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Vol. 42, No. 1, June 1987
Printed in Belgium

Power and Public Goods*

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Received February 2, 1983; revised April 29, 1986

A game-theoretic analysis using the Harsanyi–Shapley nontransferable utility value indicates that the choice of public goods in a democracy is not affected by who has voting rights. This is corroborated by an independent economic argument based on the implicit price of a vote. *Journal of Economic Literature* Classification Numbers: 020, 321, 323. © 1987 Academic Press, Inc.

1. INTRODUCTION

Taxation has two related but distinct primary[1] purposes: redistribution and the provision of public goods. Most treatments of taxation have assumed a "benevolent" central government, that seeks to maximize some social welfare function. A contrasting game-theoretic model, in which the government responds to pressures from its constituents, was introduced in [2, 3]. That analysis was set in the context of private goods, so that the main issue was redistribution. In the current study we apply a similar analysis to public goods. We assume that exclusion is ruled out; i.e., once produced, a public good is available to all.

* This work was supported by National Science Foundation Grant SES80-06654 at the Institute for Mathematical Studies in the Social Sciences, Stanford University, and by the Institute for Advanced Studies of the Hebrew University of Jerusalem.

[1] Secondary purposes include providing incentives or disincentives of various kinds, controlling inflation, and so on.

0022-0531/87 $3.00

Our main conclusion is that the distribution of voting rights has little or no relevance to the choice of public goods. This result is qualitatively different from those of the previous study, where it was shown that the vote is of central importance in redistributional questions.

The framework within which we work is that of a *public goods economy*, defined by a set of agents, a collection of public goods, a collection of nonconsumable resources, and a technology (enabling public goods to be produced from resources); moreover, each agent has a utility function for public goods, an initial endowment of resources, and a voting weight. It will be assumed that the agents form a nonatomic continuum, i.e., that there are many agents, each of whom is individually insignificant.

Private consumption goods are totally absent from this model. It represents the opposite extreme from the model of [2, 3] and indeed from all the classical general equilibrium models (e.g., Debreu [6]), in which *public* consumption goods are totally absent.[2]

The explicit introduction of voting weights into an economic model of this kind is a new feature of this study. It enables one to consider situations in which certain agents, while perhaps possessing economic power, are not citizens; or in which some agents have more influence on the political process than others.

The question at issue is which bundle of public goods will be produced; this depends on the decision process, or constitution, that is in force. One simple and natural framework is the *voting game*, studied in [4], in which any coalition (set of agents) with a majority of the vote may produce any bundle of public goods that it wishes, using its own resources only; minority coalitions may produce nothing. It is, however, desirable to have a more general framework. For example, we may wish to allow the minority to produce certain public goods (or bads) on its own; or we may wish to allow the majority to expropriate certain of the minority's resources, such as land. To this end, consider the class of economies obtainable from a given public goods economy by varying the system v of voting weights, but keeping all the other specifications fixed. A given coalition S may thus be a majority in some economies of the class, a minority in others. In defining a *public goods game*, we do not specify the rules, like in the voting game. Rather, we postulate that to each coalition S there is available a set X_v^S of strategies; the choice of a pair of strategies by a coalition and its complement determines which public goods are produced, which are then enjoyed by all agents. Two assumptions are made: First, a monotonicity assumption according to which any coalition has at least those strategic options available to any of its sub-coalitions (i.e., $S \supset U$ implies $X_v^S \supset X_v^U$). Second, that the only way in which the voting weights

[2] See Section 6b.

can affect the strategies available to a coalition S is in determining whether or not S is a majority (i.e., if ν and ζ are different weighting systems, and S is a majority under both ν and ζ, or a minority under both ν and ζ, then $X_\nu^S = X_\zeta^S$).

To this game we shall apply the solution notion known as the *Harsanyi–Shapley Non-Transferable-Utility* (NTU) *Value*;[3] the resulting outcomes (i.e., bundles of public goods produced) will be called *value outcomes*. We obtain the following:

THEOREM. *In any public goods game, the value outcomes are independent of the voting weights.*

As an illustration, think of a public goods economy in which there are two groups of people, one preferring public libraries and the other television. . It seems almost obvious that if the television fans may vote but the book lovers may not, then one may expect more television programming and fewer books than when the voting rights situation is reversed. But this is not what value theory predicts; by our theorem, the value outcomes in the two situations are identical. An intuitive discussion of this phenomenon is found in Section 6.

The paper is organized as follows. In Section 2 we formally describe public goods economies and set forth our assumptions. Section 3 contains the formal description of public goods games. Section 4 specifies the variant of the Harsanyi–Shapley NTU value used in this paper, thus completing the formal specification of all terms appearing in the above statement of the theorem. In Section 5 we demonstrate the theorem informally, stressing the intuitive background. Section 6 is devoted to intuitive discussion, Section 7 to the formal proof of the theorem, and Section 8 to some technical comments.

The paper is constructed so that readers who are not interested in the formal treatment can avoid it entirely. Such readers, after completing the Introduction, should go immediately to Sections 5 and 6. Conversely, readers interested *only* in the formal proofs may omit Sections 5, 6, and 8.

This paper is a companion piece to [4], which was devoted to the voting game mentioned above. The voting game is an instance of a public goods game, and its particular nature enabled us to obtain a result that is more specific (stronger) than the general result obtained here. The proof in [4] relies heavily on the proof in this paper (cited in [4] as AKN); this paper, however, may be read independently. Though the papers are related, there

[3] Sometimes called the "λ-transfer" value. See Shapley [15]; also Sections 4 of [2] or [3]. There is a considerable literature on the NTU value, including discussion, applications, and critical evaluation. Of course, like any other single solution concept, the NTU value does not capture all the strategic aspects of the games under consideration.

is little overlap between them. Thus, readers interested in additional background material and comment, or in numerical illustrations and examples, are referred to [4].

2. Public Goods Economies

The real line is represented by $\mathbb{R}$, the euclidean space of dimension n by E^n, its nonnegative orthant by E^n_+ (i.e., $E^N_+ = \{x \in E^n \colon x^j \geqslant 0 \text{ for all } j\}$).

A nonatomic *public goods economy* consists of

(i) A measure space $(T, \mathscr{C}, \mu)$ (T is the space of *agents* or *players*, $\mathscr{C}$ the family of *coalitions*, and μ the *population measure*); we assume that $\mu(T) = 1$ and that μ is σ-additive, nonatomic and nonnegative.

