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PAYOFFS IN NONATOMIC ECONOMIES: AN AXIOMATIC APPROACH

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1. INTRODUCTION

IT HAS BEEN OFTEN REMARKED that different solution concepts become equivalent in the setting of "perfectly competitive" economies (i.e., to use the modern idiom [**2**], economies in which the agents form a nonatomic continuum). The conjecture that the core and competitive (Walras) allocations coincide was broached as far back as 1881 by Edgeworth [**10**]. This insight has been vindicated in increasing generality in a spate of articles[2] [**21, 9, 2, 14, 13, 7, 8, 1**] over the last two decades. More recently, it was shown by Aumann [**3**] that—with a smoothness assumption on the preferences—the "value allocations"[3] also coincide with the above two.

If we restrict ourselves to the case of smooth, *transferable* utilities, then the equivalence phenomenon turns out to be even more striking: not only do these solutions coincide but they are also unique, i.e., consist of a single payoff. Our aim here is to give another view of this "coincident payoff" by putting it on an axiomatic foundation. As an upshot of our approach we get a "meta-equivalence" theorem, by way of a categorization: any solution coincides with this payoff if, and only if, it satisfies our axioms.

The transferable-utility assumption is undoubtedly restrictive. But we are encouraged by its good track record of being the precursor of the general analysis (e.g. [**6**] before [**3**]; [**18**] before [**19**]; [**21**] before [**9**]). And we hope that our approach can be extended to the nontransferable case.

Denote by M the class of nonatomic economies with transferable and differentiable utilities. Any such economy can also be viewed as a productive economy with a single consumable output. (See the discussion in Chapter 6 of [**6**].) We will, in fact, adopt the production interpretation for most of our discussion. But by thinking of the output as "utiles" everything we say can be translated into the exchange version as well.

The problem of determining payoffs (final distributions of the output) in these economies has been approached from many sides. Let us briefly recount some of

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[2] This list is only meant to be indicative, and by no means exhaustive.

[3] Which are based on the Shapley value [**18, 19**], a game-theoretic concept quite different from the core.

them. First there is the classical notion of a competitive payoff which depends on prices that clear all markets, i.e., equate supply and demand. Equally well known is the concept of the core. It is defined by the condition that no coalition of agents in the economy can, on its own, improve upon what it gets. (See Chapter 6 of **[6]** for a historical survey and detailed discussion of these concepts.) Other solutions from game theory have also been applied to the economic model. The bargaining set **[4]**, which contains the core, is based upon a weaker notion of stability: each "objection" can be ruled out by a "counter-objection." (In this terminology there can be, *a fortiori*, no objection to any payoff in the core.) Then there is the concept of the nucleolus **[17]**. Roughly, what is involved is minimizing the "dissatisfaction" of the most dissatisfied coalition, where dissatisfaction is measured by the difference between what a coalition "could" get and what it is getting. Finally, we have the Shapley value **[18]**, which has been the focus of active, recent research (and was the starting point of this inquiry as well). It is a mapping that assigns to each player of a game a number that purports to represent what he would be willing to pay in order to participate; and is uniquely determined by certain plausible conditions for all finite games **[18]**, and a large class of nonatomic games (pNA) which include the economies in M **[6]**. The value thus obtained can also be interpreted from a complementary standpoint: it assigns to a player the average of his marginal contributions to coalitions he may join (in a model of random ordering of the players).

For any economy in M, all these solutions coincide and consist of a single payoff [**6**, Chap. 6; **11**]. We would like to explicate certain underlying principles which lead to this distinguished payoff. To set the stage for this, we take a map from economies to sets of payoffs and look for a minimal list of plausible axioms that will uniquely characterize the map. Four axioms are presented which accomplish the job. The map they lead to is that of the above-mentioned distinguished payoff.

This may be viewed as a meta-equivalence theorem. For instance the equivalence of core and competitive payoffs follows from our result by simply checking that both the map which takes each m in M to its core and the map which takes each m in M to its competitive payoffs satisfy our axioms. That the value also coincides with the core and competitive payoff is immediate because it, in fact, satisfies even stronger axioms [**16**, Chap. 1]. In general: if any solution is a candidate for equivalence, it is both necessary and sufficient that it satisfy our axioms.

In our axioms, we try to identify the minimal common characteristics of solutions that hold not only on M but also when the set of agents is finite, or when utilities are not differentiable (or both). The axioms are therefore cast in as weak a form as possible. They turn out to be categorical (i.e., imply a unique solution) only on M (i.e., only in conjunction with nonatomicity and differentiability). Besides the aesthetic value of the weak form, there is also a pragmatic reason involved. In this form the truth of the axioms can be easily verified for a given solution concept, and thus it provides an effective format for discussing equivalence. Of course once a solution obeys the axioms, then an implication of our theorem is that on M it coincides with the value and therefore in fact must obey the much stronger axioms of the value. But these strong axioms are often not easy to verify directly. An example is provided by the core for which additivity on M is not obvious (and in fact is not true outside of M) but separability is.

The axioms will be spelled out precisely in Section 3, but let us present them at an intuitive level now. Denote the space of agents by $[T, \mathscr{C}, \mu]$. Here T is the set of agents, $\mathscr{C}$ the sigma-algebra of coalitions, and μ a nonatomic population measure on $[T, \mathscr{C}]$. An economy is a pair of measurable functions $(\boldsymbol{a}, \boldsymbol{u})$ where $\boldsymbol{a} : T \rightarrow R^n_+$ specifies the initial endowment of the n resource commodities, and $\boldsymbol{u} : T \times R^n_+ \rightarrow R_+$ the production—alternatively, utility—functions. M is the set of all pairs $(\boldsymbol{a}, \boldsymbol{u})$, subject to certain conditions on $\boldsymbol{a}$ and $\boldsymbol{u}$ (see Section 2). For any m in M we can define an associated *characteristic function* (or *game*) $v_m : \mathscr{C} \rightarrow R_+$, which assigns to each coalition the maximum output that it could achieve by a reallocation of the resources of its own members, i.e.,

$$v_m(S) = \max\left\{ \int_S \boldsymbol{u}(t, \boldsymbol{x}(t))\, d\mu(t) : \int_S \boldsymbol{x}(t)\, d\mu = \int_S \boldsymbol{a}(t)\, d\mu,\ \boldsymbol{x} : T \rightarrow R^n_+ \right\}.$$

Payoffs in $m \in M$ can be thought of as integrable functions from T to R_+, and in turn can be identified with nonnegative countably additive measures on $(T, \mathscr{C})$ which are absolutely continuous with respect to μ. But let us make only the assumption that they lie in FA, the collection of functions from $\mathscr{C}$ to R which are finitely additive and bounded. Let $P(FA)$ be the set of all subsets of FA. Then any assignment of payoffs to economies may be represented by a map:

$$\phi : M \rightarrow P(FA).$$

We will impose four axioms on ϕ: "inessential economy," "anonymity," "separability," and "continuity." Our main result is that there is one, and only one, map which satisfies these axioms: it maps m into the (unique) coincident payoff of m. There is an immediate corollary to this which is useful to keep in mind for the meta-equivalence aspect of the result. Given two solutions $\phi_1 : M \rightarrow P(FA)$ and $\phi_2 : M \rightarrow P(FA)$, call ϕ_2 a "cover" of ϕ_1 if $\phi_1(m) \subset \phi_2(m)$ for all m in M. Our result implies that if a solution is nonempty-valued and has a cover which satisfies the axioms, then it must also agree on M with the coincident payoff. Thus even though competitive payoffs, in general, violate continuity and the nucleolus violates separability (when the agent space is finite), their equivalence on M is assured because the core is readily seen to be a satisfactory cover for both.

The inessential economy axiom has to do with economies in which agents have no motivation to collude in order to increase the output. Indeed, suppose that m in M is such that *each* coalition $S \in \mathscr{C}$ achieves its maximum $v_m(S)$ *uniquely* by sticking to its allocation of resources. Then one would expect that no exchange, either of the inputs or the output, will occur. And this is just what the axiom says.

The anonymity axiom asserts that the labels of the agents do not matter. If we were to relabel them, this would have the effect of relabelling their payoffs accordingly.

These two axioms hold widely for most solutions, not only on M, but in general.

