g_t \circ \mu)(g_s \circ \nu)]\begin{cases} g_t(1)\cdot g_s(1)\,\dfrac{\mu+\nu}{2}, & s=t,\\ g_s(1)\cdot\nu, & s>t,\\ g_t(1)\cdot\mu, & t>s.\end{cases}$$

(*The notations are described in the beginning of* §5. b.)

PROOF. Every function in bv' can be approximated by functions of the form $f^c + \sum_{i=1}^n f_{t_i}$ where f^c is in cbv and f_{t_i}, $1 \leqslant i \leqslant n$, are single jump functions. Since $\|u\cdot v\| \leqslant \|u\|\cdot\|v\|$ for each u and v in BV, each game in $bv'\mathrm{NA} * bv'\mathrm{NA}$ can be approximated by games of the form

$$\sum_{i=1}^{n} (g_{t_i} \circ \mu_i)(g_{s_i} \circ \nu_i) + v$$

where μ_i, ν_i, $1 \leqslant i \leqslant n$, are in NA^1 and $v \in (bv'\mathrm{NA} * A) \cap (bv'\mathrm{NA} * bv'\mathrm{NA})$.

Let M_1 be the set of all games of the form $(g_t \circ \mu)(f_s \cdot \nu)$, where μ and ν are in NA^1. $bv'\mathrm{NA} * bv'\mathrm{NA}$ is contained in the linear closed subspace generated by $M_1 \cup M$. The rest of the proof is based upon

PROPOSITION 5.17. *For each partition π, $S \in \mathfrak{B}$, $v \in M_1$. If $v = (g_t \circ \mu)\cdot(g_s \circ \nu)$, then for each $\pi_k \in \pi(k, (\mu, \nu), S)$*

$$\lim_{k\to\infty} (\psi v_{\pi_k})(\bar{S}) = \begin{cases} g_t(1)\cdot g_s(1)\cdot\dfrac{\mu+\nu}{2}, & s=t,\\ g_t(1)\cdot\mu, & t>s:\\ g_s(1)\cdot\nu, & s>t.\end{cases}$$

Assuming proposition 5.17 (which we will be proved later), the proof of theorem 5.16 is completed as follows.

Let $\gamma : M \cup M_1$ be the operator which coincides on M with the partition value on $A * bv'\mathrm{NA}$ and which defined on M_1 by

$$\gamma((g_t \circ \mu)\cdot(g_s \circ \nu)) = \begin{cases} g_t(1)\cdot g_s(1)\cdot\dfrac{\mu+\nu}{2}, & s=t,\\ g_t(1)\cdot\mu, & t>s,\\ g_s(1)\cdot\nu, & t<s.\end{cases}$$

A similar argument to the one used in the proofs of theorems 5.1 and 5.4 yields that γ is a well-defined operator.

Let π be a partition, $S \in \mathfrak{B}$, n a positive integer and $v^1, v^2, \ldots, v^n$ in $M \cup M_1$. Let μ be a vector measure consisting of all the measures involved in $v^1, \ldots, v^n$: then according to proposition 5.17 and 5.10, for each $\pi_k \in \pi(k, \mu, S)$

$$\lim_{k\to\infty} (\psi v^t_{\pi_k})(\bar{S}) = (\gamma v^t)(S) \quad \text{for each } 1 \leqslant t \leqslant n.$$

Hence, the proof of theorem 5.16 now follows from lemma 4.2.

PROOF OF PROPOSITION 5.17. We will first prove the proposition for $v = (g_t \circ \mu)\cdot(g_s \circ \nu)$ $\mu, \nu \in \mathrm{NA}^1$ in the case where $s \neq t$. Assume, for instance, that $t > s$. Denote $\delta = (t-s)/2$ and $\eta = (\mu, \nu)$. Let π be a partition, $S \in \mathfrak{B}$ and $\pi_k \in \pi(k, \eta, S)$. Denote $U^\delta = \{(x, y) \mid |x-y| < \delta,\ 0 \leqslant x, y \leqslant 1\}$. U^δ is a neighborhood of the diagonal $[0, \eta(I)]$ of the vector measure η. Moreover for each $S \in \mathfrak{B}$ satisfying $\eta(S) \in U^\delta$

$$v(S) = [(g_t \circ \mu)\cdot(g_s \circ \nu)](S) = (g_t \circ \mu)(S).$$

Hence, $v = (g_t \circ \mu) + w$ where $w \in \mathrm{DIAG}$. The same proof, given to corollary 18.10 in [1] and the fact that $w \in \mathrm{DIAG}$, yields that the asymptotic value of w is 0 for each $w \in \mathrm{DIAG}$, hence

$$\lim_{k\to\infty} (\psi w_{\pi_k})(\bar{S}) = 0. \tag{5.18}$$

Proposition 5.2 implies that

$$\lim_{k\to\infty} \psi(g_t \circ \mu)_{\pi_k}(\bar{S}) = g_t(1)\cdot\mu(S) \tag{5.19}$$

and (5.18), (5.19) imply that

$$\lim_{k\to\infty} (\psi v_{\pi_k})(\bar{S}) = g_t(1)\cdot\mu(S),$$

which completes the proof of the proposition for the case $s \neq t$. The proof for the case where $s = t$ is much more complicated.

Let $v = (g_t \circ \mu)\cdot(g_t \circ \nu)$ (where $\mu, \nu \in \mathrm{NA}^1$), $S \in \mathfrak{B}$, π a partition, and $\pi_k \in \pi(k, \eta, S)$ where $\eta = (\mu, \nu)$. We have to show that

$$\lim_{k\to\infty} (\psi v_{\pi_k})(\bar{S}) = \left[g_t(1)\cdot \frac{\mu+\nu}{2} \right](S).$$

Denote by $a = a(\mathfrak{R})$ the member of π_k for which

$$v(P_a^{\mathfrak{R}} \cup a) - v(P_a^{\mathfrak{R}}) = 1.$$

Define the random variable χ^k on Ω_k (the probability space consisting of all orders $\mathfrak{R}$ on π_k, each having the same probability $1/|\pi_k|!$) by

$$\chi^k(\mathfrak{R}) = \begin{cases} 1, & a(\mathfrak{R}) \in S: \\ 0, & a(\mathfrak{R}) \notin S. \end{cases}$$