(ii) Positive integers l (the number of different kinds of resources) and m (the number of different kinds of public goods).

(iii) A correspondence G from E^l_+ to E^m_+ (the *production correspondence*).

(iv) For each t in T, a member $\mathbf{e}(t)$ of E^l_+ ($\mathbf{e}(t)\,\mu(dt)$ is dt's *endowment* of resources).

(v) For each t in T, a function $u_t \colon E^m_+ \to \mathbb{R}$ (dt's von Neumann–Morgenstern *utility*).

(vi) A σ-additive, nonatomic, nonnegative measure ν on $(T, \mathscr{C})$ (the *voting measure*); we assume $\nu(T) = 1$.

Note that the total endowment of a coalition S—its input into the production technology if it wishes to produce public goods by itself—is $\int_S \mathbf{e}(t)\,\mu(dt)$; for simplicity, this vector is sometimes denoted $e(S)$. A public goods bundle is called *jointly producible* if it is in $G(e(T))$, i.e., can be produced by all of society.

We assume that the measurable space $(T, \mathscr{C})$ is isomorphic[4] to the unit interval $[0, 1]$ with the Borel sets. This assumption is less restrictive than it sounds; any non-denumerable Borel subset of any euclidean space (or, indeed, of any complete separable metric space) is isomorphic to $[0, 1]$. We also assume:

(2.1) $u_t(y)$ is Borel measurable simultaneously in t and y, continuous in y for each fixed t, and bounded uniformly in t and y.

(2.2) G has compact and nonempty values.

[4] An *isomorphism* is a one-to-one correspondence that is measurable in both directions.

3. Public Goods Games

Recall that a strategic game[5] with player space $(T, \mathscr{C}, \mu)$ is defined by specifying, for each coalition S, a set X^S of *strategies*, and for each pair (σ, τ) of strategies belonging respectively to a coalition S and its complement $T\backslash S$, a payoff function $\mathbf{h}^S_{\sigma\tau}$ from T to $\mathbb{R}$.

In formally defining public goods games, we shall describe pure strategies only; but it is to be understood that arbitrary mixtures of pure strategies are also available to the players. The pure strategies we shall describe will have a natural Borel structure, and mixed strategies should be understood as random variables whose values are pure strategies.

As indicated in the Introduction, we consider a class of public goods economies in which all the specifications except for the voting measure ν are fixed. We assume that the set X^S_ν of pure strategies of a coalition S in the economy with voting measure ν is a compact metric space, such that

$$S \supset U \qquad \text{implies} \quad X^S_\nu \supset X^U_\nu \tag{3.1}$$

and

$$(\nu(S) - \tfrac{1}{2})(\zeta(S) - \tfrac{1}{2}) > 0 \qquad \text{implies} \quad X^S_\nu = X^S_\zeta \tag{3.2}$$

(the intuitive meaning of (3.1) and (3.2) is given in the Introduction). From (3.2), it follows that X^T_ν is independent of ν, so that $X^T_\nu = X^T$; and from (3.1), it follows that $X^S_\nu \subset X^T$ for all ν, i.e., X^T contains all strategies of all coalitions. Now we assume given a continuous function that associates with each pair (σ, τ) in $X^T \times X^T$ a public goods bundle $y(\sigma, \tau)$ in E^m_+. Intuitively, if a coalition S chooses σ and its complement $T\backslash S$ chooses τ, then the public goods bundle produced is $y(\sigma, \tau)$. (It may of course happen that a (σ, τ) in $X^T \times X^T$ is not "feasible" as a strategy pair of a coalition and its complement, i.e., there are no S and ν with $(\sigma, \tau) \in X^S_\nu \times X^{T\backslash S}_\nu$; in that case it simply does not matter how $y(\sigma, \tau)$ is defined.) Finally, we define

$$\mathbf{h}^S_{\sigma\tau}(t) = u_t(y(\sigma, \tau)) \tag{3.3}$$

for each S, ν, σ in X^S_ν, and τ in $X^{T\backslash S}_\nu$.

Note that the *feasible* public goods bundles—those that can actually arise as outcomes of a public goods game—are contained in a compact set (the image of $X^T \times X^T$ under the mapping $(\sigma, \tau) \to y(\sigma, \tau)$), and hence constitute a bounded set.

[5] See Section 4 of [3].

4. Value Outcomes

As in [4], we shall be working here with the asymptotic value,[6] an analogue of the finite-game Shapley value for games with a continuum of players, obtained by taking limits of finite approximants. Let Γ be a public goods game. A *comparison function* is a nonnegative valued μ-integrable function λ on T that is positive on a set of agents of positive measure;[7] the corresponding *comparison measure* λ is defined by $\lambda(dt) = \lambda(t)\,\mu(dt)$, i.e., $\lambda(S) = \int_S \lambda(t)\,\mu(dt)$. A *value outcome* in Γ is then a random bundle of public goods associated with the Harsanyi–Shapley NTU value based on ϕ; i.e., a random variable $\underset{\sim}{y}$ with values in $G(e(T))$, for which there exists a comparison function λ such that the Harsanyi coalitional form[8] v_λ^Γ of the game $\lambda\Gamma$ is defined and has an asymptotic value, and

$$(\phi v_\lambda^\Gamma)(S) = \int_S Eu_t(\underset{\sim}{y})\,\lambda(dt) \qquad \text{for all} \quad S \in \mathscr{C}, \tag{4.1}$$

where $Eu_t(\underset{\sim}{y})$ is the expected utility of $\underset{\sim}{y}$.

5. An Informal Demonstration of the Theorem

We start by motivating and describing in more detail than above the concept of "value outcome." Let us use the word *outcome* for a bundle of public goods.[9] Let λ be a comparison measure, i.e., a nonnegative measure on the agent space; $\lambda(dt)$ is interpreted as an infinitesimal "exchange rate" that enables comparison of agent dt's utility u_t with that of other agents. For each coalition S and each outcome y, write

$$U^y(S) = \int_S u_t(y)\,\lambda(dt).$$

$U^y(S)$ represents the "total" payoff to S when the exchange rates $\lambda(dt)$ are used and y is the total bundle of public goods produced by all coalitions;

[6] Kannai [9]; see also [5, Sect. 18], or [3, Sect. 3].