The separability axiom considers an economy made up of two separate, noninteracting parts. Take m' and m'' in M. Let us construct the economy m by, as it were, "collating" m' and m''. Each agent in m possesses the same initial resources that he had in m' and in m''; also he has access to both his production functions from m' and m''. However, suppose that the input commodities of m' and m'' are completely disjoint: those in m' cannot be used for production in m'', and vice versa (though the two economies produce, of course, the same output). Now consider any coalition S that forms in m. Each agent in S can send his black-hatted (white-hatted) representative to m' (m''). If these two types of representatives separately maximize the output in m' and m'', then the sum of what they get back is precisely what S can obtain in m, i.e., $v_m(S) = v_{m'}(S) + v_{m''}(S)$ for all $S \in \mathscr{C}$. Thus m, in essence, consists of operating in m' and in m'' independently of each other. We require that in this case if we put together a payoff in m' with one in m'', the outcome should be feasible in m. However, we do not exclude the possibility that other payoffs may also be obtained in m. In symbols: $\phi(m') + \phi(m'') \subset \phi(m)$. This is related to the additivity axiom for the value but it is so watered down as to apply to the core even when the economy is finite (in which case additivity, $\phi(m) = \phi(m') + \phi(m'')$, no longer holds). Clearly it applies always to the value and to competitive payoffs; indeed, both these satisfy additivity.

The continuity axioms say that if the distance between two economies is small, then so is that between their sets of payoffs. It is, of course, intimately bound up with the notion of distance. The one we employ declares the distance between two economies to be zero if they yield the same characteristic function. Thus *the payoffs depend on the characteristic function alone*, i.e., they depend on the data $(\boldsymbol{a}, \boldsymbol{u})$ of the economy only insofar as it shows up in the net production of the coalitions (if $v_m = v_{m'}$, $\phi(m) = \phi(m')$). Modulo this, however, our continuity requirement is weak. We choose a "large" norm on the characteristic functions (the bounded variation norm) and a "small" one on $P(FA)$ (the Hausdorff distance in the bounded variation norm, which is equivalent in FA to the maximum norm).

Our axiomatic approach is akin to that of **[6]** and invites immediate comparison. We begin with a point-to-*set* map (from M to FA). That $\phi(m)$ is a nonempty one-element set of FA is a deduction, not a postulate, in our case. Also note that we do *not* require that $\phi(m)$ consist of efficient payoffs—this, too, is deduced. Separability reduces to additivity if the solution is single-valued but not otherwise. Clearly separability is weaker than the additivity required in **[6]**. Continuity is closely related to the positivity axioms of **[6]**. (See Proposition 6.14 and Corollary 6.15 of this paper and Propositions 4.6 and 4.15 of **[6]**.) Finally, we emphasize that the axioms are invoked on the set of games that arise from M *alone*. This set is much smaller than the general space pNA of [**6**]. (Its complement in pNA is open and dense.) Thus the uniqueness of ϕ does become an issue. (Existence on the other hand is no problem: simply restrict the value on pNA to our domain.) The very question we set out with, "What are payoffs in nonatomic economies?", makes it desirable that we exclude any reference to games that do not arise from M. Thus we stay within M throughout and give a self-contained analysis of it. Each axiom is cast in an economic framework and can be interpreted therein. It is fortunate that even though the scope of the axioms is diminished by this restriction of the domain, they nevertheless are sufficiently far-reaching to determine a unique map.

The paper is organized as follows. In Sections 2 and 3 we develop the precise statement of the theorem. Section 4 is taken up with the preparations for the proof. For the most part this consists of collating results from [**6**] for the convenience of the reader. The proof is in Section 5. Finally in Section 6 we establish the "tightness" of our theorem. Even in the presence of additional axioms, enlarging the domain or dropping any one of the axioms makes the theorem break down, i.e., ϕ is no longer unique.

2. NONATOMIC ECONOMIES WITH TRANSFERABLE, DIFFERENTIABLE UTILITIES

Let us recall more precisely the economic model presented in Chapter 6 of [**6**].

We begin with a measure space $[T, \mathscr{C}, \mu]$. T is the set of agents, $\mathscr{C}$ the σ-algebra of coalitions, and μ the population measure. $[T, \mathscr{C}]$ is assumed to be isomorphic to the closed unit interval $[0,1]$ with its Borel sets. μ is a finite, σ-additive, nonnegative and nonatomic measure, and we assume (w.l.o.g.) that $\mu(T) = 1$.

Each agent $t \in T$ is characterized by an initial endowment of resources, $\boldsymbol{a}^t \in R^n_+$, and a production (utility) function $\boldsymbol{u}^t : R^n_+ \to R$. Here R^n_+ is the nonnegative orthant of the Euclidean space R^n, and n is the number of (resource) commodities. Denoting the jth component of $x \in R^n$ by x_j, $\boldsymbol{a}^t_j$ is the quantity of the jth commodity held by agent t, and $\boldsymbol{u}^t(x)$ the amount of output he can produce using x. Thus the economy consists of the pair of functions $(\boldsymbol{a}, \boldsymbol{u})$, where $\boldsymbol{a} : T \to R^n_+$, $\boldsymbol{u} : T \times R^n_+ \to R$ (note the identifications $\boldsymbol{a}(t) \equiv \boldsymbol{a}^t$; $\boldsymbol{u}(t,x) \equiv \boldsymbol{u}^t(x)$).

To spell out the conditions on $(\boldsymbol{a}, \boldsymbol{u})$, we need some additional notation. For x, y in R^n_+, say $x = y$ ($x \geqq y$, $x > y$) when $x_j = y_j$ ($x_j \geqq y_j$, $x_j > y_j$) for all $1 \le j \le n$; $x \ge y$ when $x \geqq y$, but not $x = y$. Put $\|x\| = \max\{|x_j| : 1 \le j \le n\}$. Also note that R^n_+ can be regarded as a measurable space with its Borel sets. We will require that $(\boldsymbol{a}, \boldsymbol{u})$ satisfy:

(2.1) $\boldsymbol{a} : T \to R^n_+$ is integrable;

(2.2) $\boldsymbol{u} : T \times R^n_+ \to R$ is measurable where $T \times R^n_+$ is equipped with the product σ-field $\mathscr{C} \times \mathscr{B}$ where $\mathscr{B}$ denotes the Borel sets of R^n_+;

(2.3) $u(x) = o(\|x\|)$, as $\|x\| \to \infty$, integrably in t; i.e., for every $\epsilon > 0$ there is an integrable function $\eta : T \to R$ such that $|u^t(x)| \leq \epsilon \|x\|$ whenever $\|x\| \geq \eta(t)$.

For almost all[4] $t \in T$:

(2.4) $a^t > 0$ (where, without confusion, 0 also stands for the origin of R^n_+);

(2.5) x^t is continuous and increasing (i.e., $x \geq y$ implies $u^t(x) > u^t(y)$);

(2.6) $u^t(0) = 0$;

(2.7) the partial derivative $\partial u^t / \partial x_j$ exists and is continuous at each point where $x_j > 0$.

The collection of all pairs (a, u) which satisfy (2.1)–(2.7) will be called M, i.e., we keep the space $[T, \mathscr{C}, \mu]$ of agents fixed but vary their characteristics (a, u); in particular, the number n of resource commodities can be any positive integer $1, 2, 3, \ldots$. As we said already in the introduction, to each $m = (a, u) \in M$, we associate a game or characteristic function, $v_m : \mathscr{C} \to R$ by:

(2.8) $$v_m(S) = \max\left\{ \int_S u^t(x^t)\, d\mu(t) : \mathrm{x} : T \to R^n_+ ,\, x(S) = a(S) \right\}.$$

(For an integrable function $y : T \to R^n_+$, $y(S)$ abbreviates $\int_S y\, d\mu$.) That this max is attained is essentially the main theorem in **[5]**.

FA is the collection of all functions from $\mathscr{C}$ to R that are finitely additive and bounded, and $P(FA)$ is the set of all subsets of FA. We are going to characterize a map $\phi : M \to P(FA)$ via axioms. It will turn out that, for any $m \in M$, $\phi(m)$ is the set of competitive payoffs in m. To remind the reader: a pair (p, x) [where $x : T \to R^n_+$ is an integrable function with $x(T) = a(T)$ and p a price vector in R^n_+] is called a transferable utility competitive equilibrium (t.u.c.e.) of the economy (a, u) if, for almost all $t \in T$,

$$u^t(y) - p \cdot (y - a^t) \leqslant u^t(x^t) - p \cdot (x^t - a^t)$$

for any y in R^n_+; the corresponding competitive payoff is the measure $\nu_{p,x}$ defined by

$$\nu_{p,x}(S) = \int_S \left[u^t(x^t) - p \cdot (x^t - a^t) \right] d\mu$$

for $S \in \mathscr{C}$. If we denote by $\psi(m)$ the set of competitive payoffs in m, then under the assumptions (2.1)–(2.7), $\psi(m)$ is a singleton for any $m \in M$. (See Proposition 32.3 in **[6]**.)