Similarly, $a_1 = a_1(\mathfrak{R})$ is the member of π_k for which $(g_t \circ \mu)(P_{a_1}^{\mathfrak{R}} \cup a_1) - (g_t \circ \mu)\cdot(P_{a_1}^{\mathfrak{R}}) = 1$ and $a_2 = a_2(\mathfrak{R})$ is the member of π_k for which $(g_t \circ \nu)(P_{a_2}^{\mathfrak{R}} \cup a_2) - (g_t \circ \nu)\cdot(P_{a_2}^{\mathfrak{R}}) = 1$. Define

$$L^k(\mathfrak{R}) = \begin{cases} 1, & a_1(\mathfrak{R}) \in S, \\ 0, & a_1(\mathfrak{R}) \notin S \end{cases}$$

and

$$H^k(\mathfrak{R}) = \begin{cases} 1, & a_2(\mathfrak{R}) \in S, \\ 0, & a_2(\mathfrak{R}) \notin S. \end{cases}$$

From the definitions of χ^k, L^k and H^k, together with proposition 5.2, it is obvious that

$$E(\chi^k) = (\psi v_{\pi_k})(\bar{S}),$$

$$E(L^k) = \psi(g_t \circ \mu)_{\pi_k}(\bar{S}) \underset{k\to\infty}{\to} g_t(1)\cdot\mu(S),$$

$$E(H^k) = \psi(g_t \circ \nu)_{\pi_k}(\bar{S}) \underset{k\to\infty}{\to} g_t(1)\cdot\nu(S).$$

To complete the proof of the proposition we thus have to prove that

$$\lim_{k\to\infty} E(\chi^k) = \lim_{k\to\infty} \frac{E(L^k) + E(H^k)}{2}. \tag{5.20}$$

For this purpose assume that $S_1, S_2, \ldots, S_M$ $(M = |\pi|)$ are the members of π. Denote $\mu(S_i) = a_i$ and $\nu(S_i) = b_i$, for each order $\mathfrak{R} \in \Omega_k$, $B_m^k(\mathfrak{R})$ will be the mth member of π_k in the order $\mathfrak{R}$. $B_m^k = B_m^k(\mathfrak{R})$ is defined on Ω_k. For each k and $1 \leqslant m \leqslant M \cdot k$ and for each order $\mathfrak{R}$ denote

$$Q_m^k(\mathfrak{R}) = B_1^k(\mathfrak{R}) \cup \cdots \cup B_m^k(\mathfrak{R}).$$

Define

$$x_i^k = (\mu - \nu)(B_i^k), \qquad 1 \leqslant i \leqslant M \cdot k,$$

$$S_n^k = \sum_{i=1}^{n} x_i^k \qquad 1 \leqslant n \leqslant M \cdot k,$$

$$\bar{x}_n^k = \frac{1}{n} \cdot S_n^k \qquad 1 \leqslant n \leqslant M \cdot k.$$

For each $1 \leqslant i, n \leqslant M \cdot k$

$$E(x_i^k) = E(S_n^k) = E(\bar{x}_n^k) = 0.$$

Denote

$$V_1 = \frac{1}{M} \sum_{i=1}^{M} (a_i - b_i)^2,$$

$$V_2 = \frac{1}{M} \sum_{i=1}^{M} \left(a_i - \frac{1}{M}\right)^2,$$

$$V_3 = \frac{1}{M} \sum_{i=1}^{M} \left(b_i - \frac{1}{M}\right)^2.$$

Then

$$\sigma_1^2 = \mathrm{Var}(x_i^k) = \frac{1}{M \cdot k} \sum_{i=1}^{M} k\left(\frac{a_i - b_i}{k}\right)^2 = \frac{1}{k^2} \cdot \frac{1}{M} \sum_{i=1}^{M} (a_i - b_i)^2 = \frac{V_1}{k^2} \tag{5.21}$$

where Var means "variance." From Rosén's theorem 1.1 [10] we get

$$\begin{aligned} \mathrm{Var}(S_m^k) &= \left(\frac{m}{M \cdot k - 1}\right)\left(1 - \frac{m}{M \cdot k}\right) \sum_{i=1}^{M} k \cdot \left(\frac{a_i - b_i}{k}\right)^2 \\ &= \left(\frac{m}{M \cdot k - 1}\right)\left(1 - \frac{m}{M \cdot k}\right) \cdot \frac{M}{k} \cdot V_1 \leqslant \frac{M}{k} V_1 \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} &(1) \quad \mathrm{Var}(\mu(Q_m^k)) \leqslant \frac{M}{k} V_2, \\ &(2) \quad \mathrm{Var}(\nu(Q_m^k)) \leqslant \frac{M}{k} V_3. \end{aligned} \tag{5.23}$$

According to Rosén's lemma 4.2 [10], we have

$$\mathrm{Prob}\left(\max_{M \cdot k \geqslant r \geqslant n} |\bar{x}_r^k| \geqslant \epsilon\right) \leqslant \frac{2\sigma_1^2}{\epsilon^2}\left(\frac{1}{n} - \frac{1}{M \cdot k}\right). \tag{5.24}$$

From (5.24) we can obtain the following:

LEMMA 5.25.

$$\operatorname{Prob}(\exists m \leqslant n \quad s.t. \ |S_m^k| \geqslant \epsilon) \leqslant \frac{2\sigma_1^2}{\left(\frac{\epsilon}{Mk}\right)^2}\left(\frac{1}{M\cdot k-\eta}-\frac{1}{M\cdot k}\right).$$

PROOF. Since $\sum_{i=1}^{Mk} x_i^k = 0$, then

$$\begin{aligned}&\operatorname{Prob}(\exists m \leqslant n, \text{ s.t. } |S_m^k| \geqslant \epsilon)\\ &\quad = \operatorname{Prob}(\exists r \geqslant Mk-n, \text{ s.t. } |S_r^k| \geqslant \epsilon)\\ &\quad \leqslant \operatorname{Prob}\left(\max_{Mk \geqslant r \geqslant Mk-n} |\bar{x}_r^k| \geqslant \frac{\epsilon}{M\cdot k}\right).\end{aligned}$$

And from (5.24) we obtain that

$$\operatorname{Prob}(\exists m \leqslant n, \text{ s.t. } |S_m^k| \geqslant \epsilon) \leqslant \frac{2\sigma_1^2}{\left(\frac{\epsilon}{M\cdot k}\right)^2}\left(\frac{1}{Mk-n}-\frac{1}{M\cdot k}\right),$$

which completes the proof of the lemma.