[7] This is a slight variant of previous definitions; see Section 8c.

[8] The Harsanyi coalitional form was first explicitly defined (for finite games and $\lambda \equiv 1$) by Selten [13, p. 592], who called it "the $\frac{1}{2}$-characteristic of Γ" and denoted it $c_{1/2}$; but its roots go back further (see Footnote 11). The formal definition of v_λ^Γ is reviewed at the beginning of Section 7, and an informal description and motivation is given in the next section.

[9] In general, the outcomes arising in the analysis of a public goods game are "mixed," i.e., random variables whose values are pure outcomes; but for simplicity, the informal discussion of this section is restricted to pure outcomes. For the general case, one need only replace pure outcomes y by mixed outcomes $\underset{\sim}{y}$, and the utilities $u_t(y)$ by expected utilities $Eu_t(\underset{\sim}{y})$.

this follows from the fact that all agents can enjoy all public goods produced by anybody. Note that $U^y(S)$ is a measure (i.e., additive) in S for each fixed y.

Now let Γ be a public goods game. Suppose that each coalition S announces a "threat" strategy ξ^S, representing what it would do if it forms. Then if a specific coalition S and its complement $T\backslash S$ actually form, their threats ξ^S and $\xi^{T\backslash S}$ will lead to a specific outcome $y(\xi^S, \xi^{T\backslash S})$; total payoff to S will then be

$$V(S) = U^{y(\xi^S, \xi^{T\backslash S})}(S).$$

Presumably, each agent dt will wish each coalition S to which he belongs to announce its threat ξ^S in such a way that his value $(\phi V)(dt)$ in the resulting game V is maximized. A priori, it would seem likely that different members of the same coalition S would wish S to announce different threats. It can be shown, though [3, Sect. 4], that this is not the case, i.e., that all members of S wish S to announce the same strategy ξ_0^S; and that for each S, the pair $(\xi_0^S, \xi_0^{T\backslash S})$ of "optimal" threats constitutes a saddle point of $H^{y(\xi^S, \xi^{T\backslash S})}(S)$, where[10]

$$H^y(S) = U^y(S) - U^y(T\backslash S) = 2U^y(S) - U^y(T) \tag{5.1}$$

for all outcomes y. Denoting by V_0 the game V resulting from the announcement of the optimal threats ξ_0^S, we define

$$q_\lambda^\Gamma(S) = V_0(S) = U^{y(\xi_0^S, \xi_0^{T\backslash S})}(S).$$

In words, $q_\lambda^\Gamma(S)$ is the total payoff to S when the coalitions S and $T\backslash S$ form and use their optimal threats.[11] A value outcome in the game Γ is a feasible outcome y for which there exists a comparison measure λ such that for all agents dt,

$$(\phi q_\lambda^\Gamma)(dt) = u_t(y)\, \lambda(dt), \tag{5.2}$$

where ϕ is the Shapley value; in words, it is a feasible outcome whose utility for each player, in terms of the exchange rates $\lambda(dt)$, is precisely his value in the coalitional game q_λ^Γ.

[10] $H^{y(\xi^S, \xi^{T\backslash S})}(S)$ here corresponds to $H^S(\xi_S, \xi_{T\backslash S})$ in [3, Sect. 4].

[11] $q_\lambda^\Gamma(S)$ was first defined (for finite games) by Harsanyi [7, p. 354], where it was denoted (for $\lambda \equiv 1$) by α^S.

Define

$$\begin{aligned} w_\lambda^\Gamma(S) &= \max_{\xi^S} \min_{\xi^{T\setminus S}} H^{y(\xi^S, \xi^{T\setminus S})}(S) \\ &= H^{y(\xi_0^S, \xi_0^{T\setminus S})}(S) \\ &= q_\lambda^\Gamma(S) - q_\lambda^\Gamma(T\setminus S). \end{aligned} \tag{5.3}$$

The formal definition of value outcomes, at the end of Section 4, is in terms of the Harsanyi coalitional form v, defined by

$$\begin{aligned} v_\lambda^\Gamma(s) &= \frac{(w_\lambda^\Gamma(S) + w_\lambda^\Gamma(T))}{2} \\ &= \frac{(q_\lambda^\Gamma(S) + q_\lambda^\Gamma(T) - q_\lambda^\Gamma(T\setminus S))}{2}. \end{aligned} \tag{5.4}$$

From (5.4) it follows (cf. [2, p. 1154, Step 3]) that $\phi v_\lambda^\Gamma = \phi q_\lambda^\Gamma$, so that (4.1) and (5.2) are equivalent. The v_λ^Γ are mathematically better behaved, and therefore more convenient for the demonstration in this section; whereas the q_λ^Γ are conceptually more transparent, and so were used in discussing the examples in [4, Section 6]. Briefly, $w_\lambda^\Gamma(S)$ measures S's bargaining strength, or ability to threaten; $v_\lambda^\Gamma(S)$, the total utility that S can expect from the resulting efficient compromise. Cf. [3, Section 4], and [2, p. 1144].

Intuitively, it is not surprising that the coalition S must choose its threat to maximize the *difference* $H^y(S) = U^y(S) - U^y(T\setminus S)$ between its own payoff and that of its complement, and not its own payoff. One must remember that once the final compromise is agreed upon, these threats are not carried out; their only function is to influence the final compromise. And for *this* purpose, lowering the conflict payoff of the threatened party is as important as raising that of the threatener.

We proceed now to demonstrate our result. For each θ with $0 \leqslant \theta \leqslant 1$, let θT denote a "diagonal coalition of size θ." Intuitively, θT may be thought of as a perfect sample of the population T of all agents, containing a proportion θ of the agents. A precise formal definition may be given in terms of "ideal coalitions;" see [5, Chap. IV]. The crucial property of diagonal coalitions is that $\xi(\theta T) = \theta \xi(T)$ for nonatomic measures ξ. In particular, for any outcome y,

$$U^y(\theta T) = \int_{\theta T} u_t(y)\, \lambda(dt) = \theta \int_T u_t(y)\, \lambda(dt) = \theta U^y(T). \tag{5.5}$$