[4] I.e., this is true for all except perhaps a μ-null set of agents.

3. STATEMENT OF THE THEOREM

In this section we prepare for and state the four axioms, as well as our main result.

AXIOM 1 (Inessential Economy): *Suppose* $m = (\boldsymbol{a}, \boldsymbol{u})$ *in* M *is such that, for each nonnull set* $S \in \mathscr{C}$, $v_m(S)$ *is achieved uniquely by* $\boldsymbol{a} : S \to R_+^n$ *(i.e.,* $\boldsymbol{a} : S \to R_+^n$ *is the unique solution to the maximization problem* (2.8)). *Then* $\phi(m)$ *consists of just the payoff* γ *given by*: $\gamma(S) = \int_S \boldsymbol{u}^t(\boldsymbol{a}^t)\, d\mu(t)$, *for* $S \in C$.

Let Q_μ be the set of all automorphisms of $[T, \mathscr{C}]$ which preserve the measure μ, i.e., Q_μ consists of bi-measurable bijections $\theta : T \to T$ such that $\mu(\theta(S)) = \mu(S)$ for all $S \in \mathscr{C}$. For $m = (\boldsymbol{a}, \boldsymbol{u}) \in M$ and $\theta \in Q_\mu$, define $\theta m = (\theta \boldsymbol{a}, \theta \boldsymbol{u})$ by $(\theta \boldsymbol{a})(t) = \boldsymbol{a}(\theta(t))$, $(\theta \boldsymbol{u})(t, x) = \boldsymbol{u}(\theta(t), x)$. Also, for $v \in BV$ and $\theta \in Q_\mu$ define, $\theta v : \mathscr{C} \to R$ by $(\theta v)(S) = v(\theta(S))$; and for $A \subset BV$, define $\theta A = \{\theta v : v \in A\}$.

AXIOM 2 (Anonymity): *For any* m *in* M, *and* θ *in* Q_μ, $\phi(\theta m) = \theta \phi(m)$.

Since $A \subset FA$ implies $\theta A \subset FA$, and $m \in M$ implies $\theta m \in M$, the axiom makes sense.

For the separability axiom, we need to define the disjoint sum of two economies. Take $m = (\boldsymbol{a}, \boldsymbol{u})$, $m' = (\boldsymbol{a}', \boldsymbol{u}')$ where $\boldsymbol{a} : T \to R_+^l$ and $\boldsymbol{a}' : T \to R_+^k$. Put $m \oplus m' = (\boldsymbol{a} \oplus \boldsymbol{a}', \boldsymbol{u} \oplus \boldsymbol{u}')$, where $(\boldsymbol{a} \oplus \boldsymbol{a}') : T \to R_+^{l+k}$ and $(\boldsymbol{u} \oplus \boldsymbol{u}') : T \times R_+^{l+k} \to R$ are given by:

$$(\boldsymbol{a} \oplus \boldsymbol{a}')(t) = (\boldsymbol{a}(t), \boldsymbol{a}'(t)),$$

$$(\boldsymbol{u} \otimes \boldsymbol{u}')(t, (x, y)) = \boldsymbol{u}(t, x) + \boldsymbol{u}'(t, y).$$

[For $x \in R_+^l$ and $y \in R_+^k$, (x, y) is the vector R_+^{l+k} whose first l components are according to x, and the last k according to y.] Note that $m \oplus m' \in M$ if $m \in M$ and $m' \in M$. Also note that $A + B \in P(FA)$ if $A \in P(FA)$ and $B \in P(FA)$, where we define $A + B = \{\alpha + \beta : \alpha \in A, \beta \in B\}$.

AXIOM 3 (Separability): *For any* m *and* m' *in* M, $\phi(m) + \phi(m') \subset \phi(m \oplus m')$.

The continuity axiom is stated in terms of the bounded variation norm on set functions. A set-function v is a map from $\mathscr{C}$ to R such that $v(\emptyset) = 0$. It is called monotonic if $T \subset S$ implies $v(S) \geq v(T)$. The difference between two monotonic set functions is said to be of bounded variation. Let BV be the real vector space of all set-functions of bounded variation. For $v \in BV$, define the norm $\|v\|$ of v by:

$$\|v\| = \inf\{u(T) + w(T)\}$$

where the infinum ranges over all monotonic functions u and w such that $v = u - w$.

Each characteristic function v_m of an economy m in M is monotonic and thus is in BV. So we can introduce the distance d on M by $d(m,m') = \|v_m - v_{m'}\|$. Also observe that $FA \subset BV$. For A and B in $P(FA)$, let $h(A,B)$ be the Hausdorff distance between A and B, i.e., $h(A,B) = \inf\{\epsilon \in R_+ : A \subset B^\epsilon \text{ and } B \subset A^\epsilon\}$, where A^ϵ is the set $\{\alpha' \in FA : \|\alpha - \alpha'\| < \epsilon \text{ for some } \alpha \in A\}$ etc.; and $\inf \emptyset = \infty$. We are ready for Axiom 4.

AXIOM 4 (Continuity): *There is a constant K such that $h(\phi(m),\phi(m')) \leqslant Kd(m,m')$.*

Our main result is given by:

THEOREM: *There is one, and only one, map $\phi: M \to P(FA)$ that satisfies Axioms* 1, 2, 3, *and* 4. *It assigns to each m in M the set consisting of the competitive payoff of m.*

4. PREPARATIONS

Let F denote the set of all real-valued functions f on R^n_+ that are continuous, increasing, and satisfy $f(x) = o(\|x\|)$ as $\|x\| \to \infty$. For any f in the vector space generated by F (i.e., $f \in F - F$) let

$$\|f\| = \sup\left\{|f(x)| \Big/ \left(1 + \sum x\right) : x \in R^n_+ \right\} \tag{4.1}$$

where $\sum x$ denotes $\sum_{i=1}^n x_i$. Then $\|\ \|$ constitutes a norm. Let

$$F^1 = \{f \in F : \partial f/\partial x_j \text{ exists and is continuous at each } x \in R^n_+ \text{ for which } x_j > 0\},$$

$$F^0 = \{f \in F : f(0) = 0\}, \qquad F^c = \{f \in F : f \text{ is concave}\}.$$

Denote the ith unit vector $(0, \ldots, 0, 1, \ldots, 0)$ as e^i, and denote the sum of the unit vectors $\sum e^i = (1, \ldots, 1)$ in R^n_+ by e; the dimension will be clear from the context. For $\delta > 0$, let $I(x,\delta) = \{y : x \leqq y \leqq x + \delta e\}$. We define an operator $A^\delta : F \to F$, where for $f \in F$, the function $A^\delta f: R^n_+ \to R$ is given by

$$(A^\delta f)(x) = \delta^{-n} \int_{I(0,\delta)} f_y(x)\, dy$$

where $f_y(x) = f(x+y)$. Note that $A^\delta f$ is an average of translates of f.

LEMMA 4.2: *For any f in F, $A^\delta f \in F^1$ for every $\delta > 0$, and $\|A^\delta f - f\| \to 0$ as $\delta \to 0$. If $f \in F^c$, then $A^\delta f \in F^c$.*

PROOF: The translates f_y of f are increasing (and concave if $f \in F^c$) functions and thus $A^\delta f$ is an average of increasing (and concave if $f \in F^c$) functions and therefore $A^{\delta}f$ is increasing (and concave if $f \in F^c$). To show that $A^{\delta}f \in F^1$, let $x \in R^n_+$ with $x_i > 0$. Note that then $((A^{\delta}f)(x + \Delta e^i) - (A^{\delta}f)(x))/\Delta$ converges as $\Delta \to 0$ to

$$(4.3) \qquad \delta^{-n}\int_{I^i(x+\delta e^i,\delta)} f(y)\,dy - \delta^{-n}\int_{I^i(x,\delta)} f(y)\,dy$$

where $I^i(z,\delta)$ is the face $\{y \in R^n_+ : z \leqq y \leqq z + \delta(e - e^i)\}$ of the cube $I(z,\delta)$ and dy here means $dy_1, \ldots, dy_{i-1}dy_{i+1} \ldots dy_n$. Continuity of f implies that both summands of (4.3) are continuous in x, which shows that $A^{\delta}f \in F^1$.