Let ϵ_1 be a small positive number and let

$$N = [(t-\epsilon_1)\cdot Mk + 1] \quad \text{and} \quad N' = [(t+\epsilon_1)\cdot Mk],$$

where $[x]$ is the integral part of x.

$$\begin{aligned}P_1 &= \operatorname{Prob}\left(\exists N'', N \leqslant N'' \leqslant N', |S_{N''}^k - S_N^k| \geqslant \epsilon_1^{1/3}\cdot\sqrt{\frac{MV_1}{k}}\right)\\ &= \operatorname{Prob}\left(\exists N'', N'' \leqslant N'-N, |S_{N''}^k| \geqslant \epsilon_1^{1/3}\cdot\sqrt{\frac{MV_1}{k}}\right).\end{aligned}$$

Lemma 5.25 and (5.21) imply that

$$P_1 \leqslant 2\cdot\frac{\frac{V_1}{K^2}\cdot(Mk)^2}{\frac{\epsilon_1^{2/3}\cdot MV_1}{k}}\cdot\left(\frac{1}{Mk-(N'-N)}-\frac{1}{Mk}\right) \leqslant \frac{2(N'-N)}{\epsilon_1^{2/3}(Mk-(N'-N))}.$$

Hence

$$P_1 \leqslant \frac{4\cdot\epsilon_1 Mk}{\epsilon_1^{2/3}(1-2\epsilon_1)Mk} \leqslant \frac{4\epsilon_1^{1/3}}{1-2\epsilon_1} \underset{\epsilon_1\to 0}{\longrightarrow} 0.$$

Denote by $o(1)$ any function of ϵ_1 which tends to zero as $\epsilon_1 \to 0$. Then $P_1 \leqslant o(1)$ and thus we have proved that

$$\operatorname{Prob}\left(\forall N'', N \leqslant N'' \leqslant N', |S_{N''}^k - S_N^k| < \epsilon_1^{1/3}\cdot\sqrt{\frac{MV_1}{k}}\right) > 1 - o(1). \quad (5.26)$$

Since

$$E(\mu(Q_{N'}^k)) = E(\nu(Q_{N'}^k)) \geqslant \frac{[(t+\epsilon_1)Mk]}{Mk} \geqslant t + \frac{3}{4}\epsilon_1$$

then

$$\mu(Q_{N'}^k) - E(\mu(Q_{N'}^k)) < \mu(Q_{N'}^k) - \left(t + \frac{3}{4}\epsilon_1\right)$$

and therefore

$$\operatorname{Prob}\left(\mu(Q_{N'}^k) < t + \frac{\epsilon_1}{2}\right) \leqslant \operatorname{Prob}\left(\mu(Q_{N'}^k) - E(\mu(Q_{N'}^k)) > -\frac{\epsilon_1}{4}\right)$$

$$\leqslant \operatorname{Prob}\left(|\mu(Q_{N'}^k) - E(\mu(Q_{N'}^k))| > \frac{\epsilon_1}{4}\right) \leqslant \frac{mV_2}{k} \Big/ \left(\frac{\epsilon_1}{4}\right)^2.$$

Therefore for a given $\epsilon_1 > 0$

$$\lim_{k\to\infty} \operatorname{Prob}\left(\mu(Q_{N'}^k) < t + \frac{\epsilon_1}{2}\right) = 0. \tag{5.27}$$

In the same manner we can get that

$$\lim_{k\to\infty} \operatorname{Prob}\left(\nu(Q_{N'}^k) < t + \frac{\epsilon_1}{2}\right) = 0, \tag{5.28}$$

$$\lim_{k\to\infty} \operatorname{Prob}\left(\mu(Q_N^k) \geqslant t - \frac{\epsilon_1}{2}\right) = 0, \tag{5.29}$$

$$\lim_{k\to\infty} \operatorname{Prob}\left(\nu(Q_N^k) \geqslant t - \frac{\epsilon_1}{2}\right) = 0. \tag{5.30}$$

The sequence $(S_N^k)_{k=1}^\infty$ satisfies the conditions of Erdös and Rényi's Central Limit Theorem for samples-without-replacement from a finite population [2]. Therefore for each $\epsilon > 0$

$$\lim_{k\to\infty} \operatorname{Prob}\left(\frac{S_N^k}{\sigma_N^k} \geqslant \epsilon\right) = 1 - \Phi(\epsilon)$$

where

$$\sigma_N^k = \sqrt{\operatorname{Var} S_N^k} \quad \text{and} \quad \Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-t^2/2}dt$$

for each $x \in E^1$.

Let $\epsilon_1 > 0$ be an arbitrarily small number

$$\operatorname{Var}(S_N^k) = \left(\frac{M}{Mk-1}\right)\left(1 - \frac{N}{Mk}\right)\frac{MV_1}{k}$$

$$\leqslant \frac{(t - \frac{1}{2}\epsilon_1)Mk}{Mk-1} \cdot \left(1 - \frac{(t - \frac{3}{2}\epsilon_1)Mk}{Mk}\right)\frac{MV_1}{k}$$

$$\leqslant \frac{MV_1}{k},$$

$$\operatorname{Var}(S_N^k) \geqslant \frac{(t-\epsilon_1)Mk}{Mk-1}\left(1 - \frac{(t - \frac{1}{2}\epsilon_1)Mk}{Mk}\right) \cdot \frac{MV_1}{k} \geqslant \frac{1}{4}\,\frac{MV_1}{k}.$$

Hence for large k's

$$\operatorname{Prob}\left(S_N^k \geqslant \epsilon_1^{1/3}\frac{MV_1}{k}\right) \leqslant \operatorname{Prob}\left(\frac{S_N^k}{\sigma_N^k} \geqslant \epsilon_1^{1/3}\right) \underset{k\to\infty}{\longrightarrow} 1 - \Phi(\epsilon_1^{1/3})$$

and

$$\operatorname{Prob}\left(S_N^k \geqslant \epsilon_1^{1/3}\frac{MV_1}{k}\right) \geqslant \operatorname{Prob}\left(\frac{S_N^k}{\sigma_N^k} \geqslant 2\epsilon_1^{1/3}\right) \underset{k\to\infty}{\to} 1-\Phi(2\epsilon_1^{1/3}).$$

Therefore for large k's

$$1-\Phi(4\epsilon_1^{1/3}) \leqslant \operatorname{Prob}\left(S_N^k \geqslant \epsilon_1^{1/3}\frac{MV_1}{k}\right) \leqslant 1-\Phi(\tfrac{1}{2}\epsilon_1^{1/3}). \tag{5.31}$$

In the same way we can obtain that for large k's

$$1-\Phi(4\epsilon_1^{1/3}) \leqslant \operatorname{Prob}\left(-S_N^k \geqslant \epsilon_1^{1/3}\frac{MV_1}{k}\right) \leqslant 1-\Phi(\tfrac{1}{2}\epsilon_1^{1/3}). \tag{5.32}$$