The importance of diagonal coalitions stems from the formula for the

Shapley value of a player t in a coalitional game r with a finite number n of players. This may be written[12]

$$(\phi r)(t)=\sum_{j=0}^{n-1} E(r(\underset{\sim}{S}_j \cup \{t\})-r(\underset{\sim}{S}_j))\frac{1}{n}, \tag{5.6}$$

where E stands for "expectation," and $\underset{\sim}{S}_j$ is a coalition chosen at random from among all those not containing t and having exactly j members, all such coalitions receiving the same probability. If now r is a nonatomic game and dt an infinitesimal agent, then by passing to the limit in (5.6) we obtain[13]

$$(\phi r)(dt)=\int_0^1 (r(\theta T \cup dt)-r(\theta T))\, d\theta. \tag{5.7}$$

Let ν and ζ be two different vote measures; denote the corresponding variants of Γ by Γ^ν and Γ^ζ, and write v_λ^ν for $v_\lambda^{\Gamma^\nu}$, etc. By the definition of value outcome, it is sufficient to demonstrate

$$\phi v_\lambda^\nu = \phi v_\lambda^\zeta$$

for all comparison measures λ. In the remainder of the section, therefore, we consider λ fixed, and suppress the subscript throughout (i.e., write v^ν for v_λ^ν, etc.).

Let $r=v^\nu - v^\zeta$; we wish to demonstrate that $\phi r=0$. For given dt, set $\delta=\max(\nu(dt), \zeta(dt))$. Let us call a coalition *even* if it is either a majority under both ν and ζ, or a minority under both ν and ζ. If S is even, then its complement $T\backslash S$ is also even; therefore the strategic options of S and $T\backslash S$ are the same under ν and under ζ, therefore $v^\nu(S)=v^\zeta(S)$, and so $r(S)=0$. Now θT is even for all θ, and $\theta T\cup dt$ is even for $\theta>\frac{1}{2}$ and $\theta<\frac{1}{2}-\delta$. Hence (5.7), (5.4) and $w^\nu(T)=w^\zeta(T)$ yield

$$\begin{aligned}(\phi r)(dt)&=\int_{(1/2)-\delta}^{1/2} r(\theta T\cup dt)\, d\theta\\ &=\int_{(1/2)-\delta}^{1/2} (v^\nu(\theta T\cup dt)-v^\zeta(\theta T\cup dt))\, d\theta\\ &=\int_{(1/2)-\delta}^{1/2} \tfrac{1}{2}(w^\nu(\theta T\cup dt)-w^\zeta(\theta T\cup dt))\, d\theta. \end{aligned} \tag{5.8}$$

[12] Shapley [14, Formula (13)].

[13] A precise formulation and proof of (5.7) are given in [5], Theorem H, for a restricted class of games; and for a much wider class in Mertens [10].

Let $\frac{1}{2}-\delta<\theta<\frac{1}{2}$. Suppose $w^v(\theta T\cup dt)$ is achieved at the outcome y, i.e.,

$$w^v(\theta T\cup dt)=H^y(\theta T\cup dt)=2U^y(\theta T)+2U^y(dt)-U^y(T)$$
$$=(2\theta-1)\,U^y(T)+2u_t(y)\,\lambda(dt)$$

(by (5.1) and (5.5)). Now $2\theta-1$ is infinitesimal, since $|2\theta-1|<2\delta$; also $\lambda(dt)$ is infinitesimal. Hence $w^v(\theta T\cup dt)$ is infinitesimal, and by similar reasoning, so is $w^\zeta(\theta T\cup dt)$. But then the bottom line of (5.8) is the integral of an infinitesimal over an infinitesimal range, which is an infinitesimal of the second order and so may be ignored. Thus $(\phi r)(dt)=0$, as was to be shown.

The demonstration hinges on the fact that $w^v(\theta T\cup dt)$ is infinitesimal when $\frac{1}{2}-\theta$ is infinitesimal; i.e., that $w^v(S)$ is small when S is near[14] $\frac{1}{2}T$, that w^v *is continuous at* $\frac{1}{2}T$ (where it vanishes). Indeed, if S is near $\frac{1}{2}T$, then it is also near its complement $T\backslash S$; hence no matter what the outcome is, S and $T\backslash S$ must have approximately the same utility, since all agents enjoy the same public goods. Note specifically that this breaks down in the case of redistribution of private goods [2, 3], or in a public goods voting game in which the majority may exclude the minority from enjoying the public goods [4, p. 682, Example 2]. In both cases, the minority may be prevented from consuming anything, whereas the majority can at least use its own resources; so even when S has only a slight v-majority, $w^v(S)$ will in general be very far from 0. The theorem is actually false in these cases.

6. Conceptual Discussion

a. *The Result*

To focus our thoughts, consider again the public goods economy discussed at the end of the Introduction, consisting of television fans and book lovers; by our theorem, the outcome is independent of who has the vote. To understand this intuitively, consider first the transferable utility (TU) context, in which we allow individuals to trade their votes for money if they wish. Thus the process of vote trading will continue while alternative proposals of television programming and libraries are debated. When an equilibrium of the vote trading is achieved, the winning public goods program is voted in. The power and influence of every individual thus has two elements: his material endowment (money or resources), and the market value of his vote. The effect that different voting rights have must be understood by studying the market price of a vote in equilibrium.

For concreteness, assume that there are as many television fans as book

[14] Two coalitions are "near" each other if they have similar profiles of characteristics, are *statistically* not too different. In particular, both their average utilities and their sizes are close, and so also their total utilities.

lovers, but only television fans may vote; that there is only one resource, of which all agents have the same endowment; that the utility functions are strictly concave and are "mirror images" of each other (i.e., $u_1(y^1, y^2) = u_2(y^2, y^1)$, where u_1 and u_2 are the utilities of book lovers and television fans, respectively, and y^1 and y^2 are quantities of libraries and television programming, measured in some appropriate units); that from an amount x of the resource one can produce any combination of public goods that total x; that the total amount of resource is 1; that the decision rule is as in the "voting game" of the Introduction; and that the marginal utility of money is 1.