To prove that $\|A^{\delta}f - f\| \to 0$ as $\delta \to 0$, note that for all $\delta > 0$ and f in F $0 \leq A^{\delta}f(x) - f(x) \leq f_{\delta e}(x) - f(x)$. As $f_{\delta e}$ converges to f uniformly on compact sets, we deduce that

$$\begin{aligned}\limsup_{\delta\to 0} \|A^{\delta}f - f\| &\leq \limsup_{\delta \to 0} \|f_{\delta e} - f\| \\ &\leq \max\{(f(x+e) - f(x))/(1 + \textstyle\sum x) : \|x\| \geq K\} \\ &\xrightarrow[K\to\infty]{} 0.\end{aligned}$$

Finite-type Economies

An economy $(\boldsymbol{a},\boldsymbol{u})$ in M is called finite utility (endowment)-type if there is a finite subfield $\mathscr{H}$ of $\mathscr{C}$ such that $\boldsymbol{u} : T \times R^n_+ \to R$ ($\boldsymbol{a} : T \to R^n_+$) is measurable with respect to the product σ-field $\mathscr{H} \times \mathscr{B}$ (the field $\mathscr{H}$). We then say for short that u is measurable with respect to $\mathscr{H}$. Note that every finite subfield $\mathscr{H}$ of $\mathscr{C}$ is identified with a partition of T into finitely many measurable sets $T_1, \ldots, T_p$ (the atoms of the finite subfield $\mathscr{H}$). A *uniform field* is a finite subfield $\mathscr{H}$ of $\mathscr{C}$ such that for every atom S of $\mathscr{H}$, $\mu(S) = 1/|\mathscr{H}|$, where $|\mathscr{H}|$ denotes the number of atoms of the field $\mathscr{H}$.

Approximation of Utilities

Let U_1 be the set of all functions from $T \times R^n_+$ to R which satisfy conditions (2.2), (2.3), (2.5)–(2.7). (Note that if $\boldsymbol{u} \in U_1$, then $\boldsymbol{u}^t \in F^0 \cap F^1$ for all $t \in T$). For $\delta > 0$, a δ-approximation of $\boldsymbol{u}$ is defined to be a member $\hat{\boldsymbol{u}}$ of U_1, such that $\|\hat{\boldsymbol{u}}^t - \boldsymbol{u}^t\| \leq \delta$ for all t except possibly a set of μ-measure $\leq \delta$, in which $\hat{\boldsymbol{u}}^t = \sqrt{\sum x}$. (Here $\|\ \|$ is the norm defined in (4.1).)

LEMMA 4.4: *Let $(\boldsymbol{a},\boldsymbol{u})$ be an economy in M. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that if $\hat{\boldsymbol{u}} \in U_1$ is a δ-approximation to $\boldsymbol{u}$, then*

$$\|v_{(\boldsymbol{a},\boldsymbol{u})} - v_{(\boldsymbol{a},\hat{\boldsymbol{u}})}\| < \epsilon.$$

PROOF: This is Proposition 40.24 of [**6**].

The next proposition is a simple variation of Proposition 35.6 of [**6**].

LEMMA 4.5: *For every $\delta > 0$ and $\boldsymbol{u} \in U_1$, there is a $\hat{\boldsymbol{u}} \in U_1$ that is a δ-approximation to $\boldsymbol{u}$ and is measurable with respect to a uniform field $\mathscr{H}$.*

PROOF: Proposition 35.6 of [**6**] asserts that there is $\bar{\boldsymbol{u}}$ in U_1 that is a $\delta/2$ approximation to $\boldsymbol{u}$ and is measurable with respect to a finite subfield $\mathscr{H}_0$. Let $|\mathscr{H}_0| = s$ and let r be a sufficiently large integer, such that $s/r < \delta/2$. For every atom T_i, $1 \leq i \leq s$, of $\mathscr{H}$ let $\overline{T}_i$ be a measurable subset of T_i with $\mu(T_i \setminus \overline{T}_i) < 1/r$ and $r\mu(\overline{T}_i)$ an integer. Set $\overline{T}_0 = T \setminus \bigcup_{i=1}^{s} \overline{T}_i$. Then $\mu(T_0) = k/r$ for some integer $k \leq s$ and thus $\mu(T_0) < \delta/2$. Define

$$\hat{\boldsymbol{u}}^t = \begin{cases} \bar{\boldsymbol{u}}^t & \text{if } t \notin \overline{T}_0, \\ \sqrt{\sum x} & \text{if } t \in \overline{T}_0. \end{cases}$$

Then, obviously $\hat{\boldsymbol{u}}$ is a δ-approximation to $\boldsymbol{u}$. Let $\mathscr{H}$ be a uniform field with $|\mathscr{H}| = r$ and $\overline{T}_i \in \mathscr{H}$, $0 \leq i \leq s$. Then $\hat{\boldsymbol{u}}$ is measurable with respect to $\mathscr{H}$ which completes the proof of Lemma 4.5.

For each $n > 0$, a real-valued function f on R_+^n is called a *market function* if it is concave, continuous, 1-homogeneous[5] and nondecreasing; *smooth*, if $\partial f/\partial x_i$ exists and is continuous whenever $x_i > 0$.

If $J = \{T_1, \ldots, T_q\}$ is an ordered measurable partition of T, then the vector measure μ^J induced by μ and J is defined by

$$\mu_i^J(S) = \mu(T_i \cap S) \qquad (i = 1, \ldots, q).$$

LEMMA 4.6: *Let $J = \{T_1, \ldots, T_q\}$ be an ordered measure partition of T, and let $u \in U_1$ be measurable with respect to J. Then there is a smooth market function $g : R_+^q \times R_+^n \to R$ such that for all integrable $\boldsymbol{a} : T \to R_+^n$*

$$v_{(\boldsymbol{a},\boldsymbol{u})}(S) = g\big(\mu^J(S), \boldsymbol{a}(S)\big).$$

Proof is accomplished by piecing together results from Propositions 36.3, 39.13, and Lemmas 39.8, 39.9 of [**6**]; or alternatively, by Lemma 39.16 of [**6**] and its proof (specifically 39.18).

5. PROOF OF THE THEOREM

First, we show (Proposition 5.1) that there are economies of a special kind that are dense in M, and then (Proposition 5.12) that for any such economy m, $\phi(m)$ is a nonempty one-element set that is uniquely determined by Axioms 1, 2, 3, and 4. Uniqueness of ϕ on all of M follows from the continuity Axiom 4.

[5]This means homogeneous of degree 1.

Define a subset M^* of M by: $m \in M^*$ iff there are mutually singular probability measures $\mu_1, \ldots, \mu_p$ with $\sum \mu_i = p\mu$ and a function $f:[0,1]^p \to R$ such that (i) $f \circ (\mu_1, \ldots, \mu_p) = v_m$; (ii) f is linear on a conical neighborhood of the diagonal, i.e., there are $\beta > 0$ and $\alpha_1, \ldots, \alpha_p$ in R such that $f(x) = \sum \alpha_i x_i$ on $\{x \in [0,1]^p : |x_i - x_j| < \beta \sum x\}$.

PROPOSITION 5.1: *M^* is dense in M (i.e., for any m in M and $\epsilon > 0$ there is m^* in M^* with $d(m^*, m) < \epsilon$).*

PROOF: Let $m = (\boldsymbol{a}, \boldsymbol{u}) \in M$. By Lemma 4.4 and 4.5, there is a $\hat{\boldsymbol{u}}$ (a δ-approximation to $\boldsymbol{u}$) that is measurable with respect to a uniform field $\mathscr{H}_0$, such that $d(m, (\boldsymbol{a}, \hat{\boldsymbol{u}})) < \epsilon/3$. The proof of Proposition 5.1 proceeds via two steps. First, we approximate $(\boldsymbol{a}, \hat{\boldsymbol{u}})$ by an economy $m = (\hat{\boldsymbol{a}}, \hat{\boldsymbol{u}})$ where $\hat{\boldsymbol{a}}$ is measurable with respect to a uniform field $\mathscr{H}_1 \supset \mathscr{H}_0$. We then approximate $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{u}})$ with $m^* = (\hat{\boldsymbol{a}}, \boldsymbol{u}^*)$ where $\boldsymbol{u}^*$ is measurable with respect to $\mathscr{H}_0$ and satisfies some additional properties that guarantee that m^* is in M^*.