PROPOSITION 5.33.

$$\lim_{k\to\infty} \left|E\left(\chi^k - H^k \,|\, S_N^k \geqslant \epsilon_1^{1/3}\frac{MV_1}{k}\right)\right| \leqslant o(1)$$

and

$$\lim_{k\to\infty} \left|E\left(\chi^k - L^k| - S_N^k \geqslant \epsilon_1^{1/3}\frac{MV_1}{k}\right)\right| \leqslant o(1).$$

PROOF. Denote

$$P = \operatorname{Prob}\left(\chi^k - H^k \neq 0 \mid S_N^k \geqslant \epsilon_1^{1/3}\sqrt{\frac{MV_1}{k}}\right),$$

$$P_1 = \operatorname{Prob}\left(\chi^k = 0, H^k = 1 \mid S_N^k \geqslant \epsilon_1^{1/3}\sqrt{\frac{MV_1}{k}}\right),$$

$$P_2 = \operatorname{Prob}\left(\chi^k = 1, H^k = 0 \mid S_N^k \geqslant \epsilon_1^{1/3}\sqrt{\frac{MV_1}{k}}\right),$$

$$L_1 = \{\mathfrak{R} \mid H^k = 1, \chi^k = 0\},$$

$$L_2 = \left\{\mathfrak{R} \mid S_N^k \geqslant \epsilon_1^{1/3}\sqrt{\frac{MV_1}{k}}\right\},$$

$$L_3 = \left\{\mathfrak{R} \mid \forall N'', N \leqslant N'' \leqslant N', |S_{N''}^k - S_N^k| < \epsilon_1 \cdot \sqrt{\frac{MV_2}{k}}\right\},$$

$$L_4 = \left\{\mathfrak{R} \mid \nu(Q_{N'}^k) \geqslant t + \frac{\epsilon_1}{2}\right\},$$

$$P(L_1 \mid L_2) \leqslant \frac{P(L_1 \cap L_2)}{P(L_2 \cap L_3 \cap L_4)}.$$

Since for each two events A and B, $P(A \cap B) \geqslant 1 - P(A^c) - P(B^c)$,

$$P\left(\bigcap_{i=1}^{4} L_i\right) \geqslant P(L_1 \cap L_2) - 1 + P(L_3 \cap L_4) \geqslant P(L_1 \cap L_2) - P(L_3^c) - P(L_4^c).$$

From (5.26) and (5.28) we get that for large k's

$$P\left(\bigcap_{i=1}^{4} L_i\right) \geqslant P(L_1 \cap L_2) - o(1).$$

Hence

$$\begin{aligned} P(L_1 \mid L_2) &\leqslant \frac{P\left(\bigcap_{i=1}^{4} L_i\right) + o(1)}{P(L_2 \cap L_3 \cap L_4)} \\ &\leqslant P(L_1 \mid L_2 \cap L_3 \cap L_4) \\ &\quad + \frac{o(1)}{P(L_2 \cap L_3 \cap L_4)}. \end{aligned} \tag{5.34}$$

But

$$P(L_2 \cap L_3 \cap L_4) \geqslant 1 - P(L_2^c) - P(L_3^c) - P(L_4^c).$$

Therefore (5.26), (5.28) and (5.31) imply that for each $\epsilon_1 > 0$ small enough and for large k's

$$P(L_2 \cap L_3 \cap L_4) \geqslant 1 - \Phi(4\epsilon_1^{1/3}) + o(1) \geqslant \tfrac{1}{4}.$$

Combining it with (5.34) we have

$$P(L_1 \mid L_2) \leqslant P(L_1 \mid L_2 \cap L_3 \cap L_4) + o(1).$$

But $P(L_1 \mid L_2 \cap L_3 \cap L_4) = 0$. Thus for k large enough

$$P_1 = P(L_1 \mid L_2) \leqslant o(1).$$

In the same way we get

$$P_2 \leqslant o(1).$$

Hence for large k's, $P \leqslant o(1)$. Now since $|\chi^k - H^k| \leqslant 1$, we obtain that for large k's

$$E(\chi^k - H^k \mid L_2) \leqslant o(1).$$

With the same arguments we can prove that for large k's

$$E(\chi^k - L^k \mid L_2) \leqslant o(1)$$

and the proof of proposition 5.33 is completed.

Denote

$$A^k = \left\{ \mathfrak{R} \mid S_N^k \geqslant \epsilon_1^{1/3} \cdot \sqrt{\frac{MV_1}{k}} \right\},$$

$$B^k = \left\{ \mathfrak{R} \mid -S_N^k \geqslant \epsilon_1^{1/3} \cdot \sqrt{\frac{MV_1}{k}} \right\},$$

$$C^k = \left\{ \mathfrak{R} \mid |S_N^k| \leqslant \epsilon_1^{1/3} \cdot \sqrt{\frac{MV_1}{k}} \right\},$$

$$E(\chi^k) = E(\chi^k \mid A^k)P(A^k) + E(\chi^k \mid B^k)P(B^k) + E(\chi^k \mid C^k)P(C^k).$$

Proposition (5.33) asserts that

$$\lim_{k\to\infty} |E(\chi^k \mid A^k) - E(H^k \mid A^k)| \leqslant o(1). \tag{5.35}$$

We will prove that $|\lim_{k\to\infty} E(\chi^k \mid A^k) - \nu(S)\cdot g_t(1)| \leqslant o(1)$ by proving that

$$\lim_{k\to\infty} E(H^k \mid A^k) = \nu(S)\cdot g_t(1).$$

In order to do that, define

$$D^k = \{\mathfrak{R} \mid \nu(Q_N^k) < t - \epsilon_1/2\}.$$

Assume that $F^k \subset \Omega^k$ is defined (F^k will be defined later) then on $W^k = A^k \cap D^k \cap F^k$ we define an equivalence relation $\sim$ by $\mathfrak{R}_1 \sim \mathfrak{R}_2$ iff $Q_N^k(\mathfrak{R}_1) = Q_N^k(\mathfrak{R}_2)$. Divide $A^k \cap D^k \cap F^k$ into equivalence classes $W_1, \ldots, W_{n(k)}$ with respect to $\sim$. For each k and $1 \leqslant s \leqslant n(k)$ and for each $\mathfrak{R}$ in W_s^k denote

$$T_s^k = Q_N^k(\mathfrak{R}).$$

Let $\nu|_{I\setminus T_s^k}$ be the measure ν restricted to $I\setminus T_s^k$. Define

$$\hat{\nu} = \frac{1}{\nu(I\setminus T_s^k)}(\nu|_{I\setminus T_s^k}),$$

obviously $\hat{\nu} \in \mathrm{NA}^1$. Denote $t_s^k = t - \epsilon_1/2 - \nu(T_s^k)\cdot f_{t_s^k}$ is the single jump function with a jump at t_s^k, satisfying $f_{t_s^k}(1) = g_t(1)$.