An outcome of the TU game has two components: A vector y of public goods, and a schedule of accompanying side payments. We wish to make out an intuitive case for our contention that the vector of public goods "actually" produced is $(\frac{1}{2}, \frac{1}{2})$, and that there will be no accompanying side payments. Indeed, suppose first that more television programming than libraries are produced, as one might expect from the voting rights situation; for example, let us consider the vector $(\frac{1}{3}, \frac{2}{3})$. The strict concavity of the utilities implies that such an outcome cannot be Pareto optimal: the book lovers, by appropriate side payments, could make it worthwhile for all the television fans to vote for more books and less television. Indeed, they could make it worthwhile for them to vote for $(\frac{1}{2}, \frac{1}{2})$, since what is gained by the book lovers in going from $(\frac{1}{3}, \frac{2}{3})$ to $(\frac{1}{2}, \frac{1}{2})$ is more than what is lost by the television fans $(u_1(\frac{1}{2}, \frac{1}{2}) - u_1(\frac{1}{3}, \frac{2}{3}) > u_2(\frac{1}{3}, \frac{2}{3}) - u_2(\frac{1}{2}, \frac{1}{2})$, by the "mirror symmetry" and the strict concavity).

In fact, though it is worthwhile for the book lovers to make sidepayments to *all* television fans to get them to vote for $(\frac{1}{2}, \frac{1}{2})$, this is not al all necessary; only slightly more than half the television fans are needed. It is obvious that this enables the book lovers to cut their expenses for side payments by almost 50%. Only slightly less obvious, though, is the fact that the book lovers can get much more out of the situation: they can play the television fans off against each other. The television fans know that the book lovers will pick some 51% of them and get them to vote for $(\frac{1}{2}, \frac{1}{2})$ by appropriate side payments. Naturally, they would like to be among the 51% who receive side payments, not among the 49% who do not. So they will bid against each other, offering to accept lower and lower amounts; and in the end, the equilibrium side payment will be practically zero.

The argument can be summed up as follows:

(1) The actual outcome must be Pareto optimal.

(2) The only Pareto optimal outcome involves a production of $(\frac{1}{2}, \frac{1}{2})$, possibly with an accompanying schedule of side payments from book lovers to television fans.

(3) As only slightly more than half the television fans are needed to

approve this outcome, competition between them will drive the side payments down to zero.

In brief, by playing the television fans off against each other, the book lovers can achieve parity in public goods for only a pittance in bribes.

This argument sounds almost too simple, and we would like to examine it from several viewpoints. First, when the issue is redistribution rather than choice of public goods, as in [2, 3], then the vote has a very important effect. Suppose, for example, that there are two groups of people, citizens and noncitizens; each is endowed with a single unit of a consumption good, but only citizens are allowed to vote. Any redistribution of the good may be decided on by a vote of the majority, but the minority has the right to destroy some or all of its goods. Then the citizens will always get a considerably larger share of the pie; when the utilities are linear (in the relevant range), citizens will get three times as much as noncitizens.

The vote also has an important effect when one is dealing with public goods with exclusion allowed. Specifically, consider the above television–library game, modified by the proviso that the majority may exclude the minority from use of any or all of the public goods produced. In that case, again, the value allocation calls for considerably more television than libraries.

When this result was presented at Professor E. Malinvaud's seminar in Paris, Professor K. Shell asked, why doesn't the argument about Pareto optimality and competition for side payments work in the cases of redistribution and exclusive public goods? What makes these cases different from that of nonexclusive public goods?

The answer is as follows. In the voting game with nonexclusive public goods, each voter will enjoy the public goods eventually produced no matter how he votes. He is a *free rider*; therefore he sells his vote for whatever it will fetch, producing cutthroat competition which drives its market value down to zero.

However, in the case of redistribution, the payoff of each player depends on how he votes. The man who votes with the majority will usually get all of his own initial bundle plus a part of the minority's; whereas the man who votes with the minority will not even get all of his own initial bundle. In the case of exclusive public goods, enjoyment of the public goods depends on voting the right way. The ride is not free in either of these cases; and a man will not sell his vote as lightly as in the voting game of this paper, as voting "wrong" is liable to cost him personally dearly.

As this contrast is crucial to an understanding of our results, we would like to dwell on it a little longer. It is not true that in our voting game, there is no cost at all to switching one's vote. Every player's vote does have some influence, since when all is said and done, all the players together do determine which public goods are produced. If there are 10^8 televition fans, then roughly speaking, each one's vote can be expected to affect the amount of television programming by something of the order of magnitude of 10^{-8}, on the average. So to him, the cost of selling his vote is approximately one one-hundred millionth of his television viewing, a quite negligible amount. Competition therefore drives the price of the vote to 0. But in the case of redistribution or exclusive public goods, the cost of vote switching to the individual voter may be all of his television viewing, by all odds a considerable cost; and so the price of his vote will also be considerable.

The arguments adduced up to the present appear to depend critically on the TU assumption. However, the theorem of this paper is set in an NTU (nontransferable utility) context. Can we make economic sense of our result in that context as well?

To understand the situation, note that the TU assumption has two levels. The first may be called the *ordinal* level—simply that side payments are permitted, that an agent can act so that others gain while he loses. In our context, this is achieved if there are nonexpropriable, desirable private goods that may be transferred among the agents at will; specifically, if $u_t(y, x)$ is monotonic in t's private goods bundle x, where u_t is t's utility, and y is the public goods bundle.

The second level is the *cardinal* one—that an agent can act so that (for an appropriate choice of utilities), the total utility gained by others precisely equals the utility lost by him. This is achieved if there is just one private good, and $u_t(y, x) = u_t(y, 0) + x$.

To express a game in coalitional[15] form, as in (5.4), one needs the cardinal TU assumption. This is sometimes considered excessively strong, e.g., because it implies a complete lack of income effects. The NTU ("λ-transfer") value was developed for situations in which this strong assumption does not hold. In particular, it is needed whenever only the weak, ordinal form of the TU assumption holds.

The arguments in this section can be modified so as to use only this weak, ordinal form. At this level the TU assumption is intuitively very plausible; in most real situations at least some small amount of private goods is available for transfer. Perhaps it would have been methodologically preferable to put these private goods explicitly into the model. We did not do so because it would have cluttered up both the description and the analysis of the model without adding much insight. Indeed, the result is the same as before: both the choice of public goods, and the schedule of transfers (if any), are independent of the voting weights. In the television–library game, under appropriate symmetry conditions, the outcome remains $\frac{1}{2}-\frac{1}{2}$, and there are no side payments.

[15] Or "characteristic function."