For f defined on R^n_+ and $b \in R^n_+$ write

$$\|f\|_b = \sup \sum_{j=1}^{k} |f(x_j) - f(x_{j-1})|,$$

where the sup is over all finite sequences of points

$$0 = x^0 \leq x^1 \leq \cdots \leq x^k = b;$$

note that for all vectors ζ of nonnegative measures,

$$\text{(5.2)} \qquad \|f \circ \zeta\| \leq \|f\|_b \qquad \text{where} \quad b = \zeta(T).$$

We use the C^1-norm $\|f\|_b^1$ for f that are C^1 on $\{y \in R^n_+ : 0 \leqq y \leqq b\}$ (cf. [**6**, p. 42]). Note that

$$\text{(5.3)} \qquad \|f\|_b \leq \|f\|_b^1 (\textstyle\sum b)$$

(cf. (7.5) of [**6**]). If ξ is a vector measure, write $\|\xi\|$ for $\max_i \|\xi_i\|$ where $\| \ \|$ denotes the variation norm.

LEMMA 5.4: *Let $\xi, \xi^1, \xi^2 \ldots$ be q-dimensional vectors of nonnegative measures such that $\|\xi^\beta - \xi\| \to 0$ as $\beta \to \infty$ and $\xi^\beta(T) = \xi(T)$ for all $\beta = 1, 2, \ldots$. Let f be a smooth market function on R^q_+. Then $\|f \circ \xi^\beta - f \circ \xi\| \to 0$ as $\beta \to \infty$.*

PROOF: W.l.o.g. $\xi_i(T) \leqq 1$ and $\xi_i^k(T) \leqq 1$, for all i and k, and let $b = \xi(T)$. Let $\epsilon > 0$ be given. For each $\delta > 0$ and $x \in R^q_+$, define $B^\delta x = (x + \delta e)/(1 + 2\delta)$ and $f^\delta(x) = f(B^\delta x) - f(B^\delta 0)$. By Proposition 10.7 of [**6**], $\|f^\delta - f\|_b \to 0$ as $\delta \to 0$; in particular, there is a $\delta > 0$ with

$$\|f^\delta - f\|_b < \epsilon.$$

In general, f^δ is not 1-homogeneous, and hence not a market function, but it is C^1 on R^q_+, and in particular on $[0,1]^q$. Now in the C^1-norm, the polynomials are dense in the C^1 functions on $[0,1]^q$ (cf. Lemma 7.4 of **[6]**); hence by (5.3) there is a polynomial f^* such that

$$\|f^* - f^\delta\|_b \leqq q\|f^* - f^\delta\|_b^1 < \epsilon.$$

Since addition and multiplication are continuous in the variation norm (Proposition 4.5 of **[6]**),

$$\|f^* \circ \xi^\delta - f^* \circ \xi\| \to 0 \qquad \text{as} \quad \beta \to \infty.$$

Using the triangle inequality, (5.2), and the three above displays, we find

$$\limsup_{\beta\to\infty} \|f \circ \xi^\delta - f \circ \xi\| < 4\epsilon.$$

As ϵ may be chosen arbitrarily, the proof of the Lemma is complete.

Let $(\mathscr{H}_i)_{i=1}^\infty$ be an increasing sequence of uniform fields that generates $\mathscr{C}$. Then[6] $\|E(\boldsymbol{a} \mid \mathscr{H}_i) - \boldsymbol{a}\| \to 0$ as $i \to \infty$, and therefore by Lemmas 4.6 and 5.4 there is a uniform field $\mathscr{H} \supset \mathscr{H}_0$ such that $d((\boldsymbol{a}, \hat{\boldsymbol{u}}), (E(\boldsymbol{a} \mid \mathscr{H}), \hat{\boldsymbol{u}})) < \epsilon/3$. Without loss of generality we may assume that $\boldsymbol{u}^t$ is concave (cf. **[6**, Proposition 36.3]) for all $t \in T$. Let $p = |\mathscr{H}|$ and let $T_1, \dots, T_p$ be the atoms of the uniform field $\mathscr{H}$. Let f^i, $1 \le i \le p$, be the utility $\boldsymbol{u}^t$ where $t \in T_i$. By Proposition 36.4 and Lemma 39.8 of **[6]**, there are $z^1, \dots, z^p$ in R^n_+ and $c > 0$ in R^n_+ such that

$$f^i(x) \le f^i(z^i) + c \cdot (x - z) \qquad \text{for all } x \text{ in } R^n_+$$

and

$$\sum_{i=1}^n \mu(T_i) z^i = \hat{\boldsymbol{a}}(T) = \sum_{i=1}^n \mu(T_i) a^i \qquad \text{where} \quad a^i = \hat{\boldsymbol{a}}(t) \quad \text{for} \quad t \in T_i.$$

Let $\delta > 0$ be such that for any 4δ-approximation $\boldsymbol{u}^*$ to $\hat{\boldsymbol{u}}$, $d((\hat{\boldsymbol{a}}, \boldsymbol{u}^*), (\hat{\boldsymbol{a}}, \hat{\boldsymbol{u}})) < \epsilon/3$. Define $h^i_0 : R^n_+ \to R$ by

$$h^i_\delta(x) = \min\{f^i(x), f^i(z^i) + c(x - z) - \delta\}.$$

As h^i_δ is the minimum of finitely many increasing concave continuous functions, $h^i_\delta \in F^c$. The inequalities $0 \le f^i(x) - h^i_\delta(x) \le \delta$ imply that $\|f^i - h^i_\delta\| < \delta$ ($\|f\| \le \sup_x\{|f(x)|\}$). The continuity of both $f^i(x)$ and $f^i(z^i) - \delta + c(x - z^i)$ (in x) and the strict inequality $f^i(z^i) > f^i(z^i) - \delta + c(z^i - z^i)$ imply that $h^i(x) = f^i(z^i) - \delta + c(x - z^i)$ in a neighborhood N^i of z^i in R^n_+. Therefore for sufficiently small $\epsilon > 0$, $A^\epsilon h^i_\delta(x) = f(z^i) - \delta + c(x + \epsilon e/2 - z^i)$ in a neighborhood $\overline{N}^i \subset N^i$ of z^i in R^n_+ and $\|A^\epsilon h^i_\delta - h^i_\delta\| < \delta$. For such an $\epsilon > 0$ set g^i

[6] $E(\boldsymbol{a} \mid \mathscr{H})$ is the conditional expectation of $\boldsymbol{a}$ with respect to $\mathscr{H}$, i.e., a function from T into R^n_+ that is measurable with respect to $\mathscr{H}$ and for which $\int_S E(\boldsymbol{a} \mid \mathscr{H})\, d\mu = \int_S \boldsymbol{a}\, d\mu$ for all S in $\mathscr{H}$.

$= A^{\epsilon}h^i_{\delta} - A^{\epsilon}h^i_{\delta}(0)$. Then $g^i \in F^1 \cap F^c \cap F^0$ and $\| g^i - f^i \| < 4\delta$. Also

$$(*) \qquad g^i(x) \leq g^i(y^i) + c(x - y^i) \qquad \text{for all } y^i \text{ in } \overline{N}^i \text{ and all } \quad x \in R^n_+ .$$

In particular $g^i(y^i) = g^i(z^i) + c(y^i - z^i)$ for all $y^i \in \overline{N}^i$. Set $\boldsymbol{u}^* : T \times R^n_+ \to R$ by $\boldsymbol{u}(t,x) = g^i(x)$ if $x \in T_i$. Then $\boldsymbol{u}^*$ is a 4δ-approximation to $\hat{\boldsymbol{u}}$ and thus letting $m^* = (\hat{a}, \boldsymbol{u}^*)$, we have

$$d(m^*, m) < d(m^*, \hat{m}) + d(\hat{m}, m) < \epsilon.$$

Denote by J the ordered partition $\{T_1, \ldots, T_p\}$ where $T_1, \ldots, T_p$ are the atoms of $\mathscr{H}$. As $0 < \hat{a}(T) = \sum_{i=1}^p (1/p) a^i \in \sum_{i=1}^p (1/p) \overline{N}^i$ it follows that there is $\beta > 0$ such that for all $S \subset T$ with $|\mu_i^J(S) - 1/p| < \beta$ there are y^i in $\overline{N}^i$, $1 \leqq i \leqq p$ with

$$\sum \mu_i^J(S) y^i = \sum \mu_i^J(S) a^i = \hat{\boldsymbol{a}}(S).$$

Thus by Proposition 36.4 of [**6**] and (*)

$$\begin{aligned} v_{m^*}(S) &= \sum_{i=1}^p \mu_i^J(S) g^i(y^i) \\ &= \sum_{i=1}^p \mu_i^J(S)\big(g^i(z^i) + c(y^i - z^i)\big) \\ &= \left(\sum_{i=1}^p \mu_i^J(S) g^i(z^i)\right) + c(\hat{\boldsymbol{a}}(S)) - \sum \mu_i^J(S)(c \cdot z^i) \\ &= \sum_{i=1}^p \mu_i^J(S)\big[g^i(z^i) - cz^i\big] + \sum_{i=1}^p c a^i \mu_i^J(S) \\ &= \sum_{i=1}^p \mu_i^J(S)\big[g^i(z^i) - cz^i + ca^i\big]. \end{aligned}$$

This completes the proof of Proposition 5.1.