Denote by $\tilde{\pi}_k^s$ the set of all members of π_k contained in $I\setminus T_s^k$. Corollary 4.11 implies that for each $\epsilon_2 > 0$ there exists K such that for each $k > K$ and $1 \leqslant s \leqslant n(k)$

$$|\psi(f_{t_s^k} \circ \tilde{\nu})_{\tilde{\pi}_k^s}(\bar{S}) - g_t(1)\cdot\tilde{\nu}(S)| < \epsilon_2$$

or for each $k > K$ and for each $1 \leqslant s \leqslant n(k)$

$$\left|\psi(f_{t_s^k} \circ \tilde{\nu})_{\tilde{\pi}_k^s}(\bar{S}) - g_t(1)\cdot\frac{\nu(S\setminus T_s^k)}{\nu(I\setminus T_s^k)}\right| < \epsilon_2 \tag{5.36}$$

and

$$E(H^k \mid Q_N^k = T_s^k) = \psi(f_{t_s^k} \circ \tilde{\nu})_{\tilde{\pi}_k^s}(\bar{S}). \tag{5.37}$$

Let $\nu|_S$ be the measure ν restricted to S and m the vector measure defined by $m = (\nu|_S/\nu(S), \nu)$. For each $\delta > 0$ denote

$$U^\delta = \{(x, y) \mid |x - y| < \delta\}.$$

Corollary 18.10 of [1] asserts that

$$\lim_{k\to\infty} \mathrm{Prob}(m(Q_N^k) \in U^\delta) = 1.$$

Let ϵ_3 be a small positive number, and let F^k be defined by

$$F^k = \left\{\mathfrak{R} \;\middle|\; \left|\frac{\nu|_S(I\setminus Q_N^k)}{\nu(S)} - \frac{\nu(I\setminus Q_N^k)}{\nu(I)}\right| < \epsilon_3\right\}$$

again, from corollary 18.10 in [1]

$$\lim_{k\to\infty} \mathrm{Prob}(F^k) = 1. \tag{5.38}$$

Since $W_s^k \subset F^k$, then by the definition of F^k and together with (5.37), the inequality

(5.36) can be replaced by

$$|E(H^k \mid W_s^k) - g_t(1)\nu(S)| < \epsilon_2 + \epsilon_3$$

for $k > K$ and $1 \leqslant s \leqslant n(k)$.

(5.30) asserts that $\lim_{k\to\infty}$ Prob $D^k = 1$ and together with (5.38) we obtain that

$$\lim_{k\to\infty} E(H^k \mid A^k) = \lim_{k\to\infty} E(H^k \mid W^k)$$

and

$$E(H^k \mid W^k) = \sum_{s=1}^{n(k)} E(H^k \mid W_s^k) \cdot \frac{P(W_s^k)}{P(W^k)}$$

$$= \sum_{s=1}^{n(k)} \left\{ \left[E(H^k \mid W_s^k) - g_t(1)\nu(S) \right] \frac{P(W_s^k)}{P(W^k)} + g_t(1)\nu(S) \cdot \frac{P(W_s^k)}{P(W^k)} \right\}.$$

From (5.39) we get, for $k > K$

$$-(\epsilon_2 + \epsilon_3) + g_t(1)\nu(S) \leqslant E(H^k \mid W^k) \leqslant \epsilon_2 + \epsilon_3 + g_t(1)\nu(S).$$

Since ϵ_2 and ϵ_3 are arbitrary positive numbers,

$$\lim_{k\to\infty} (H^k \mid A^k) = \lim_{k\to\infty} E(H^k \mid W^k) = g_t(1)\nu(S)$$

and hence, by (5.35)

$$\lim_{k\to\infty} E(\chi^k \mid A^k) = g_t(1)\nu(S). \tag{5.40}$$

The same arguments lead to the conclusion that

$$\lim_{k\to\infty} E(\chi^k \mid B^k) = g_t(1) \cdot \mu(S). \tag{5.41}$$

But (5.31) and (5.32) imply that for large k's $P(C^k) \leqslant o(1)$, and since $|\chi^k| \leqslant 1$

$$E(\chi^k \mid C^k) \cdot P(C^k) \leqslant o(1).$$

$$E(\chi^k) = E(\chi^k \mid A^k)P(A^k) + E(\chi^k \mid B^k)P(B^k)$$
$$+ E(\chi^k \mid C^k)P(C^k).$$

Again, by (5.31) and (5.32) together with (5.40) and (5.41)

$$E(\chi^k) \leqslant g_t(1)\nu(S)\left[1 - \Phi\left(\tfrac{1}{2}\epsilon_1^{1/3}\right)\right] + g_t(1)\,\mu(S)\left(1 - \Phi\left(\tfrac{1}{2}\epsilon_1^{1/3}\right)\right) + o(1)$$

and

$$E(\chi^k) \geqslant g_t(1)\nu(S)\left[1 - \Phi(4\epsilon_1^{1/3})\right] + g_t(1)\,\mu(S)\left[1 - \Phi(4\epsilon_1^{1/3})\right].$$

Hence

$$\limsup_k E(\chi^k) \leqslant \left[1 - \Phi\left(\tfrac{1}{2}\epsilon_1^{1/3}\right)\right]\left[g_t(1)\nu(S) + g_t(1) \cdot \mu(S)\right] + o(1),$$

$$\liminf_k E(\chi^k) \geqslant \left[1 - \Phi(4\epsilon_1^{1/3})\right]\left[g_t(1)\nu(S) + g_t(1) \cdot \mu(S)\right].$$

The last two inequalities hold for each $\epsilon_1 > 0$ small enough, therefore

$$\limsup_k E(\chi^k) \leqslant \tfrac{1}{2}\left[g_t(1)\nu(S) + g_t(1)\mu(S)\right]$$

and

$$\liminf_k E(\chi^k) \geqslant \tfrac{1}{2}[\, g_t(1)\nu(S) + g_t(1)\mu(S)]$$

which imply that

$$\lim_k E(\chi^k) = g_t(1) \cdot \frac{\nu(S) + \mu(S)}{2}$$

and the proof of proposition 5.17 is completed.

PROPOSITION 5.42. *bv'NA * bv'NA is not contained in* ASYMP.

PROOF. Let f be the single jump function defined by

$$f(x) = \begin{cases} 0, & x \leqslant 1/4, \\ 1, & 1 \geqslant x > 1/4. \end{cases}$$

Let $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. λ is Lesbegue measure on $[0, 1]$, μ_1 and μ_2 are the measures defined by

$$\mu_1(S) = 2\lambda(S \cap I_1) \quad \text{and} \quad \mu_2(S) = 2\lambda(S \cap I_2).$$

We will show that if $v = (f \circ \mu_1) \cdot (f \circ \mu_2)$ then $v \notin$ ASYMP. Since the proof is very similar to the one given in [1, chapter III, example 19.2] to prove that the game defined there is not in ASYMP, we will not write the formal proof in detail. A sketch of the proof is given below.

Let $I = [0, 1]$ be partitioned into $2k + 1$ elements as follows: I_2 is divided into k equal intervals of length $1/2k$. I_1 is divided into a single interval J_ϵ of length $\epsilon/2$ and k equal intervals of length $(1 - \epsilon)/2k$. Here ϵ is meant to be small, but $1/k$ is much smaller.

In a random ordering, we will have approximately equal numbers of intervals from each of I_1, I_2, up to any point. However, their total length (which we will denote by α_1 and α_2) will not be equal. *Before* the appearance of J_ϵ, α_1 is approximately $(1 - \epsilon)\alpha_2$, and *after* its appearance, $\alpha_1 \approx (1 - \epsilon)(\alpha_2 - \epsilon)$. Thus, if l_1 (see figure) is the line through $(0, 0)$ and $(1, 1 - \epsilon)$, and l_2 is the line through $(0, \epsilon)$ and $(1, 1)$, then almost every order $\mathfrak{R}$ induces a discrete path from $(0, 0)$ to $(1, 1)$ which is close to l_1 up to the

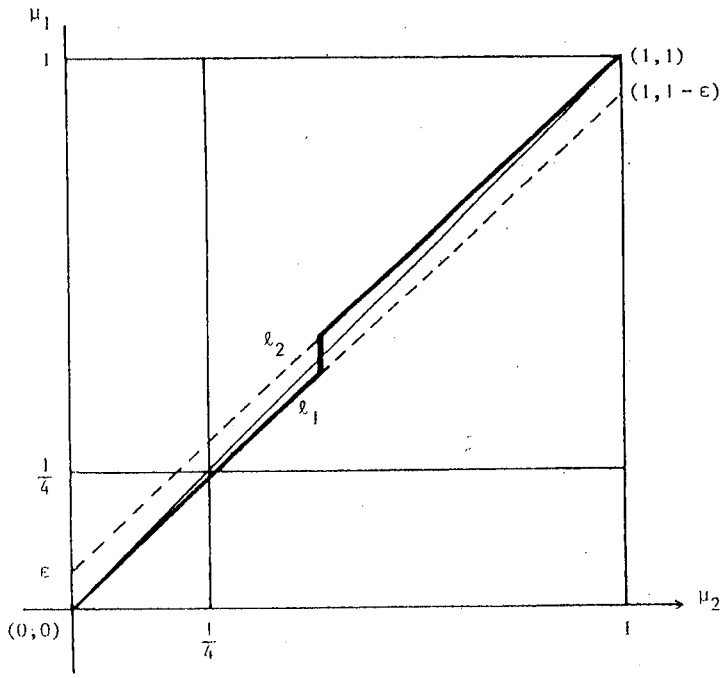

FIGURE 1. Almost every order $\mathfrak{R}$ induces a path (a discrete one) from $(0, 0)$ to $(1, 1)$ which is close to the line l_1 up to the appearance of J_ϵ and from there on is close to the line l_2.

appearance of J_ϵ and from there on is close to l_2. Therefore (with high probability) the point $(\mu_1(P_{J_\epsilon}^{\mathfrak{R}}), \mu_2(P_{J_\epsilon}^{\mathfrak{R}}))$ is very close to the line l_1. Thus, if $\mu_2(P_{J_\epsilon}^{\mathfrak{R}}) > 1/4$ the jump occurs according to μ_1, i.e., if $\bar{a} = \bar{a}(\mathfrak{R})$ is the interval for which $v(P_{\bar{a}}^{\mathfrak{R}} \cup \bar{a}) - v(P_{\bar{a}}^{\mathfrak{R}}) = 1$ then, $\mu_2(P_{J_\epsilon}^{\mathfrak{R}}) > 1/4$ implies that $\bar{a} \subset I_2$ (with high probability). Prob$(\mu_2(P_{J_\epsilon}^{\mathfrak{R}}) > 1/4)$ is approximately $3/4$, therefore the assumption that $v \in$ ASYMP leads to the conclusion that $(\varphi v)(I_1) = 3/4$ (where φv is the asymptotic value of v).

In similar fashion by choosing J_ϵ from I_2 it would follow that if $v \in$ ASYMP then $(\varphi v)(I_2) = 3/4$ which implies that $(\varphi v)(I) = 1\frac{1}{2}$ contradicting $(\varphi v)(I) = v(I) = 1$.

THEOREM 5.43. *A partition value φ exists on the space $A * bv'$NA $* bv'$NA, satisfying*

$$\varphi\left[(g_t \circ \mu)(g_s \circ \nu)\cdot \prod_{i=1}^{n}(f_i \circ \mu_i)\right]$$

$$= \begin{cases} \left(\prod_{i=1}^{n} f_i\right)(t) g_t(1)\cdot g_s(1)\cdot \mu + g_t(1)\cdot g_s(1)\varphi_t\left(\prod_{i=1}^{n} f_i \circ \mu_i\right), & t > s, \\ \left(\prod_{i=1}^{n} f_i\right)(t) g_t^2(1)\cdot \dfrac{\mu+\nu}{2} + g_t^2(1)\varphi_t\left(\prod_{i=1}^{n} f_i \circ \mu_i\right), & t = s, \end{cases} \tag{5.44}$$

where f_i, $1 \leqslant i \leqslant n$, are continuous functions in bv', g_t and g_s are single jump functions; $\mu, \nu, \mu_1, \ldots, \mu_n$ are in NA^1 *and φ_t, as defined in the beginning of* §5.b, *satisfies*

$$\varphi_t\left(\prod_{i=1}^{n}(f_i \circ \mu_i)\right) = \sum_{j=1}^{n} \mu_j \int_t^1 \left(\prod_{i \neq j} f_i\right) df_j.$$

PROOF. Each game in $A * bv'$NA $* bv'$NA can be approximated by games of the form $W_1 + W_2 + \prod_{i=1}^{n}(f_i \circ \mu_i)\sum_{i=1}^{m}(g_{t_i} \circ \eta_i)\cdot(g_{s_i} \circ \nu_i)$ where f_i, $1 \leqslant i \leqslant n$, is in cbv (the set of all continuous functions in bv'); g_{t_i}, g_{s_i}, $1 \leqslant i \leqslant m$, are single jump functions which have a jump at t_i and s_i, respectively; μ_j $(1 \leqslant j \leqslant n)$, η_i, ν_i $(1 \leqslant i \leqslant m)$ are in NA^1, and W_1 and W_2 are in $[A * bv'\text{NA}] \cup A$.

Let M_2 be the set of all games of the form

$$\left[\prod_{i=1}^{n}(f_i \circ \mu_i)\right](g_t \circ \mu)\cdot(g_s \circ \nu) \quad \text{where } f_i \in cbv,\ 1 \leqslant i \leqslant n.$$

$A * bv'$NA $* bv'$NA is the closure of the linear subspace generated by $B \cup M \cup M_2$. The rest of the proof is based upon:

PROPOSITION 5.45. *Let π be a partition, $S \in B$ and let $v \in M_2$ be given by*

$$v = (g_t \circ \mu)\cdot(g_s \circ \nu)\prod_{i=1}^{n}(f_i \circ \mu_i).$$

For each $\pi_k \in \pi(k, \eta, S)$ where $\eta = (\mu, \nu, \mu_1, \ldots, \mu_n)$, if $t > s$ then

$$\lim_{k\to\infty}(\psi v_{\pi_k})(\bar{S}) = \left(\prod_{i=1}^{n} f_i\right)(t)\, g_t(1)\, g_s(1)\, \mu + g_t(1)\, g_s(1)\varphi_t\left(\prod_{i=1}^{n}(f_i \circ \mu_i)\right)$$

and if $t = s$, then

$$\lim_{k\to\infty}(\psi v_{\pi_k})(\bar{S}) = \left(\prod_{i=1}^{n} f_i\right)(t)\, g_t^2(1)\cdot \frac{\mu+\nu}{2} + g_t^2(1)\varphi_t\left(\prod_{i=1}^{n}(f_i \circ \mu_i)\right).$$

Assuming proposition 5.45, the proof of theorem 5.44 is completed as follows.

Let $\gamma : M_2 \cup M \cup B$, be the operator which coincides with the partition values of $A * bv'$NA and A, on M and B, respectively, and is defined on M_2 as φ is defined in (5.44). An argument, analogous to the ones used in theorems 5.1 and 5.4, leads to the conclusion that γ is a well-defined operator.

Let π be a partition, $S \in B$, n a positive integer and $v^1, v^2, \ldots, v^n$ in $M_2 \cup M \cup B$. Let ξ be a vector measure consisting of all measures involved in $v^1, \ldots, v^n$. Then for each $\pi_k \in \pi(k, \xi, S)$

$$\lim_{k\to\infty} (\psi v^t_{\pi_k})(\bar{S}) = (\gamma v^t)(S) \quad \text{for each } 1 \leqslant t \leqslant n.$$

(The above equation follows from proposition 5.10 and from proposition 5.45.) Hence by lemma 4.2 the proof of theorem 5.44 is completed.

We will prove now proposition 5.45. The proof is very similar to the proof of proposition 5.10.

PROOF OF PROPOSITION 5.45. Let π be any measurable partition, $S \in \mathfrak{B}$ and $v \in M$ is given by $v = (g_t \circ \mu)(g_s \circ \nu)\prod_{i=1}^n (f_i \circ \mu_i)$. W.l.o.g. let us assume that for each $1 \leqslant i \leqslant n$, f_i is a monotonic function. Let $\pi_k \in \pi(k, \xi, S)$ where $\xi = (\mu_1, \ldots, \mu_n, \nu, \mu)$. Let $\delta > 0$ be given with $2\delta < \min(s, t, 1-s, 1-t)$. Define g_t^δ as follows:

$$g_t^\delta(x) = \begin{cases} 0, & 0 \leqslant x \leqslant t - \dfrac{\delta}{2}, \\ \dfrac{2g_t(1)}{\delta}\left(x - t - \dfrac{\delta}{2}\right), & t - \dfrac{\delta}{2} \leqslant x \leqslant t, \\ g_t(1), & t \leqslant x \leqslant 1. \end{cases}$$

g_s^δ is defined as g_t^δ replacing each t by s. Denote $u^\delta_{t,s} = (g_t^\delta \circ \mu)\cdot(g_s^\delta \circ \nu)$, $u_{t,s} = (g_t \circ \mu)(g_s \circ \nu)$, $u = \prod_{i=1}^n f_i \circ \mu_i$, $w = u_{t,s} - u^\delta_{t,s}$. Obviously $v = u^\delta_{t,s}\cdot u + w \cdot u$. Since $u^\delta_{t,s}\cdot u$ is in A, proposition 5.5 implies that

$$\lim_{k\to\infty} \psi(u^\delta_{t,s}\cdot u)_{\pi_k}(\bar{S}) = \sum_{j=1}^n \mu_j(S)\int_0^1 \left[\left(\prod_{i\neq j} f_i\right) g_t^\delta g_s^\delta\right] df_j$$
$$+ \mu(S)\int_0^1 \left[\left(\prod_{i=1}^n f_i\right) g_s^\delta\right] dg_t^\delta + \nu(S)\int_0^1 \left[\left(\prod_{i=1}^n f_i\right) g_t^\delta\right] dg_s^\delta.$$

From the definition of g_t^δ and g_s^δ it is easy to verify that for $t \geqslant s$

$$\lim_{\delta\to 0} \int_0^1 \left[\left(\prod_{i\neq j} f_i\right) g_t^\delta g_s^\delta\right] df_j = g_t(1)\cdot g_s(1)\int_t^1 \left(\prod_{i\neq j} f_i\right) df_j \tag{5.46}$$

and

$$\lim_{\delta\to 0} \int_0^1 \left[\left(\prod_{i=1}^n f_i\right) g_s^\delta\right] dg_t^\delta = g_t(1)\cdot g_s(1)\cdot \left(\prod_{i=1}^n f_i\right)(t) \tag{5.47}$$

and for $t > s$

$$\lim_{\delta\to 0} \int_0^1 \left[\left(\prod_{i=1}^n f_i\right) g_t^\delta\right] dg_s^\delta = 0. \tag{5.48}$$

From (5.46), (5.47) and (5.48), to complete the proof of the proposition it is sufficient to prove that $\lim_{k\to\infty} \psi(w\cdot u)_{\pi_k}(S) = 0$. Let

$$\Omega_k^\delta = \left\{ \mathfrak{R} \mid |(\nu - \mu_i)(P_a^{\mathfrak{R}})| < \delta, |(\mu - \mu_i)(P_a^{\mathfrak{R}})| < \delta \ \forall a \in \pi_k \text{ and } \forall i \ 1 \leqslant i \leqslant n \right\}.$$

According to Aumann-Shapley's corollary 18.10 [1], we have

$$\lim_{k\to\infty} \operatorname{Prob}(\Omega_k^\delta) = 1. \tag{5.49}$$

Define the random variables x_k, y_k, z_k on Ω_k by

$$x_k = \sum \left[u(P_a^{\mathfrak{R}} \cup a) - u(P_a^{\mathfrak{R}})\right] \cdot w(P_a^{\mathfrak{R}} \cup a), \tag{5.50}$$

$$y_k = \sum \left[w(P_a^{\mathfrak{R}} \cup a) - w(P_a^{\mathfrak{R}})\right] \cdot \left[u(P_a^{\mathfrak{R}}) - \left(\prod_{i=1}^n f_i\right)(t)\right], \tag{5.51}$$

$$z_k = \sum \left[w(P_a^{\mathfrak{R}} \cup a) - w(P_a^{\mathfrak{R}})\right] \cdot \left(\prod_{i=1}^n f_i\right)(t),$$

where each of the summations is taken over all $a \in \pi_k$ with $a \subset S$

$$\psi(u \cdot w)_{\pi_k}(\bar{S}) = E(x_k + y_k + z_k).$$

Since $w \in bv'\mathrm{NA} * bv'\mathrm{NA}$, propositions 5.5 and 5.17 imply that

$$\lim_{k\to\infty} (\psi w_{\pi_k})(\bar{S}) = 0$$

and hence that $\lim_{k\to\infty} E(z_k) = 0$. Thus we will complete the proof by proving that $\lim_{k\to\infty} E(x_k + y_k) = 0$.

Since $w(T)$ vanishes for each $T \in \mathfrak{B}$ satisfying

$$\nu(T) \notin [s - \delta/2, s] \quad \text{and} \quad \mu(T) \in [t - \delta/2, t]$$

we can assume that for large k's, on Ω_k^δ if $w(P_a^{\mathfrak{R}} \cup a) - w(P_a^{\mathfrak{R}}) \neq 0$, then the summation appearing in (5.50) and (5.51) is taken over all $a \in \pi_k$ and $a \subset S$ satisfying

$$t - 2\delta < \mu_i(P_a^{\mathfrak{R}}) < t + \delta$$

for each $1 \leqslant i \leqslant n$. Hence, the monotonicity of u and the inequality $|w| \leqslant 2g_t(1) \cdot g_s(1)$ imply that for sufficiently large k

$$|x_k|_{\Omega_k^\delta}| \leqslant \left[\left(\prod_{i=1}^n f_i\right)(t + \delta) - \left(\prod_{i=1}^n f_i\right)(t - 2\delta)\right] g_t(1)\, g_s(1)$$

and

$$|y_k|_{\Omega_k^\delta}| \leqslant 2g_t(1) \cdot g_s(1)\left[\left(\prod_{i=1}^n f_i\right)(t + \delta) - \left(\prod_{i=1}^n f_i\right)(t - 2\delta)\right].$$

Combining the last two inequalities with (5.49) we obtain by proposition 5.3 that

$$\lim_{k\to\infty} \left[E(x_k + y_k) - E((x_k + y_k)|_{\Omega_k^\delta})\right] = 0$$

and

$$\lim_{k\to\infty} E((x_k + y_k)|_{\Omega_k^\delta}) \leqslant 3g_t(1) \cdot g_s(1)\left[\left(\prod_{i=1}^n f_i\right)(t + \delta) - \left(\prod_{i=1}^n f_i\right)(t - 2\delta)\right] \underset{\delta\to 0}{\to} 0$$

which imply that $\lim_{k\to\infty} E(x_k + y_k) = 0$, and the proof is completed.

THEOREM 5.52. *A partition value φ exists on the space W. (The space W is the closed linear subspace of bv generated by the spaces* ASYMP, bv'NA, A, $A * bv'$NA, bv'NA $* bv'$NA *and* $A * bv'$NA $* bv'$NA.)

PROOF. It can be shown that $A * bv'\mathrm{NA} * bv'\mathrm{NA}$ contains each of the spaces $bv'\mathrm{NA}$, A, $A * bv'\mathrm{NA}$ and $bv'\mathrm{NA} * bv'\mathrm{NA}$. Hence, the proof follows immediately from proposition 3.6.

Another way to prove the theorem is to look at $M_3 = B_0 \cup B \cup M \cup M_1 \cup M_2 \cup$ ASYMP, where B_0 is the set of all games of the form $f \circ \mu$, $f \in bv'$, $\mu \in \mathrm{NA}^1$. It is clear that W is the closed linear subspace generated by M_3.

Let $\gamma : M_3 \to \mathrm{NA}$ be the operator which coincides with the partition value of $bv'\mathrm{NA}$, A, $A * bv'\mathrm{NA}$, $bv'\mathrm{NA} * bv'\mathrm{NA}$ and $A * bv'\mathrm{NA} * bv'\mathrm{NA}$ on B_0, B, M, M_1, M_2, respectively, and which coincides with the asymptotic value on ASYMP. Let π be any partition, $S \in \mathfrak{B}$ and let $v^1, v^2, \ldots, v^n$ be in M_3. If ξ is a vector measure consisting of all measures involved in $v^1, \ldots, v^n$, then propositions 5.2, 5.5, 5.10, 5.17 and 5.45 imply that for each $\pi_k \in \pi(k, \xi, S)$

$$\lim_{k\to\infty} (\psi v^t_{\pi_k})(\bar{S}) = (\gamma v^t)(S), \qquad 1 \leqslant t \leqslant n.$$

An argument, analogous to the ones used in theorems 5.1 and 5.4, leads to the conclusion that γ is a well-defined operator and the proof of the theorem is now following from lemma 4.2.

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