To sum up: Strictly speaking, in our NTU result there are no private goods; intuitively, it can be considered the limit of what happens when the amount of private goods goes to 0. More broadly, the same intuitive ideas, and a similar theorem, apply when transfers of private goods are possible in any amount.

b. *The Model*

In the third paragraph of this paper we made the point that a public goods economy represents an extreme, idealized model of a certain politico–economic phenomenon. Studying such phenomena in isolation is typical of economic theory.[16] Pure exchange economies, pure monopolies, purely constant returns to scale, pure competition, etc., are all idealizations. We study them because they are associated with certain phenomena that are found (or sought) in a mixed, attenuated manner in the "real world"; they enable us to try to see some order, some regularity, in the chaos.

An example of a "real" system similar to the model of this paper is a commune like a kibbutz. But this paper is not meant to be about communes; rather, it is about the public goods *aspect* of complex politico–economic systems.

7. Formal Proof of the Theorem

Throughout this section, the word "measure" means "signed σ-additive measure on (T, C)." The symbol $\| \|$ denotes the max norm ($\|x\| = \max_i |x_i|$) when applied to points x in a euclidean space, and the variation norm ($\|\xi\| = \max_{S \in \mathscr{C}} (|\xi(S) + |\xi(T \backslash S)|)$) when applied to measures ξ. Sets of measures are always endowed with the metric induced by the variation norm. K denotes a uniform bound on $|u_t(y)|$ (Assumption 2.1), and C an m-dimensional hypercube containing all feasible public goods bundles (see the end of Sect. 3).

Let Γ be a public goods game and λ a comparison measure. "Value outcomes" for Γ were defined in Section 4, in terms of the Harsanyi coalitional form v_λ^Γ of the game $\lambda\Gamma$, whose explicit definition we now recall. For each public goods bundle y and coalition S, define

$$U_\lambda^y(S) = U^y(S) = \int_S u_t(y)\, \lambda(dt), \tag{7.1}$$

$$H_\lambda^y(S) = H^y(S) = U^y(S) - U^y(T \backslash S), \tag{7.2}$$

$$w_\lambda^\Gamma(S) = w^\Gamma(S) = \min \max EH^{y(\sigma,\tau)}(S) = \max \min EH^{y(\sigma,\tau)}(S), \tag{7.3}$$

$$v_\lambda^\Gamma(S) = v^\Gamma(S) = (w^\Gamma(S) + w^\Gamma(T))/2; \tag{7.4}$$

[16] And of much theory in the physical sciences as well.

in (7.3), E is the expectation operator, the max is over mixed strategies $\underset{\sim}{\sigma}$ of the coalition S (i.e., random variables with values in the pure strategy space X^S), and similarly the min is over mixed strategies $\underset{\sim}{\tau}$ of $T \backslash S$. To see that the min max in (7.3) is attained and equals the max min, note that Lebesgue's dominated convergence theorem and the continuity of $y(\sigma, \tau)$ in pairs (σ, τ) of pure strategies imply that $H^{y(\sigma,\tau)}(S)$ is continuous in (σ, τ), and then use the minimax theorem on arbitrary compact strategy spaces.

LEMMA 7.5. *Let Γ and Δ be two public goods games with the same player space $(T, \mathscr{C}, \mu)$, the same utilities u_t, and the same set $G(e(T))$ of jointly producible public goods bundles. Assume that for every comparison function λ, the asymptotic value of $v_\lambda^\Gamma - v_\lambda^\Delta$ exists and vanishes identically. Then Γ and Δ have the same value outcomes.*

Proof. Follows from the definition of value outcome.

In what follows, λ will be a fixed integrable comparison function, λ the corresponding measure. We usually suppress the subscript λ, e.g., write v^Γ for v_λ^Γ. Also, we assume, as we may, that $\lambda(T) = 1$.

Before proceeding it is useful to recall some definitions. A coalitional game v is *monotonic* if $v(S) \geqslant v(T)$ whenever $S \supset T$. The difference of two monotonic games is *of bounded variation*; the linear space of all such games (on $(T, \mathscr{C})$) is denoted BV. A nondecreasing sequence Ω of coalitions $S_1 \subset S_2 \subset \cdots \subset S_k$ is a *chain*. The *variation* of a coalitional game v over the chain Ω is defined by $\|v\|_\Omega = \sum_{i=1}^{k-1} |v(S_{i+1}) - v(S_i)|$; of course $\|v\|_\Omega$ is a seminorm on BV. For $\mathscr{D} \subset \mathscr{C}$ the seminorm $\|v\|_\mathscr{D}$ is defined by $\|v\|_\mathscr{D} = \sup \|v\|_\Omega$ where the sup is taken over all chains $S_1 \subset S_2 \subset \cdots \subset S_k$ in $\mathscr{D}$, i.e., with all $S_i \in \mathscr{D}$. If $\varepsilon > 0$ and Ψ is a collection of nonatomic probability measures on $(T, \mathscr{C})$, define $\mathscr{U}(\Psi, \varepsilon)$ to consist of all coalitions S such that $(\psi(S) - \psi'(S)) < \varepsilon$ whenever ψ and ψ' are in Ψ. A *diagonal neighborhood* is a family of coalitions that includes some $\mathscr{U}(\Psi, \varepsilon)$ in which Ψ is finite (i.e., is essentially a finite-dimensional vector measure).

LEMMA 7.6. *Given a game r in BV, suppose that for every $\varepsilon > 0$ there is a diagonal neighborhood $\mathscr{D}$ such that $\|r\|_\mathscr{D} < \varepsilon$. Then the asymptotic value of r exists and vanishes identically.*

Proof. For given ε, let $\mathscr{D}$ be as in the hypothesis. Let Π be a partition of the player space. Construct a chain $\underset{\sim}{\Omega}$ by taking successive unions of the elements of Π, one at a time and in a random order. Corollary 18.10 in [5] asserts that if Π is sufficiently "fine," the entire chain $\underset{\sim}{\Omega}$ will with arbitrarily high probability be in $\mathscr{D}$, and therefore $\|r\|_{\underset{\sim}{\Omega}} < \varepsilon$. But $\|r\|_{\underset{\sim}{\Omega}}$ is bounded even if $\underset{\sim}{\Omega}$ is not in $\mathscr{D}$, since $v \in BV$; since the residual probability may be made arbitrarily small, it follows that $E\|r\|_{\underset{\sim}{\Omega}} < 2\varepsilon$ (where E stands for "expectation"). Hence the Shapley value of the finite approximant to r corresponding to Π has (variation) norm $<2\varepsilon$. Since ε can be made arbitrarily small, the lemma follows from standard arguments.