LEMMA 5.5: *Let $\bar{\mu} = (\mu_1, \ldots, \mu_p)$ be a vector of mutually singular probability measures with $\sum \mu_i = p\mu$. Let $h : R^p_+ \to R$ be a 1-homogeneous function with $\|h \circ \bar{\mu}\| < \infty$, and suppose that h vanishes on a conical neighborhood of the diagonal $\{te : 0 \leq t \leq 1\}$, i.e., there is $\epsilon > 0$ such that $h(x) = 0$ on $\{x \in [0,1]^P : |x_i - x_j| < \epsilon \sum x\}$. Then there is a μ-measure preserving automorphism θ such that*

$$\sup \left\| \sum_{i=l} a_i \theta^i (h \circ \bar{\mu}) \right\| \Big/ \sqrt{l} \to 0, \qquad \textit{as} \quad l \to \infty$$

where the supremum is taken over all sets of numbers a_i, $1 \leq i \leq l$, with $|a_i| \leq 1$.

PROOF: Without loss of generality, $\epsilon < 1$. Set $\beta = 1 - \epsilon/4$. Let $N > 0$ be the smallest positive integer with $N^3 > l$. Let Ω be a fixed chain. We proceed to bound from above the variation of $\sum_{i<l} a_i \theta^i (h \circ \bar{\mu})$ over Ω where $|a_i| \leqslant 1$. As $\| \|_\Omega$ is monotonic in Ω, we may assume without loss of generality (by possibly adding additional elements to Ω) that for every $0 \leq m \leq N$ there is S^m in Ω with $\mu(S^m) = \beta^m \mu(T) = \beta^m$ and $S^0 = T$. For $0 \leq m < N$, we denote by Ω_m the subchain of Ω of all S in Ω with $S^{m+1} \subseteq S \subseteq S^m$ and by Ω_N the subchain of all S in Ω with $S \subseteq S^N$. Then

$$(5.6) \qquad \left\| \sum_{i<l} a_i \theta^i (h \circ \bar{\mu}) \right\|_\Omega = \sum_{m \leq N} \left\| \sum_{i<l} a_i \theta^i (h \circ \bar{\mu}) \right\|_{\Omega_m}$$
$$\leq \sum_{m \leq N} \sum_{i<l} \| a_i \theta^i (h \circ \bar{\mu}) \|_{\Omega_m} \leq \sum_{m \leq N} \sum_{i<l} \| h \circ \bar{\mu} \|_{\theta^i \Omega_m} .$$

For all subchains Ω', $\| h \circ \bar{\mu} \|_{\Omega'} \leq \| h \|_{\bar{\mu}(\hat{S})}$ where $\hat{S}$ is the maximal element in the subchain Ω'. As $\| h \|_b$ is nondecreasing in b, the vector inequality $\bar{\mu}(\hat{S}) \leq p\mu(\hat{S})$ $\bar{\mu}(T)$ implies that $\| h \circ \bar{\mu} \|_{\Omega'} \leq \| h \|_{p\mu(\hat{S})\bar{\mu}(T)}$. As h is 1-homogeneous, so is $\| h \|_b$ (as a function of b) and therefore $\| h \circ \bar{\mu} \|_{\Omega'} \leq p\mu(\hat{S}) \| h \|_{\bar{\mu}(T)}$, for all subchains Ω'. In particular, if θ is a μ-measure preserving automorphism,

$$(5.7) \qquad \| h \circ \bar{\mu} \|_{\theta^k \Omega_m} \leq p\mu(S^m) \| h \|_{\bar{\mu}(T)} = p\beta^m \| h \|_{\bar{\mu}(T)} .$$

For $\alpha > 0$, let

$$D_\alpha = \left\{ S \in \mathscr{C} : | \mu_i(S) - \mu_j(S) | \leq \alpha \sum_{k=1}^{p} \mu_k(S) = \alpha p \mu(S) \right.$$
$$\left. \text{for all } 1 \leq i,j \leq p \right\}.$$

Then $S \in D_\epsilon$ implies that $(h \circ \bar{\mu})(S) = 0$ and thus, for each fixed $m < N$ and any k, if $\theta^k \Omega_m \subset D_\epsilon$, then $\| h \circ \bar{\mu} \|_{\theta^k \Omega_m} = 0$. Together with (5.7) we deduce that

$$(5.8) \qquad \sum_{m \leq N} \sum_{i<l} \| h \circ \bar{\mu} \|_{\theta^i \Omega_m} \leq l p \beta^N \| h \|_{\bar{\mu}(T)} + \sum_{m<N} p\beta^m \| h \|_{\bar{\mu}(T)} \sum_{i<l} I(\theta^i \Omega_m \not\subset D_\epsilon)$$

where $I(\theta^i \Omega_m \not\subset D_\epsilon) = 1$ if $\theta^i \Omega_m \not\subset D_\epsilon$ and 0 otherwise. Note that for a μ-measure preserving automorphism θ, if $\theta^k S^m \in D_{\epsilon/2}$ for some $m < N$, then for every S in Ω_m and all i, j, $1 \leq i < j \leq p$,

$$| \mu_i(\theta^k S) - \mu_j(\theta^k S) | \leq | \mu_i(\theta^k S^m) - \mu_j(\theta^k S^m) | + p\mu(S)(\beta^{-1} - 1)$$
$$\leq (\epsilon/2) p\mu(S^m) + p\mu(S)(\beta^{-1} - 1)$$
$$\leq (\epsilon/2) p\beta^{-1}\mu(S) + p\mu(S)(\beta^{-1} - 1)$$
$$= p\mu(S)(\epsilon\beta^{-1}/2 + \beta^{-1} - 1) < \epsilon p \mu(S)$$

and thus $\theta^k \Omega_m \subset D_\epsilon$. Therefore, for any $m < N$ and a μ-measure preserving automorphism θ,

$$\sum_{i<l} I(\theta^i \Omega_m \not\subset D_\epsilon) \leq \#\{k : \theta^k S^m \notin D_{\epsilon/2}\}. \tag{5.9}$$

We proceed to show that there is a μ-measure preserving transformation θ for which the right hand side of (5.9) is at most $4(p-1)\beta^{-m}/\epsilon^2$. Set $\nu = p\mu = \mu_1 + \cdots + \mu_p$. For f and g in the Hilbert space, $L_2(\nu)$, we denote by $\langle f, g\rangle$ the inner product $\int f(t)g(t)\,d\nu(t)$. Let $f_i = d\mu_i/d\nu$. Then $0 \leq f_i \leq 1$ a.e. and thus $f_i \in L_2(\nu)$ and $\|f_i\|_2^2 = \langle f_i, f_i\rangle = 1$ and $\|f_i - f_j\|_2^2 = 2$ for $i \neq j$. Let θ be a μ-measure preserving transformation such that for all $1 \leq i < j \leq p$ $((1/\sqrt{2})(f_i - f_j) \circ \theta^k)_{k=0}^{\infty}$ is an orthonormal sequence in $L_2(\nu)$. To see that such a μ-measure preserving transformation exists identify $(T, \mathscr{C}, \nu)$ with the product measure space Ω^Z (Z-the integers, Ω a finite measure space with p elements and the counting measure) and identify f_i with $I(\omega(0) = i)$, and let θ be the shift operator on Ω^Z. Observe that $S \in D_\alpha$ iff $|\langle \chi_S, f_i - f_j\rangle| \leq \alpha\nu(S)$ for all $1 \leq i < j \leq p$ and that $\theta^k S \in D_\alpha$ iff $|\langle \chi_S, (f_i - f_j) \circ \theta^k\rangle| \leq \alpha\nu(S)$. By Parseval's inequality

$$\sum_{0 \leq k} \langle \chi_S, (f_i - f_j) \circ \theta^k\rangle^2 \leq 2\|\chi_S\|_2^2 = 2\nu(S).$$

Thus, for any $1 \leq i < j \leq p$,

$$\#\{k : |\langle \chi_S, (f_i - f_j) \circ \theta^k\rangle| > \epsilon\nu(S)/2\} \leq 8\epsilon^{-2}/\nu(S),$$

and therefore

$$\#\{k : \theta^k S \notin D_{\epsilon/2}\} \leq \left(\frac{p(p-1)}{2}\right) 8\epsilon^{-2}/\nu(S), \tag{5.10}$$

which in particular implies that for every $m < N$

$$\#\{k : \theta^k S^m \notin D_{\epsilon/2}\} \leq 4(p-1)\beta^{-m}/\epsilon^2. \tag{5.11}$$

Combining (5.6), (5.8), (5.9) and (5.11) we deduce that

$$\left\|\sum_{i<l} a_i \theta^i (h \circ \bar{\mu})\right\|_\Omega \leq lp\beta^N \|h\|_{\bar{\mu}(T)} + \sum_{m<N} 4p(p-1)\epsilon^{-2}\|h\|_{\bar{\mu}(T)}$$
$$= \left(lp\beta^N + 4p(p-1)\epsilon^{-2}N\right)\|h\|_{\bar{\mu}(T)}.$$

As this bound is independent of Ω, $(l\beta^N + n)/\sqrt{l} \to 0$ as $l \to \infty$, completing the proof of Lemma 5.5.

PROPOSITION 5.12: *Assume $m \in M^*$. Let $\bar{\mu} = (\mu_1, \ldots, \mu_p)$ be a vector of mutually singular probability measures with $\sum \mu_i = p\mu$ and let $f: [0,1]^p \to R$ be such that $f \circ \bar{\mu} = v_m$ and assume that $f(x) = \sum \alpha_i x_i$ on a conical neighborhood of the diagonal. Then for any $\phi: M^* \to P(FA)$ satisfying Axioms* 1, 2, 3, *and* 4, $\phi(m) = \{\sum \alpha_i \mu_i\}$.

PROOF: Let $u \in \phi(m)$ and let $w = \sum \alpha_i \mu_i$. By **[16]** there is a universal constant K such that for every automorphism θ of the measurable space and every positive integer l there are real numbers a_i^l with $|a_i^l| = 1$, $1 \leq i \leq l$ and $\|\sum_1^l a_i \theta^i (u - w)\| \geq 2K\sqrt{l}\,\|u - w\|$. By the triangle inequality it follows that there are numbers $a_i^l \in \{0, 1\}$, $1 \leq i \leq l$, with

$$(5.13) \qquad \left\| \sum_1^l a_i \theta^i (u - w) \right\| \geq K\sqrt{l}\,\|u - w\|.$$

The vector measure $\bar{\mu}$ and the function $h : R_+^p \to R$ given by $h(x) = f(x) - \sum \alpha_i x_i$ obeys the conditions of Lemma 5.5 and $h \circ \bar{\mu} = f \circ \bar{\mu} - w$ and thus there is a μ-measure preserving automorphism θ for which

$$(5.14) \qquad \left\| \sum_{i=1}^l a_i^l \theta^i (f \circ \bar{\mu} - w) \right\| \Big/ \sqrt{l} \to 0 \qquad \text{as} \quad l \to \infty.$$

Consider the two set functions:

$$v_l = \sum_{k<l} a_k^l \theta^k (f \circ \bar{\mu})$$

and

$$w_l = \sum_{k<l} a_k^l \theta^k w.$$

It is easy to verify that w_l is the characteristic function of some inessential economy $\overline{m}_l$ and thus $\phi(\overline{m}_l) = \{w_l\}$. It is also easy to verify that v_l is the characteristic function of the market

$$m_l = \sum_{k\,:\,a_k^l = 1} \oplus \theta^k m.$$

By the separability and anonymity axioms

$$(5.15) \qquad \sum_{k=1}^l a_k^l \theta^k u \in \phi(m_l).$$

By (5.14), it follows that

$$(5.16) \qquad d(m_l, \overline{m}_l)/\sqrt{l} \to 0 \qquad \text{as} \quad l \to \infty$$

while from (5.13) and (5.15), it follows that

$$d(\phi(m_l), \phi(\overline{m}_l)) \geq \left\| \sum a_k^l \theta^k u - w_l \right\| \geq K\sqrt{l}\,\|u - w\|$$

which together with (5.16), contradicts the continuity axiom unless $u = w$ which completes the proof of Proposition 5.12.

6. TIGHTNESS OF THE ASSUMPTIONS

In this section we discuss the tightness of the assumptions of our theorem, and variants of our theorem.

Since a part of the continuity axiom requires that the map ϕ from M to $P(FA)$ satisfy:

(CF) $\phi(m)$ depends only on v_m,

any such ψ is factorized with the map v from M to $E = \{v_m : m \in M\}$ which sends m to v_m. In other words, there is $\psi : E \to P(FA)$ such that $\phi(m) = \psi(v_m)$. Note that for w in E and θ in Q_μ the set function $\theta w : \mathscr{C} \to R$ that is given by $\theta w(S) = w(\theta S)$ is in E. For m and m' in M, $v_{m \oplus m'} = v_{m'} + v_m$, hence E is closed under addition. Also, any finitely additive set function in E is the characteristic function of an inessential economy. Therefore our theorem implies (and, in fact, is implied by the fact) that there is a unique map $\psi : E \to P(FA)$ such that for any v, w in E and θ in Q_μ:

(6.1) $\psi(v) + \psi(w) = \psi(v + w)$,

(6.2) $\psi(\theta w) = \theta\psi(w)$,

(6.3) $w \in E \cap FA \Rightarrow \psi(w) = \{w\}$,

(6.4) there is a K such that $h(\psi(v), \psi(w)) \leq K\|v - w\|$.

It turns out that this unique map also satisfies the following additional properties, which are common to many solutions:

(6.5) $\nu \in \psi(v) \Rightarrow \nu(T) = v(T)$ (efficiency),

(6.6) $v - w$ monotonic (non-decreasing) $\Rightarrow \psi(v) \subset \psi(w) + FA_+$ (positivity),

(6.7) for all $\lambda > 0$, $\psi(\lambda v) = \lambda\psi(v)$ (rescaling),

(6.8) $w \in E \cap FA \Rightarrow \psi(v + w) = \psi(v) + \psi(w)$.

The map ψ satisfies also the following (value) properties:

(6.9) ψ is a point-to-point map ($|\psi(v)| = 1$);

(6.10) ψ has an extension to the linear span $E - E$ of E in such a way that (6.1)–(6.9) still hold[7] (now λ in (6.7) may be negative).

In our definition of M the population measure μ was held fixed. If we let it vary over all possible nonatomic measures to obtain the larger class of economies

[7] Obviously such an extension is unique.

N ($M \subset N$), then the same theorem holds on the domain N. (This follows trivially from our theorem.) Let G denote the set of all characteristic functions that arise from N, and $\mathscr{A}$ the set of all automorphisms (i.e., bimeasurable bijections) on $(T, \mathscr{C})$. In conjunction with the existence of a value with norm 1 on pNA [**6**, Theorem B] and the inclusion $G \subset pNA$ [**6**, Theorem J] our theorem implies that there is[8] a unique map $\psi: G - G \to FA$ which for any v, w in $G - G$ and for every θ in $\mathscr{A}$ satisfies (6.1)–(6.9). Though this is a weaker[9] result, we should point out that we are unable to prove it in a significantly simpler way.

The assumptions of our theorem fall into two categories: conditions imposed on the domain M (mainly nonatomicity and differentiability), and the axioms themselves.

The nonatomicity is clearly indispensable (in economies with finitely-many agents and concave utilities, the core and the value obey our axioms but are well known to differ). What if we drop differentiability?

Let M' (N') be the larger domain of economies that is obtained by dropping the differentiability assumption (2.7) on the utilities of the economies in M (in N). Let E' (G') be the set of characteristic functions obtained from M' (N'). Is there a map from G' into FA satisfying (6.1)–(6.9)? This was an open problem (even without the requirement (6.4)) which was only recently resolved when J. F. Mertens [**15**] showed the existence of a continuous value (actually a value of norm 1) on a large space of characteristic functions which includes G'. Consider the two maps on M':

$$m \to \text{the Mertens value of } v_m,$$

$$m \to \text{core of } v_m.$$

Both of these maps obey anonymity, separability, continuity, and the inessential economy axiom.[10] It is well known that for m in M' the core of v_m no longer necessarily consists of a single point (though it still coincides with the competitive payoffs), and thus the two differ. We conclude that our theorem breaks down on M'. Restricting the domain to M'', made up of those economies in M' with finitely many utility types and with $\boldsymbol{a}$ bounded (or, more generally, in $L^2(\mu)$), we could still obtain two different (single-valued) maps from M'' to FA that satisfy our axioms: the "Mertens value" and the μ-measure based value defined by S. Hart [**12**].

Dropping any one of our axioms makes the theorem break down as the following examples show.

EXAMPLE 6.11: The egalitarian solution $\psi: M \to FA$ that is given by

$$\Psi(m) = v_m(T)\,\mu$$

[8] Dropping (6.6), (6.8) and any one of (6.3) or (6.5) would not affect this result.
[9] We refer here to the uniqueness.
[10] They obey also the "equivalents" of (6.5)–(6.8).

obeys continuity, separability, anonymity, positivity, and efficiency but violates the inessential economy axiom.

EXAMPLE 6.12: For every m in M define $\alpha(m)$ by $\alpha(m) = \sup\{v_m(S_1 \cup S_2) - v_m(S_1) - v_m(S_2) : \text{all disjoint subsets } S_1, S_2 \in \mathscr{C}\}$. Let $\psi : E \to FA$ denote the value operator. Let $\phi : M \to P(FA)$ be given by

$$\phi(m) = \{\nu \in FA_+ : \|\nu - \psi v_m\| \leq \alpha(m), \nu(T) = v_m(T)\}.$$

Then ϕ obeys continuity, anonymity, efficiency, and the inessential economy axiom, but violates separability.

EXAMPLE 6.13: Let $\{T_1, T_2\}$ be a partition of T into two measurable sets. In this example we use the extension operator (cf. [**6**, Ch. IV]) that associates with each characteristic function v an ideal game v^*, i.e., a real valued function defined on the ideal sets (i.e., measurable function from T to $[0, 1]$) with $v^*(0) = 0$. For ideal sets f, g we define $\partial v^*(f, g)$ by

$$\partial v^*(f, g) = \lim_{h \to 0} \frac{v^*(f + hg) - v^*(f)}{h}.$$

For S in $\mathscr{C}$ we denote the ideal set χ_S by S. It is possible to prove that for almost all $0 \leq t \leq 1$, for all $S \in \mathscr{C}$, both derivatives

$$\partial v^*(tT_1, S \cap T_1) \quad \text{and}$$

$$\partial v^*(T + tT_2, S \cap T_2)$$

exist and are integrable. Define $\psi : M \to FA$ by

$$\psi(m)(S) = \int_0^1 \partial v^*(tT_1, S \cap T_1)\, dt + \int_0^1 \partial v^*(T_1 + tT_2, S \cap T_2)\, dt.$$

Then ψ obeys continuity (constant $K = 1$), positivity, efficiency, separability, and the inessential economy axiom, but violates anonymity.

The continuity axiom appears in the form of a Lipschitz condition. If we added the rescaling axiom (for all $(\boldsymbol{a}, \boldsymbol{u}) \in M$ and $\lambda > 0$, $\phi(\boldsymbol{a}, \lambda \boldsymbol{u}) = \lambda \phi(\boldsymbol{a}, \boldsymbol{u})$) it reduces to a uniform continuity which can be stated as follows: for every $\epsilon > 0$ there is $\delta > 0$ such that for all $m, \hat{m}$ in M, $d(m, \hat{m}) < \delta$ implies $h(\phi(m), \phi(\hat{m})) < \epsilon$. It would further reduce to just continuity if we were to assume additivity instead of separability. However, this would clash with our intention of using axioms expressing characteristics that are common across a wide range of solutions.

The uniform continuity is used in the proof only for pairs $m, \hat{m}$ where m is an inessential economy. This last fact, in turn, can be inferred from just continuity if we maintain the rescaling axiom and strengthen the inessential economy axiom in the following way: there are $\epsilon > 0$ and $\delta > 0$ such that for all m in M, if

$v_m(S_1 \cup S_2) - v_m(S_1) - v_m(S_2) < \delta$ whenever S_1, S_2 are in $\mathscr{C}$, and $\nu \in \phi(m)$, then for all S in $\mathscr{C}$, $|\nu(S) - v_m(S)| < \epsilon$.

The continuity axiom, in the presence of efficiency, is closely related to positivity and one should wonder whether it is possible to replace our continuity axiom by positivity and efficiency. Before taking up this inquiry we would like to first express more formally the relation between positivity and our Lipschitz form of continuity. This is established via the internal norm on the space generated by a cone Q of increasing characteristic functions: if $v, u \in Q$,

$$\|v - u\|_{\text{int}} = \inf\{\alpha : v_\alpha, u_\alpha \in Q \text{ with } v_\alpha(T) + u_\alpha(T) = \alpha \text{ such that both } u + u_\alpha - v \text{ and } v + v_\alpha - u \text{ are monotonic}\}.$$

PROPOSITION 6.14: *Let Q be a cone of monotonic characteristic functions and let $\psi : Q \to P(FA)$ be a nonempty set-valued superadditive, positive and efficient map. Then for all v, u in Q*

$$h(\psi(v), \psi(u)) \leq \|v - u\|_{\text{int}}.$$

PROOF: Let $v, u \in Q$ and $\alpha > 0$ be given. Let v_α and u_α be in Q and assume that (i) $v_\alpha + v - u$ is monotonic, (ii) $u_\alpha + u - v$ is monotonic, (iii) $v_\alpha(T) + u_\alpha(T) = \alpha$. Note that it is sufficient to prove that for every $\nu \in \psi(v)$ there is $\eta \in \psi(u)$ such that $\|\nu - \eta\| \leq 4\alpha$. Let $\nu \in \psi(v)$ and $\gamma \in \psi(v_\alpha)$. By superadditivity $\nu + \gamma \in \psi(v + v_\alpha)$. The positivity of ψ together with (i) implies the existence of η in $\psi(u)$ such that $\nu + \gamma - \eta$ is monotonic. In particular, for every $S \in \mathscr{C}$, $\gamma(S) + \nu(S) \geq \eta(S)$ and

$$\nu(T) - \nu(S) + \gamma(T'S) = \gamma(T'S) + \nu(T'S) \geq \eta(T'S)$$
$$= \eta(T) - \eta(S) \geq \nu(T) - \alpha - \eta(S)$$

which implies that

$$\alpha \geq \gamma(T) \geq \gamma(S) \geq \eta(S) - \eta(S) \geq -2\alpha.$$

As $\|\eta - \nu\| \leq 2\sup|\eta(S) - \nu(S)| \leq 4\alpha$ the result follows.

COROLLARY 6.15: *If Q is a cone for which the internal norm on $Q - Q$ is equivalent to the bounded variation norm, then any nonempty set-valued map $\psi : Q \to P(FA)$ that is efficient, positive, and superadditive is continuous.*

In spite of this close connection (between positivity and continuity) it turns out that it is impossible to replace the continuity assumption by (CF), positivity and efficiency.

EXAMPLE 6.16: The operator $\bar{\phi}: M \to FA$ that is given by

$$(\bar{\phi}(m))(S) = \int_S u(t, a(t))\, d\mu(t),$$

where $m = (a, u)$, obeys (CF), separability, anonymity, positivity, and the inessential economy axiom, but violates continuity and efficiency.

The mapping $\phi: M \to P(FA)$ given by

$$\phi(m) = \{\nu \in FA : \nu \geq \bar{\phi}(m), \nu(T) = v_m(T)\},$$

obeys separability, anonymity, inessential economy, positivity, and efficiency, but violates continuity.

To see that it is positive, let m, m' be in M with $v_m - v_{m'}$ monotonic. Let $\nu \in \phi(m)$. Then $\nu \geq \bar{\phi}(m')$ and thus for all $0 \leq t \leq 1$, $\nu \geq t\nu + (1-t)\bar{\phi}(m') \geq \bar{\phi}(m')$. For $0 \leq t \leq 1$ with $t\nu(T) + (1-t)\bar{\phi}(m')(T) = v_{m'}(T)$, $t\nu + (1-t)\bar{\phi}(m') \in \phi(m')$ which shows positivity.

An implication of the last example is that the internal norm on E (characteristic functions arising from M) is not equivalent to the bounded variation norm.

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