LEMMA 7.7. *Let C be a compact subset of the euclidean space E^m, and let g: $T\times C\to\mathbb{R}$ be a strictly positive uniformly bounded measurable function, such that for any fixed t in T, $g(t, y)$ is continuous in y. For each y in C, define a measure ψ^y by*

$$\psi^y(S)=\int_S g(t, y)\,\lambda(dt)$$

and a probability measure $\hat{\psi}^y$ by

$$\hat{\psi}^y(S)=\psi^y(S)/\psi^y(T).$$

Then the set $\{\hat{\psi}^y\colon y\in C\}$ is compact.

Proof. If $y_n\to y$, then by Lebesgue's dominated convergence theorem,

$$\|\psi^{y_n}-\psi^y\|=\int_T |g(t, y_n)-g(t, y)|\,\lambda(dt)\to 0;$$

hence the mapping $y\to\psi^y$ is continuous. Since g is strictly positive, $\psi^y(T)>0$ for each y in C; hence $\psi^y\to\hat{\psi}^y$ is continuous. Hence $y\to\hat{\psi}^y$ is continuous, and so Lemma 7.7 follows from the compactness of C.

LEMMA 7.8. *If Ψ is a compact set of nonatomic probability measures, then for every $\varepsilon>0$, $\mathscr{U}(\Psi, \varepsilon)$ is diagonal neighborhood.*

Proof. Since Ψ is compact, it has a finite subset Ψ' such that for every ψ in Ψ there is a ψ' in Ψ' with $\|\psi-\psi'\|<\varepsilon/3$. Then $\mathscr{U}(\Psi', \varepsilon/3)\subset\mathscr{U}(\Psi, \varepsilon)$, completing the proof of Lemma 7.8.

LEMMA 7.9. *Let Γ be a public goods game in which the utilities u_t are non-negative. Then v^Γ is monotonic.*

Proof. Assume $Q\supset S$. As $X^Q\supset X^S$ and $X^{T\backslash Q}\subset X^{T\backslash S}$ (see (3.1)), it is enough to show that for every y in $G(e(T))$,

$$I\equiv\int_T u_t(y)[(\chi_Q-\chi_{T\backslash Q})-(\chi_S-\chi_{T\backslash S})]\,\lambda(dt)\geqslant 0;$$

but $(\chi_Q-\chi_{T\backslash Q})-(\chi_S-\chi_{T\backslash S})=2\chi_{Q\backslash S}$ and therefore $I\geqslant 0$. This completes the proof.

Proof of the Theorem. Let Γ^ν and Γ^ζ be two variants of a public goods game corresponding to voting measures ν and ζ, respectively. Let $\varepsilon > 0$ be given. Let Ψ consist of all the measures $\hat{U}^y = U^y/U^y(T)$ with y in C (see (7.1)), together with the two voting measures ν and ζ. Set $\mathscr{D} = U(\Psi, \varepsilon)$.

For the moment, assume that $u_t(y)$ is strictly positive for all t and y; this assumption will be removed later. Then by Lemma 7.7, Ψ is compact, and hence by Lemma 7.8, $\mathscr{D}$ is a diagonal neighborhood.

Set $v^\nu = v^{\Gamma^\nu}$, $v^\zeta = v^{\Gamma^\zeta}$, $r = v^\nu - v^\zeta$, and let $\varnothing = S_0 \subset \cdots \subset S_k = T$ be a chain in $\mathscr{D}$, which we call Ω. Let i_1 be the greatest index for which $\max(\nu(S_i), \zeta(S_i)) < \frac{1}{2}$, and i_2 the smallest index for which $\min(\nu(S_i), \zeta(S_i)) > \frac{1}{2}$; clearly $i_1 < i_2$. Let Ω_1 be the chain $S_0 \subset S_1 \subset \cdots \subset S_{i_1}$, Ω_2 the chain $S_{i_1} \subset S_{i_1+1}$, Ω_3 the chain $S_{i_1+1} \subset \cdots \subset S_{i_2-1}$, Ω_4 the chain $S_{i_2-1} \subset S_{i_2}$ and Ω_5 the chain $S_{i_2} \subset \cdots \subset S_k = T$. Clearly $\|r\|_\Omega = \sum_{i=1}^5 \|r\|_{\Omega_i}$. From (3.2) it follows that for $i \leqslant i_1$ as well as for $i \geqslant i_2$, $v_\lambda^\nu(S_i) = v_\lambda^\zeta(S_i)$ and therefore $\|r\|_{\Omega_1} = \|r\|_{\Omega_5} = 0$.

Next, let $i_1 < i < i_2$. The definition of $\mathscr{D}$ then implies that $|\hat{U}^y(S_i) - \frac{1}{2}| < 2\varepsilon$ for each y, and similarly $|\hat{U}^y(T \backslash S_i) - \frac{1}{2}| < 2\varepsilon$ for each y; hence $|U^y(S_i) - U^y(T \backslash S_i)| < 4\varepsilon U^y(T) \leqslant 4\varepsilon K$ for all y. Hence by (7.2), $|H^y(S_i)| < 4\varepsilon K$ for all y, and hence $|w^\nu(S_i)| < 4\varepsilon K$, where $w^\nu = w^{\Gamma^\nu}$ (see (7.3)). Hence by the monotonicity of v^ν (Lemma 7.9), and by (7.4),

$$\|v^\nu\|_{\Omega_3} = v^\nu(S_{i_2-1}) - v^\nu(S_{i_1+1}) = w^\nu(S_{i_2+1}) - w^\nu(S_{i_1+1}) < 8\varepsilon K,$$

and similarly $\|v^\zeta\|_{\Omega_3} < 8\varepsilon K$. Hence

$$\|r\|_{\Omega_3} < \|v^\nu\|_{\Omega_3} + \|v^\zeta\|_{\Omega_3} < 16\varepsilon K.$$

Finally, setting $w^\zeta = w^{\Gamma^\zeta}$, we have

$$\|r\|_{\Omega_2} < |r(S_{i_1})| + |w^\nu(S_{i_1+1})| + |w^\zeta(S_{i_1+1})| < 0 + 4\varepsilon K = 8\varepsilon K,$$

and similarly $\|r\|_{\Omega_4} < 8\varepsilon K$. Summing up, we obtain

$$\|v^\nu - v^\zeta\|_\Omega = \|r\|_\Omega < 0 + 8\varepsilon K + 16\varepsilon K + 8\varepsilon K + 0 = 32\varepsilon K.$$

Hence by Lemmas 7.5 and 7.6, the proof of the theorem is complete when $u_t(y)$ is strictly positive.

In the general case, one may modify the utility functions by adding the constant $K+1$ to them; they will then be strictly positive. The corresponding game r is not changed by the modification; since it has vanishing asymptotic value with the modified utilities, the same holds for the original utilities, and so by Lemma 7.5 the proof is complete.

8. Technical Comments

a. *Redistribution as a Public Good*

When this result was presented at a seminar at the London School of Economics, Professor William Gorman pointed out that technically, a redistribution plan [2] may be viewed as a public good for which different agents have different utilities. This presents a paradox, since the vote counts heavily in determining redistribution but not at all in the choice of public goods.

To resolve the paradox, note that the dimension of a redistribution plan is one less than the number of agents; thus with a continuum of agents one would need an infinite-dimensional public goods space to accommodate all feasible redistribution plans, whereas the model presented in Section 2 limits the number of public goods to the finite number m.

Unfortunately, this does not quite resolve the paradox. The proof of our theorem still works when the public goods space E^m_+ is replaced by any separable metric space Y, as long as the strategy spaces X^S_v are compact (which implies that the feasible bundles are in a compact subspace of Y). The crucial point is not finite dimensionality, but compactness.

Intuitively, a compact topological space is one that can be approximated by one with finitely many points. Thus the compactness of the space of public goods bundles means that there cannot be too many dissimilar feasible outcomes, where "similar" outcomes are those considered similar by all agents. With a continuum of agents, there is a continuum of dissimilar redistribution plans, and that is the reason that the results of this paper do not work for redistribution.[17]

To sum up, for the vote not to matter, we need a large number of individually insignificant agents (the continuum), but a relatively restricted choice of feasible outcomes (the compact outcome space).

b. *Beyond the Asymptotic Value*

Our theorem continues to hold when the asymptotic value is replaced by the μ-value [8, 2], or by a partition value [12]. Like the asymptotic value, these are obtained by taking limits of values of finite approximants to the given game. Unlike the asymptotic value, they do not cover *all* finite approximants; the μ-value, for example, looks only at approximating games in which all players have the same "size," when measured by the population measure μ. It follows that these values are "stronger" than the asymptotic value, in the sense that they exist and equal the asymptotic

[17] Of course, even with a continuum of agents one could restrict oneself to a finitely parametrized family of redistribution plans. The outcome space then really is compact, and the theorem of this paper does apply. Intuitively, what is happening in that case is that the individual voter is restricted in his effectiveness because he can benefit himself only if he simultaneously benefits others.

value whenever the latter exists, but also exist for many more games. Therefore Lemma 7.6 applies to these values as well, and the rest of the proof follows without change.

The advantage of using one of these stronger values is that it may well provide value outcomes in public goods games in which there are no value outcomes based on the asymptotic value; we have proved no general existence theorem, only an equivalence theorem.

Similar remarks apply to certain other values, such as the values obtained by Mertens [10, 11], or the mixing value [5, Chap. II]. While these are not necessarily stronger than the asymptotic value, they do appear to satisfy Lemma 7.6, and so our proof carries over to them as well.

c. *Comparison Functions and Measures*

In Section 4, we defined a comparison function to ba a μ-integrable non-negative function on $(T, \mathscr{C})$ that is positive on a set of positive measure. This is in line with the original definition of Shapley [15], but differs slightly from the definition in [1] and in [3], in which $\lambda(t) > 0$ for all t, and λ is measurable but not necessarily integrable. In our case, $\lambda(t)$ may vanish for some t even under the simplest of circumstances. As for the integrability, conceptually it involves no loss of generality. Indeed, by applying appropriate linear transformations, we may obtain a bundle y in $G(e(T))$ with uniformly positive utilities (i.e., $\inf_t u_t(y) > 0$); then non-integrable λ lead to undefined (in fact, infinite) $v_\lambda^\Gamma(T)$, and hence cannot correspond to value outcomes. In [3] the uniform positivity assumption was made explicit (5.7); but there it had substantive content, since the assumptions of $u_t(0) = 0$ (not made here) and uniform boundedness do not in general permit further linear adjustment to obtain uniform positivity.

Perhaps most natural would be to dispense altogether with the comparison function λ, and define value outcomes directly in terms of the comparison measure λ. In that case one would have to start out by proving that value outcomes can correspond only to non-atomic comparison measures; this offers no particular difficulty, but is in any case avoided under our approach. In our approach, the comparison measures are in fact absolutely continuous w.r.t. μ, since $\lambda(ds) = \lambda(t)\, \mu(ds)$. Like non-atomicity, absolute continuity can be *proved* in the alternative approach; unlike non-atomicity, it is not needed to prove our results.

d. *Representative Democracy and Other Voting Schemes*

When this result was presented at the Berkeley–Stanford Value Theory conference in 1981, Professor Lloyd Shapley inquired whether it also applies to other voting schemes, e.g., when the voting is by district, and a majority of the districts is required. (Technically, this is represented by a finite number of nonatomic vote measures $\nu_1, \ldots, \nu_k$, where a coalition S "wins" if and only if more than $k/2$ of the $\nu_i(S)$ are $> \nu_i(T)/2$.)

The answer is "yes." The theorem applies whenever there are finitely many nonatomic measures $v_1, \ldots, v_k$, such that S "wins" if all the $v_i(S)$ are $> v_i(T)/2$. Roughly, this condition means that a coalition that is both a good sample of the population and a majority always wins. Any two voting schemes satisfying this condition will lead to the same choice of a public goods bundle.

The condition is of more general applicability than may at first appear. Assuming that a legislator's vote reflects the wishes of a majority of his constituents, it is satisfied even for the process of amending the constitution of the United States, which requires majorities of both houses of Congress, and of each of the legislatures of $\frac{3}{4}$ of the states; and for that of removing the president of the United States, which requires a majority of the House of Representatives and $\frac{2}{3}$ of the Senate.[18]

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[18] On the other hand, there *are* voting schemes for which the theorem is false; for example, if $\frac{2}{3}$ of the entire population is required in order to "win." In that case w_λ^v is discontinuous at the point $\frac{2}{3}T$ on the diagonal, and the jump at that point makes the vote measure important.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium