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PRICES FOR HOMOGENEOUS COST FUNCTIONS*

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The problem of allocating the production cost of a finite bundle of infinitely divisible consumption goods by means of prices is a basic problem in economics. This paper extends the recent axiomatic approach in which one considers a class of cost problems and studies the maps from the class of cost problems to prices by means of the properties these prices satisfy. The class of continuously differentiable costs functions used in previous studies is narrowed to the subclass containing non-decreasing, homogeneous of degree one and convex functions. On this subclass it is shown that there exists a unique continuous price mechanism satisfying axioms similar to those assumed in previous studies.

1. Introduction

The problem of allocating the production cost of a finite bundle of infinitely divisible consumption goods (or services) by means of per unit costs or prices is a basic problem in economics. Recently an axiomatic approach has been proposed [Billera–Heath (1982) and Mirman–Tauman (1982)] in which one considers a class of cost problems and studies the maps from that class of cost problems to prices by means of the properties these prices satisfy. In that approach a *cost function* is a function $F:R^n_+ \to R$, with $F(0)=0$, where for x in R^n_+, $F(x)$ is interpreted as the cost of producing the bundle $x=(x_1,\ldots,x_n)$ of commodities.[1] A *cost problem* is a pair (F,α) where F is a cost function and α is in R^n_{++}, the strictly positive orthant of R^n (i.e., all components α_i of α are strictly positive). The vector α is interpreted as the vector of quantities produced. A mapping $\psi(\cdot,\cdot)$ that associates with each cost problem (F,α) $(F:R^n_+ \to R,\ \alpha$ in $R^n_{++})$ from a given class of cost problems, a set of price vectors $\psi(F,\alpha)\subset R^n$ is called a *price correspondence.*

A price correspondence is *cost sharing* if for every $P=(P_1,\ldots,P_n)$ in $\psi(F,\alpha)$, $\sum P_i\alpha_i=F(\alpha)$. In some circumstances setting cost sharing prices that charge each bundle β, $0\leqq\beta\leqq\alpha$, of commodities no more than $F(\beta)$, the cost of producing β, is desirable. This leads to the definition of the *core price*

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[1]Commodities may be interpreted either as consumption goods or as services.

correspondence, $C(F,\alpha)$. A price $P=(P_1,\dots,P_n)$ is in $C(F,\alpha)$ if and only if $\sum P_i\alpha_i=F(\alpha)$ and for all $0\leqq\beta\leqq\alpha$, $\sum P_i\beta_i\leqq F(\beta)$. For a given cost problem (F,α), $C(F,\alpha)$ might be empty, might consist of a single price vector or of many (a continuum of) price vectors. When the price correspondence $\psi(\cdot,\cdot)$ assigns a single price vector to each cost problem in the class under consideration, it is called a price mechanism. In that case we denote by $P(F,\alpha)$ the unique vector in $\psi(F,\alpha)$.

Recent papers [e.g., Mirman–Tauman (1982), Billera–Heath (1982), Samet–Tauman (1982)] study the class of cost problems (F,α) in which F is any continuously differentiable function on R^n_+. For that class of cost problems several price mechanisms could be defined. Among these is the marginal cost price mechanism given by

$$P_i(F,\alpha)=\frac{\partial F}{\partial x_i}(\alpha),$$

the Aumann–Shapley price mechanism given by

$$P_i(F,\alpha)=\int_0^1\frac{\partial F}{\partial x_i}(t\alpha)\,\mathrm{d}t,$$

or more generally, for every probability measure μ on $[0,1]$ one could define the price mechanism P given by

$$P_i(F,\alpha)=\int_0^1\frac{\partial F}{\partial x_i}(t\alpha)\,\mathrm{d}\mu(t).$$

All the above price correspondences (which are in fact price mechanisms) satisfy several natural properties,[2] among which are *rescaling invariance*, i.e., independence of the units of measurement; *superadditivity*, i.e., $\psi(F,\alpha)+\psi(G,\alpha)\subset\psi(F+G,\alpha)$, for all α in R^n_{++} and all $F,G\colon R^n_+\to R$, in the class of cost functions (for a price mechanism the inclusion reduces to equality and thus superadditivity reduces to additivity); *positivity*, i.e., for all α in R^n_{++} if F and G are cost functions for which $F-G$ is non-decreasing at each $x\leqq\alpha$, then $\psi(F,\alpha)\subset\psi(G,\alpha)+R^n_+$ (for a price mechanism this reduces to $\psi(F,\alpha)\geqq\psi(G,\alpha)$); and *consistency* which (roughly speaking) means, for price mechanisms, that the prices of two commodities having the same effect on costs are the same.

Both Mirman–Tauman (1982) and Billera–Heath (1982) proved that for the class of continuously differentiable cost problems the Aumann–Shapley

[2]These properties generalize those found in Billera–Heath (1982) and Mirman–Tauman (1982). For additional discussion, see also Samet–Tauman (1982).

price mechanism is the unique[3] one which is cost sharing and satisfies the above mentioned four properties, i.e., additivity, rescaling invariance, consistency and positivity. Samet and Tauman (1982) have also characterized marginal cost prices by a similar set of axioms.

The above mentioned price correspondences are well defined on the larger class $\mathscr{D}$ of all cost problems (F,α) where $\alpha \in R^n_{++}$ and $F: R^n_+ \to R$ is any continuous cost function for which $\partial F/\partial x_i$ exists and is continuous at each y in R^n_+ with $y_i > 0$.

In the study of long run production it is natural to assume that cost functions are non-decreasing, 1-homogeneous (homogeneous of degree one) and convex. Let $\mathscr{H}^c$ be the class of cost problems (F,α) in $\mathscr{D}$ with $F: R^n_+ \to R$ non-decreasing, 1-homogeneous and convex.

For the class $\mathscr{D}$ of cost problems all[4] the price correspondences mentioned above differ. Restricting attention to $\mathscr{H}^c$, all these price correspondences coincide and consist of a single price vector. Our main aim is to put this 'coincident price' on an axiomatic foundation and to explicate certain underlying principles which lead to this unique price vector. To set the stage for this, we take a map ψ from $\mathscr{H}^c$ to sets of prices and look for a minimal list of plausible properties which are common to all these price correspondences and which uniquely characterize the 'coincident prices' for smooth convex cost functions that are homogeneous of degree one.

The formal definitions and statements of the results are presented in section 2. The proof of the main theorem is presented in section 3. A discussion of the axioms will follow in section 4.

The present contribution is inspired by a recent development in the theory of non-atomic economies, Dubey–Neyman (1982). The proof in the present paper is similar in structure but different in detail from the one in Dubey–Neyman (1982). Also the proof, part of which is closely related to the one in Mirman–Neyman (1982), is self-contained; in order to save time for readers familiar with that paper the same notation is used here.

2. Definitions and statements of results

The class of all functions $F: R^n_+ \to R$ with $F(0)=0$ is denoted by $\mathscr{F}_n$. A *cost problem* is an ordered pair (F,α) where $F \in \mathscr{F}_n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is in R^n_{++} (i.e., $\alpha_i > 0$, for all $i = 1, \ldots, n$). The set $\bigcup_{n=1}^{\infty} \mathscr{F}_n \times R^n_{++}$ of all cost problems is denoted by $\mathscr{F}$. If $\mathscr{G}$ is a subset of $\mathscr{F}$ we denote by $\mathscr{G}_n$ the set $\mathscr{G} \cap (\mathscr{F}_n \times R^n_{++})$. Let $\mathscr{G}$ be a class of cost problems, i.e., $\mathscr{G} \subset \mathscr{F}$. A *price correspondence* for $\mathscr{G}$ is

[3]Uniqueness on certain classes of piecewise continuously differenciable cost problems has been shown by Samet et al. (1983).

[4]The price mechanism, $P_i(F,\alpha) = \int_0^1 (\partial F/\partial x_i)(t\alpha)\,\mathrm{d}\mu(t)$, associated with the probability measure μ on $[0,1]$ is well defined on $\mathscr{D}$ if and only if μ does not have an atom at zero. Hence measures having an atom at 0 are excluded.

a correspondence $\psi:\mathscr{G}\to\bigcup_{n=1}^{\infty}R^n$ [i.e., for every (F,α) in $\mathscr{G}$, $\psi(F,\alpha)\subset\bigcup_{n=1}^{\infty}R^n$] such that if $(F,\alpha)\in\mathscr{G}_n$, $\psi(F,\alpha)\subset R^n$.

Each vector $\lambda=(\lambda_1,\ldots,\lambda_n)$ in R^n_{++} induces a mapping $\lambda:R^n\to R^n$ that is given by

$$\lambda x\equiv\lambda(x_1,\ldots,x_n)=(x_1/\lambda_1,\ldots,x_n/\lambda_n),$$

and also induces a mapping $\lambda:\mathscr{F}_n\to\mathscr{F}_n$ that is given by

$$(\lambda F)(x)=F(\lambda x).$$

For $A\subset R^n$ we denote by λA the set $\{\lambda a:a\in A\}$. For $\lambda=(\lambda_1,\ldots,\lambda_n)$ in R^n_{++}, λ^{-1} denotes the vector $(\lambda_1^{-1},\ldots,\lambda_n^{-1})$. A subset $\mathscr{G}$ of $\mathscr{F}$ is *rescaling invariant* if for all $n=1,2,\ldots,$and all λ in R^n_{++} and all (F,α) in $\mathscr{G}_n$, the cost problem $(\lambda F,\lambda^{-1}\alpha)$ is in $\mathscr{G}_n$. A subset $\mathscr{G}$ of $\mathscr{F}$ is *additive* if for all n and all α in R^n_{++}, if (F,α) and (G,α) are in $\mathscr{G}_n$, then $(F+G,\alpha)$ is in $\mathscr{G}_n$.

Definition 2.1. Let $\mathscr{G}$ be a rescaling invariant set of cost problems. A price correspondence for $\mathscr{G}$ is *rescaling invariant*, if for every positive integer n, every λ in R^n_{++}, and every (F,α) in $\mathscr{G}_n$,

$$\psi(\lambda F,\lambda^{-1}\alpha)=\lambda\psi(F,\alpha).$$

Definition 2.2. Let $\mathscr{G}$ be an additive set of cost problems. A price correspondence for $\mathscr{G}$ is *superadditive*, if for every positive integer n, and every (F,α), (G,α) in $\mathscr{G}_n$,

$$\psi(F+G,\alpha)\supset\psi(F,\alpha)+\psi(G,\alpha).$$

By an ordered partition $T=(T_1,\ldots,T_n)$ of $\{1,2,\ldots,m\}$ we mean an ordered tuple of non-empty disjoint subsets of $\{1,\ldots,m\}$, such that $\bigcup_{i=1}^{n}T_i=\{1,\ldots,m\}$. Each ordered partition $T=(T_1,\ldots,T_n)$ of $\{1,\ldots,m\}$ induces a mapping $T:R^m\to R^n$ that is given by

$$Tx=(x(T_1),x(T_2),\ldots,x(T_n)),$$

where for $T_i\subset\{1,\ldots,m\}$ and x in R^m, $x(T_i)=\sum_{j\in T_i}x_j$. It also induces a mapping $T:\mathscr{F}_n\to\mathscr{F}_m$ that is given by

$$TF(x)=F(Tx)\quad\text{for all}\quad F\quad\text{in}\quad\mathscr{F}_n\quad\text{and}\quad x\quad\text{in}\quad R^n_+.$$

A subset $\mathscr{G}$ of $\mathscr{F}$ is said to be *consistent* if for every positive integer n and every ordered partition $T=(T_1,\ldots,T_n)$ of $\{1,\ldots,m\}$ and every (F,α) in $\mathscr{G}_n$, the cost problem (TF,β) is in $\mathscr{G}_m$ whenever $\beta\in R^m_{++}$ is such that $T\beta=\alpha$.

Definition 2.3. Let $\mathscr{G}$ be a consistent class of cost problems. A price correspondence ψ for $\mathscr{G}$ is *consistent* if for every ordered partition $T=(T_1,\dots,T_n)$ of $\{1,\dots,m\}$ where $m \geqq n$ and every β in R^n_{++} such that $(F,T\beta)\in\mathscr{G}_n$, if $P\in\psi(F,T\beta)$ and $Q\in R^m$ is given by $Q_j=P_i$, where $j\in T_i$, then $Q\in\psi(TF,\beta)$.

A class of cost problems $\mathscr{G}$ that is additive, rescaling invariant and consistent is called *admissable.*

Definition 2.4. Let $\mathscr{G}$ be an admissable class of cost problems. A *cost allocation price correspondence*, 'C.A.P.C.', for $\mathscr{G}$ is a price correspondence ψ for $\mathscr{G}$ that it superadditive, consistent, rescaling invariant, and such that if $(F,\alpha)\in\mathscr{G}_n$ with $F(x)=\sum_{i=1}^n c_i x_i$ then $\psi(F,\alpha)=\{(c_1,\dots,c_n)\}$.

For each α in R^n_{++} we define a semimetric d_α on $\mathscr{F}_n$ by

$$d_\alpha(F,G)=\sup\sum_{i=1}^{k}\left|(F(x^i)-F(x^{i-1}))-(G(x^i)-G(x^{i-1}))\right|, \tag{2.5}$$

where the supremum is taken over all chains $0=x^0\leqq x^1\leqq\dots\leqq x^k=\alpha$.

We will also use the notation $\|F-G\|_\alpha$ for $d_\alpha(F,G)$. Note that the range of d_α is $[0,\infty]$ and that if both F and G are monotonic then $d_\alpha(F,G)$ is finite. Actually $d_\alpha(F,G)$ is finite if and only if $F-G$ is a difference $F'-G'$ where F' and G' are in $\mathscr{F}_n$ and are monotonic on $\{x\in R^n_+ : 0\leqq x\leqq\alpha\}$. An equivalent definition for the semimetric d_α is: $d_\alpha(F,G)=\inf\{a\in R_+$: there are non-decreasing functions F_a, G_a in $\mathscr{F}_n$ with $F_a(\alpha)+G_a(\alpha)=a$ and $F+F_a-G$, $G+G_a-F$ are both non-decreasing on $\{x\in R^n_+ : 0\leqq x\leqq\alpha\}$, where $\inf\phi=\infty$.

Definition 2.6. A price correspondence ψ for a set of cost problems $\mathscr{G}$ is said to be *continuous* if there exists a constant C such that for all n, all (F,α), (G,α) in $\mathscr{G}_n$ and all P in $\psi(F,\alpha)$ there is a Q in $\psi(G,\alpha)$ with

$$|(P-Q)\cdot\beta|\leqq C\|F-G\|_\alpha \quad \text{for all} \quad 0\leqq\beta\leqq\alpha.$$

A discussion of the continuity property will be found in section 4.

For each n let $\bar{\mathscr{D}}_n$ be the class of all cost problems (F,α) where $\alpha\in R^n_{++}$ and $F:R^n_+\to R$ is continuous with $F(0)=0$. Let $\mathscr{D}_n$ be the class of all cost problems (F,α) in $\bar{\mathscr{D}}_n$ for which $\partial F/\partial x_i$ exists and is continuous at each point y in R^n_+ with $y_i>0$. Let $\mathscr{H}_n$ ($\bar{\mathscr{H}}_n$) be the class of all cost problems (F,α) in $\mathscr{D}_n$ (in $\bar{\mathscr{D}}_n$) where F is 1-homogeneous, i.e., $F(ax)=aF(x)$ for all a in R_+ and all x in R^n_+. Let $\mathscr{H}^c_n$ ($\bar{\mathscr{H}}^c_n$) be the class of all cost problems (F,α) in $\mathscr{H}_n$ (in $\bar{\mathscr{H}}_n$) that are non-decreasing, i.e., $F(x)\geqq F(y)$ whenever $0\leqq y\leqq x$, and subadditive, i.e., for all x, y in R^n_+, $F(x+y)\leqq F(x)+F(y)$. Note that for cost functions that are 1-homogeneous, subadditivity is equivalent to convexity. We denote by $\mathscr{D}$, $\mathscr{H}$ and $\mathscr{H}^c$ the classes $\bigcup_{n=1}^\infty\mathscr{D}_n$, $\bigcup_{n=1}^\infty\mathscr{H}_n$ and $\bigcup_{n=1}^\infty\mathscr{H}^c_n$, respectively. It is straightforward to verify that each of the classes $\mathscr{D}$, $\mathscr{H}$ and $\mathscr{H}^c$ is admissible. Our main result is:

Theorem 2.7. There exists a unique continuous C.A.P.C. ψ on $\mathscr{H}^c$.

The assumptions of this theorem could be classified into two categories: assumptions on the domain $\mathscr{H}^c$ of the correspondence ψ, mainly homogeneity, smoothness, convexity and monotonicity and the required properties of ψ. We will show in section 4 that except for convexity and monotonicity all of these assumptions are essential for the conclusion of Theorem 2.7. Dropping either convexity or monotonicity does not affect the uniqueness and indeed we have:

Theorem 2.8. Let $\mathscr{G}$ be either the class of all (F,α) in $\mathscr{H}$ with F nondecreasing or the class of all cost problems (F,α) in $\mathscr{H}$ with F convex. Then there exists a unique continuous C.A.P.C. ψ for $\mathscr{G}$.

We would like to mention that there is no trivial way of deducing Theorem 2.7 from Theorem 2.8 and vice versa. Conceptually Theorem 2.7 is more appealing than Theorem 2.8 since both subadditivity and monotonicity are natural properties of long-run cost production functions. Therefore a detailed proof of Theoreom 2.7 will be presented (in section 3) while only an outline will be given for the proof of Theorem 2.8.

3. Proof of Theorem 2.7

For a function $F:R^n_+\to R$, and b in R^n_{++} we denote by $\|F\|_b$ the supremum of $\sum_{i=1}^l |F(x^i)-F(x^{i-1})|$ taken over all chains $0\leqq x^0\leqq\ldots\leqq x^l=b$. For y in R^n we denote by $\|y\|_1$ the norm $\sum|y_i|$.

We start with a result that is needed for the proof of Theorem 2.7 but is also of independent interest.

Theorem 3.1. Let $\mathscr{G}$ be an admissible class of cost problems. Let (F,α) and (G,α) be in $\mathscr{G}_n$ where $G(x)=\sum c_i x_i$ is a linear function. Assume that

(i) *$F(x)=G(x)$ for all $0\leqq x\leqq\alpha$ with $|x_i/\alpha_i-x_j/\alpha_j|<\delta\sum(x_i/\alpha_i)$ for some $\delta>0$; and*

(ii) *there exists an $A>0$ such that for all $0\leqq y\leqq\alpha$,*

$$\|F-G\|_y\leqq A\|y\|_1.$$

Then for every continuous C.A.P.C. ψ on $\mathscr{G}$, $\psi(F,\alpha)=\{(c_1,\ldots,c_n)\}$.

Proof. Using rescaling invariance we may assume without loss of generality that $\alpha = e$ where e denotes the vector in R^n_+ with $e_i = 1/n$ for all $1 \leqq i \leqq n$. We use the same symbol e for different values of n; no confusion should result. For x in R^n we denote by $\|x\|_1$ the norm $\sum |x_i|$ and by $\|x\|_2$ the norm $(\sum x_i^2)^{\frac{1}{2}}$.

Fix a positive integer K and let M be the set $\{1, 2, \ldots, n^K\}$. Let $T^k = (T^k_1, \ldots, T^k_n)$, $1 \leqq k \leqq K$, be K ordered partitions of M that satisfy the following properties:

(i) $|T^k_i| = n^{K-1}$ for all $1 \leqq k \leqq K$ and all $1 \leqq i \leqq n$, where for a set A we denote by $|A|$ the number of elements of A; and

(ii) if for every pair i, j with $1 \leqq i, j \leqq n$ and every $1 \leqq k \leqq K$ we denote by $f^k_{i,j}$ the function on M that is given by

$$\begin{aligned} f^k_{i,j}(m) &= 1 \quad \text{if} \quad m \in T^k_i, \\ &= -1 \quad \text{if} \quad m \in T^k_j, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

then for all $1 \leqq k < k' \leqq K$,

$$\sum_{m \in M} f^k_{i,j}(m) f^{k'}_{i,j}(m) = 0. \tag{3.2}$$

To show that such a sequence of ordered partitions does indeed exist, identify M with all functions $h: \{1, \ldots, K\} \to \{1, \ldots, n\}$, (there are exactly n^K such functions), and let $T^k_i = \{h : h(k) = i\}$.

Let ψ be a continuous C.A.P.C. for $\mathcal{G}$. Note that $\psi(G, e) = \{(c_1, \ldots, c_n)\}$ and that if $j \in T^k_i$ then $\psi_j(T^k G, e) = \{c_i\}$. We must prove that $\psi(F, e) = \psi(G, e)$. Since $\psi(G, e)$ contains only one price vector, it is enough to prove that if p is a price vector in $\psi(F, e) - \psi(G, e)$ then $p = 0$. Let $p = (p_1, \ldots, p_n) \in \psi(F, e) - \psi(G, e)$. Let p^k be the vector in R^{n^k} given by $p^k_j = p_i$ whenever $j \in T^k_i$. By consistency it follows that $p^k \in \psi(T^k F, e) - \psi(T^k G, e)$. The sequence p^k, $1 \leqq k \leqq K$, satisfies the following property: For every $1 \leqq k \leqq K$,

$$\sum_{m=1}^{n^K} |p^k_m| = \sum_{i=1}^{m} \sum_{m \in T^k_i} |p^k_m| = \sum_{i=1}^{n} n^{K-1} |p_i| = n^{K-1} \|p\|_1. \tag{3.3}$$

Q.E.D.

Lemma 3.4. There is a universal constant $C > 0$ *(i.e., independent of n and K), such that for every n, every K, every such sequence of ordered partitions*

$T^1, \dots, T^K$ and every p in R^n, there is a subset $\bar{K}$ of $\{1, 2, \dots, K\}$ such that

$$\left\| \sum_{k \in \bar{K}} p^k \right\|_1 \geqq C\sqrt{K} \left\| p \right\|_1 n^{K-1}.$$

Proof. This is Lemma 3.3 of Mirman–Neyman (1982).

For every $\eta > 0$, $\bar{\mathscr{C}}(\eta)$ will denote the set of all vectors y in R^n_+ with $|y_i - y_j| < \eta \|y\|_1$ and $\mathscr{C}(\eta)$ will denote all y in $\bar{\mathscr{C}}(\eta)$ for which $0 \leqq y \leqq e$. Note that our assumptions imply that on $\mathscr{C}(\delta)$, F and G coincide (recall that we assumed w.l.o.g. that $\alpha = e$). Let $\eta > 0$ and $0 < \beta < 1$ be such that $\eta + 1 - \beta \leqq \delta$. Q.E.D.

Lemma 3.5. For any x, y in R^{n^K} with $0 \leqq x \leqq y \leqq e$ and $\|x\|_1 \geqq \beta \|y\|_1$ and for every $1 \leqq k \leqq K$,

$$T^k x \in \mathscr{C}(\eta) \Rightarrow T^k y \in \mathscr{C}(\delta).$$

Proof. $T^k x \in \mathscr{C}(\eta)$ implies that for every $1 \leqq i, j \leqq n$,

$$|x(T_i^k) - x(T_j^k)| < \eta \sum_{i=1}^{n} x(T_i^k) = \eta \|x\|_1. \quad \text{As} \quad \|x\|_1 \geqq \beta \|y\|_1,$$

$$|y(T_i^k) - x(T_i^k)| + |y(T_j^k) - x(T_j^k)| \leqq (1-\beta) \|y\|_1.$$

Therefore,

$$|y(T_i^k) - y(T_j^k)| \leqq |x(T_i^k) - x(T_j^k)| + |y(T_i^k) - x(T_i^k)| + |y(T_j^k) - x(T_j^k)|$$

$$< \eta \|x\|_1 + (1-\beta)\|y\|_1 \leqq \eta \|y\|_1 + (1-\beta)\|y\|_1 \leqq \delta \|y\|_1. \qquad \text{Q.E.D.}$$

Lemma 3.6. For every positive integer R and every subset $\bar{K}$ of $\{1, \dots, K\}$,

$$d_e\left(\sum_{k \in \bar{K}} T^k F, \sum_{k \in \bar{K}} T^k G \right) \leqq KA\beta^R + (n-1)\eta^{-2}\beta^{-1}AR.$$

Proof. In what follows we identify with each function $f_{i,j}^k : M \to R$, $1 \leqq i < j \leqq n$, $1 \leqq k \leqq K$, the vector $(f_{i,j}^k(1), \dots, f_{i,j}^k(n^K))$ in the Euclidean space R^{n^K}. The inner product of two vectors x, y in R^{n^K} will be denoted by $\langle x, y \rangle$. Observe that for given $1 \leqq k \leqq K$ and x in R^{n^K} with $0 \leqq x \leqq e$, $T^k x \in \mathscr{C}(\eta)$ if and only if for all $1 \leqq i < j \leqq n$, $|\langle f_{i,j}^k, x \rangle| < \eta \|x\|_1$. By (3.10) of Mirman–Neyman (1982) for every x in $R_+^{n^K}$, $|\{k : 1 \leqq k \leqq K, T^k x \notin \bar{\mathscr{C}}(\eta)\}| \leqq \langle x, x \rangle n^K (n-1)\eta^{-2} / \|x\|_1^2$. For all $x \in R_+^{n^K}$, $\langle x, x \rangle = \sum x_i^2 \leqq (\max_{i=1}^{n^K} x_i) \sum x_i$ and thus for all $x \in R_+^{n^K}$ with $0 \leqq x \leqq e$, $\langle x, x \rangle \leqq \|x\|_1 n^{-K}$. Therefore

$$\left|\{k: 1 \leqq k \leqq K, T^k x \notin \mathscr{C}(\eta)\}\right| \leqq (n-1)\eta^{-2}/\|x\|_1. \tag{3.7}$$

In order to prove Lemma 3.6 we have to prove that for every subset $\bar{K}$ of $\{1,\ldots,K\}$ and every increasing sequence $0=x^0 \leqq x^1 \leqq \ldots \leqq x^l = e$ in R^{n^K},

$$\sum_{s=1}^{l}\left|\left(\sum_{k\in\bar{K}} T^k F(x^s) - \sum_{k\in\bar{K}} T^k G(x^s)\right) - \left(\sum_{k\in\bar{K}} T^k F(x^{s-1}) - \sum_{k\in\bar{K}} T^k G(x^{s-1})\right)\right|$$

$$\equiv \sum_{s=1}^{l}\left|\sum_{k\in\bar{K}}((F(T^k x^s) - G(T^k x^s)) - (F(T^k x^{s-1}) - G(T^k x^{s-1})))\right|$$

$$\leqq KA\beta^R + (n-1)\eta^{-2}\beta^{-1}AR.$$

For $1 \leqq s \leqq l$ and $1 \leqq k \leqq K$ we denote by $g(k,s)$ the difference $(F(T^k x^s) - G(T^k x^s)) - (F(T^k x^{s-1}) - G(T^k x^{s-1}))$. As the left-hand side of the last inequality is bounded from above by $\sum_{k\in\bar{K}}\sum_{s=1}^{l}|g(k,s)|$ it is enough to prove that

$$\sum_{k\in\bar{K}}\sum_{s=1}^{l}|g(k,s)| \leqq KA\beta^R + (n-1)\eta^{-2}\beta^{-1}AR. \tag{3.8}$$

By adding elements to the sequence $(x^s)_{s=0}^{l}$ we may assume without loss of generality that for every non-negative integer r, $0 \leqq r \leqq R$, there is an s_r, $1 \leqq s_r \leqq l$, with $\|x^{s_r}\|_1 = \beta^r$. The equality $\|T^k x^{s_R}\|_1 = \beta^R$ and the inequality $\sum_{s=1}^{s_R}|g(k,s)| \leqq \|F-G\|_{T^k x^{s_R}} \leqq A\|T^k x^{s_R}\|_1$ imply that

$$\sum_{k\in\bar{K}}\sum_{s=1}^{s_R}|g(k,s)| \leqq |\bar{K}|A\beta^R \leqq KA\beta^R. \tag{3.9}$$

For each given $1 \leqq r \leqq R$ and $1 \leqq k \leqq K$ with $T^k x^{s_r} \in C(\eta)$ it follows from Lemma 3.5 that $T^k x^s \in C(\delta)$ for all $s_r < s \leqq s_{r-1}$ and thus $g(k,s)=0$ which imply $\sum_{s=s_r+1}^{s_{r-1}}|g(k,s)|=0$. For every given $1 \leqq k \leqq K$ and $1 \leqq r \leqq R$ we have

$$\sum_{s=s_r+1}^{s_{r-1}}|g(k,s)| \leqq \sum_{s=1}^{s_{r-1}}|g(k,s)| \leqq \|F-G\|_{T^k x^{s_{r-1}}} \leqq A\beta^{r-1}.$$

Inequality (3.7) implies that $\left|\{k: 1 \leqq k \leqq K, T^k x^{s_r} \notin C(\eta)\}\right| \leqq (n-1)\eta^{-2}/\beta^r$. Altogether we conclude that

$$\sum_{k\in\bar{K}}\sum_{s=s_r+1}^{s_{r-1}}|g(k,s)| \leqq ((n-1)\eta^{-2}/\beta^r)A\beta^{r-1} = A(n-1)\eta^{-2}\beta^{-1}. \tag{3.10}$$

Summing (3.10) over all $1 \leqq r \leqq R$ we have

$$\sum_{k \in \bar{K}} \sum_{s=s_R+1}^{s_0} |g(k,s)| \leqq (n-1)\eta^{-2}\beta^{-1}AR, \tag{3.11}$$

which together with (3.9) implies (3.8). This completes the proof of lemma 3.6.

To prove the theorem we choose a sequence $(R(K))_{K=1}^{\infty}$ of positive integers so that $\sqrt{K}\beta^{R(K)}+(n-1)\eta^{-2}\beta^{-1}R(K)/\sqrt{K} \to 0$ as $K \to \infty$ [for instance take $R(K)$ to be the integer part of $K^{\frac{1}{3}}$]. Consider the sequence of pairs of cost problems $(\sum_{k \in \bar{K}} T^k F, e)$ and $(\sum_{k \in \bar{K}} T^k G, e)$ where $\bar{K}$ is a subset of $\{1,\ldots,K\}$ such that $\|\sum_{k \in \bar{K}} p^k\|_1 \geqq C\sqrt{K}\|p\|_1 n^K$ (the existence of such $\bar{K}$ follows by Lemma 3.4). Note that by consistency and superadditivity there is Q^K in $\psi(\sum_{k \in \bar{K}} T^k G, e)$ and $\bar{Q}^K$ in $\psi(\sum_{k \in \bar{K}} T^k F, e)$ so that $\bar{Q}^K - Q^K = \sum_{k \in \bar{K}} p^k$. As $\sum_{k \in \bar{K}} T^k G$ is a linear function, $\psi(\sum_{k \in \bar{K}} T^k G, e)$ contains a single element, Q^K, and thus continuity of ψ implies the existence of a positive constant B such that

$$\sup\{|(\bar{Q}^K - Q^K)\cdot\beta| : \beta \in R^{n^K}, 0 \leqq \beta \leqq e\} \leqq B\left\| \sum_{k \in \bar{K}} T^k F - \sum_{k \in \bar{K}} T^k G \right\|_e. \tag{3.12}$$

Note that

$$\begin{aligned}\sup\{|(\bar{Q}^K - Q^K)\cdot\beta| : \beta \in R^{n^K}, 0 \leqq \beta \leqq e\} &= \sup\left\{\left|\left(\sum_{k \in \bar{K}} p^k\right)\cdot\beta\right| : 0 \leqq \beta \leqq e\right\} \\ &\geqq \tfrac{1}{2}\left\|\sum_{k \in \bar{K}} p^k\right\|_1 n^{-K} \geqq C\sqrt{K}\|p\|_1/2.\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13),

$$C\sqrt{K}\|p\|_1 \leqq 2B\left\|\sum_{k \in \bar{K}} T^k F - \sum_{k \in \bar{K}} T^k G\right\|_e.$$

By Lemma 3.6 the right-hand side of this inequality is bounded from above by $KA\beta^{R(K)}+(n-1)\eta^{-2}\beta^{-1}AR(K)$ and thus $C\|p\|_1 \leqq 2BA(\sqrt{K}\beta^{R(K)}+(n-1)\eta^{-2}\beta^{-1}R(K)/\sqrt{K}) \to 0$, as $K \to \infty$. Thus the last inequality could hold only if $\|p\|_1 = 0$, i.e., $p=0$ which completes the proof of Theorem 3.1. Q.E.D.

Proposition 3.14. For every (F,α) in $\mathscr{H}_n^c$ and every $\varepsilon>0$ there is (F^*,α) in $\mathscr{H}_n^c$, and $c_1,\ldots,c_n$ in R_+ and $\delta>0$ satisfying

(i) $\|F^* - F\|_\alpha < \varepsilon$,

(ii) $F^*(x) = \sum c_1 x_i$ *for all* $0 \leqq x \leqq \alpha$ *with* $|(x_i/\alpha_i)$

$$-(x_j/\alpha_j)| < \delta \sum (x_i/\alpha_i).$$

Proof. Without loss of generality we may assume that $\alpha = e$. Let $G: R^n_+ \to R$ be given by

$$G(x) = F(x) + \frac{\varepsilon \|x\|_2}{3}.$$

Note that (G, e) is in $\mathscr{H}^c_n$ and that

$$\|G - F\|_e = \frac{\varepsilon \|e\|_2}{3} \leqq \frac{\varepsilon}{3}. \tag{3.15}$$

For $t > 0$, let $G^t: R^n_+ \to R$ be given by

$$G^t(x) = \max\{G(x), (G'(e) + te) \cdot x],$$

where $G'(e)$ is the gradient of G at e, i.e., $G'(e) = ((\partial G/\partial x_1)(e), \ldots, (\partial G/\partial x_n)(e))$. Note that G^t is continuous, 1-homogeneous and convex. Q.E.D.

Lemma 3.16. $\|G - G^t\|_e \to 0$ *as* $t \to 0^+$.

Proof. Denote by Q_t the subset of R^n_+ of all points $x \in R^n_+$ for which $G^t(x) > G(x)$, i.e., $(G'(e) + te) \cdot x > G(x)$. As $(G'(e) + te) \cdot x$ is concave and 1-homogeneous (it is linear) while G is convex and 1-homogeneous it follows that $Q_t \cup \{0\}$ is a cone in R^n_t. Note that for all $t > 0$, $e \in Q_t$. As $G'(e) + te$ is increasing in t, $Q_{t'} \subset Q_t$ whenever $t' \leqq t$. Let $x^k \in Q_{k-1}$ be a sequence of points that converges to y in R^n_+ with $\|y\|_1 = 1$. Then $G(y) = \lim_{k\to\infty} G(x^k) \leqq \lim_{k\to\infty} (G'(e) + k^{-1}e) \cdot x^k = G'(e) \cdot y$. Since G is strictly convex on $\{y \in R^n_+ : \|y\|_1 = 1\}$ [it is the sum of a convex function F and a strictly convex function $(\varepsilon/3)\|\cdot\|_2$], the inequality $G'(e) \cdot x \leqq G(x)$, the linearity of $G'(e) \cdot x$ and the equality $G'(e) \cdot e = G(e)$ imply that $y = e$. Therefore the cones $Q_t \cup \{0\}$, $t > 0$, converge to the cone generated by e as $t \to 0_+$. Therefore, using homogeneity of G and continuity of $G'(\cdot)$ at e, for every $\eta > 0$ there is t_0 sufficiently small so that $x, y \in Q_{t_0}$ implies that

$$|G(x) - G(y) - (x - y) \cdot G'(e)| \leqq \eta \|x - y\|_1. \tag{3.17}$$

Note that

$$\|G^t - G\|_e \leqq \sup\{|(G^t - G)(x) - (G^t - G)(y)|/\|x - y\|_1\},$$

where the supremum is taken over all x, y in R^n_+. Continuity of $G^t - G$ and the equality of G^t and G on $R^n_+ \backslash Q_t$ imply that the supremum can be taken over all x, y in Q_t. But for x, y in Q_t,

$$|(G^t - G)(x) - (G^t - G)(y)| = |(G'(e) + te)\cdot(x - y) - (G(x) - G(y))|.$$

For $t \leqq t_0$ it follows, from (3.17), that this is bounded from above by $(t + \eta)\|x - y\|_1$ and thus $\limsup_{t \to 0^+} \|G^t - G\|_e \leqq \eta$. As this is true for all $\eta > 0$ the lemma follows.

Let $\bar{t} > 0$ be such that for $0 < t \leqq 2\bar{t}$, $\|G^t - G\|_e < \varepsilon/3$ and Q_t is contained in the interior of R^n_+. Define $F^*: R^n_+ \to R$ by

$$F^*(x) = (1/\bar{t}) \int_{\bar{t}}^{2\bar{t}} G^t(x)\,\mathrm{d}t.$$

Note that F^* is an average of the continuous, 1-homogeneous convex functions G^t, $\bar{t} \leqq t \leqq 2\bar{t}$ which are uniformly bounded on compact sets and thus F^* is a well defined, continuous, 1-homogeneous, convex function. Also, as $\|\ \|_e$ is a semi-norm on $\mathscr{F}_n$, $\|F^* - G\|_e \leqq \sup_{0 < t < 2\bar{t}} \|G^t - G\|_e \leqq \varepsilon/3$ and therefore

$$\|F^* - F\|_e \leqq \|G - F\|_e + \varepsilon/3 \leqq 2\varepsilon/3 < \varepsilon.$$

In order to complete the proof of Proposition 3.14 we still must show that $\partial F^*/\partial x_i$ exists and is continuous at each point y for which $y_i > 0$.

For each such y, $\partial G^t/\partial x_i(y)$, exists for all $t \leqq 2\bar{t}$ but at most one value of t and is bounded by $\max\{\partial G/\partial x_i(y), (G'(e) + te)_i\}$ and therefore

$$\frac{\partial F^*}{\partial x_i}(y) = \frac{1}{\bar{t}} \int_{\bar{t}}^{2\bar{t}} \frac{\partial G^t}{\partial x_i}(y)\,\mathrm{d}t.$$

Let $s(y) = \max\{0 \leqq t : G^t(y) = G(y)\}$. Then we have

$$\frac{\partial F^*}{\partial x_i}(y) = \frac{1}{\bar{t}} \int_{\bar{t}}^{2\bar{t}} \frac{\partial G}{\partial x_i}(y) I(t \leqq s(y))\,\mathrm{d}t + \frac{1}{\bar{t}} \int_{\bar{t}}^{2\bar{t}} \left(\frac{\partial G}{\partial x_i}(e) + t\right) I(t > s(y))\,\mathrm{d}t,$$

where I denotes the indicator function. As $s(y)$ is continuous at each point y with $y_i > 0$, continuity of $\partial F^*/\partial x_i$ follows. This completes the proof of Proposition 3.14. Q.E.D.

Lemma 3.18. For each $F:R^n_+\to R$ that is 1-homogeneous with $\|F\|_e<\infty$, there is an $A>0$ such that

$$\|F\|_y\leqq A\|y\|_1.$$

Proof. Note that for every F in $\mathscr{F}_n$, $\|F\|_x$ is non-decreasing in x, and if F is 1-homogeneous, so is $\|F\|_x$. Therefore, as $y\leqq e\|y\|_1 n$, we deduce that

$$\|F\|_y\leqq n\|F\|_e\|y\|_1.$$

The proof of Theorem 2.7 is now easily completed. Let (F,α) be in $\mathscr{H}^c_n$. By Proposition 3.14 for every $\varepsilon>0$ there is (F^*_ε,α) in $\mathscr{H}^c_n$ with $\|F^*_\varepsilon-F\|_\alpha<\varepsilon$ and satisfying condition (i) of Theorem 3.1, where $G(x)=(F^*_\varepsilon)'(\alpha)\cdot x$. Since both F^*_ε and $((F^*_\varepsilon)'(\alpha))\cdot(\cdot)$ are 1-homogeneous and non-decreasing, $\|F^*_\varepsilon-G\|_e<\infty$. Therefore, by Lemma 3.18 F^*_ε and G also satisfy condition (ii) of Theorem 3.1. Therefore, $\psi(F^*_\varepsilon,\alpha)=\{(F^*_\varepsilon)'(\alpha)\}$ and by continuity $\psi(F,\alpha)=\{F'(\alpha)\}$, which completes the proof of Theorem 2.7. Q.E.D.

3.1. Outline of the proof of Theorem 2.8.

The proof follows along the lines of the proof of Theorem 2.7. The main difference between the proof of Theorem 2.7 and the proof of Theorem 2.8 is in proving an analog of Proposition 3.14.

Proposition 3.14. Let $\mathscr{G}$ be either the class of all (F,α) in $\mathscr{H}$ with F non-decreasing or the class of all cost problems (F,α) in $\mathscr{H}$ with F convex. Then, for every (F,α) in $\mathscr{G}_n$ and every $\varepsilon>0$ there is (F^*,α) in $\mathscr{G}_n$ and $c_1,\ldots,c_n$ in R and $\delta>0$ satisfying:*

(*i*) $\|F^*-F\|_\alpha<\varepsilon,$

(*ii*) $F^*(x)=\sum c_ix_i$ *for all* $0\leqq x\leqq\alpha$ *with*

$$|(x_i/\alpha_i)-(x_j/\alpha_j)|<\delta\sum(x_i/\alpha_i).$$

Outline of the proof. Without loss of generality we assume that $\alpha=e$. First assume that $\mathscr{G}$ is the class of *all* (F,α) in $\mathscr{H}$ with F nondecreasing. Let $(F,\alpha)\in G_n$ and let $C_t=\{x\in R^n_{++}:|(x_i/x_j)-1|<t\}$. Let F^t be the smallest non-decreasing function from R^n_+ to R such that for all x in C_t, $F^t(x)=(F'(e)+te)\cdot x$. Note that both, F^t and $\max(F^t,F)$ are 1-homogeneous, non-decreasing and continuous. For sufficiently small $\bar{t}>0$, it can be shown that the function $F^*:R^n_+\to R$ given by

$$F^*(x)=\left(\frac{1}{\bar{t}}\right)\int_{\bar{t}}^{2\bar{t}} \max(F^t(x), F(x))\,dt$$

obeys (i) and (ii) and $(F^*, e)\in\mathscr{G}$.

For the class $\mathscr{G}$ of all cost problems (F,α) in $\mathscr{H}$ with F convex, the proof is the same as that of Proposition 3.14. This completes the outline of the proof of Proposition 3.14.*

The other difference between the proof of Theorem 2.7 and the proof of Theorem 2.8 is in the demonstration of the finiteness of $\|F^*-G\|_\alpha$, where $G:R^n_n\to R$ is given by $G(x)=(F^*)'(\alpha)\cdot x$. When F^* is non-decreasing this is obvious. When F^* is convex it follows from the existence of a linear function $\bar{G}:R^n_+\to R$ with $(F^*-G-\bar{G})(x)\leqq 0$ for all x in R^n_+ and the fact that any convex, 1-homogeneous and non-positive function from R^n_+ to R is non-increasing.

4. Comments on the assumptions

In this section we discuss the tightness of the assumption of our main theorem and make some further observations.

The main theorem states that there is a unique price correspondence $\psi:\mathscr{H}^c\to\bigcup_{n=1}^\infty R^n$ satisfying:

(*4.1*) superadditivity,
(*4.2*) consistency,
(*4.3*) continuity,
(*4.4*) $\alpha\in R^n_{++}, F:R^n_+\to R, F$ linear$\Rightarrow\psi(F,\alpha)=\{F'(\alpha)\}$,
(*4.5*) rescaling invariance.

It turns out that this unique price correspondence satisfies on $\mathscr{H}^c$ the following additional properties:

(*4.6*) cost sharing,
(*4.7*) positivity: $\alpha\in R^n_{++}$, $F-G$ non-decreasing on

$$\{x\in R^n_+:0\leqq x\leqq\alpha\}\Rightarrow\psi(F,\alpha)\subset\psi(G,\alpha)+R^n_+,$$

(*4.8*) additivity,
(*4.9*) $|\psi(F,\alpha)|=1$.

We will show that each of the assumptions (4.1)–(4.4) is essential by showing in Examples 4.10 through 4.13 that uniqueness is no longer implied if any one of these requirements is dropped.

Example 4.10 (superadditivity). For every (F,α) in $\mathscr{H}_n^c$, let $a(F,\alpha)$ be defined as $\sup\{F(x)+F(y)-F(x+y):0\leqq x,y;x+y\leqq\alpha\}$. Let ψ be the unique continuous C.A.P.C. for $\mathscr{H}^c$ and define a price correspondence ζ for $\mathscr{H}^c$ by: if $(F,\alpha)\in\mathscr{H}_n^c$ then

$$\zeta(F,\alpha)=\left\{P\in R_+^n:\sum|P_i-\psi_i(F,\alpha)|\alpha_i\leqq a(F,\alpha),\ P\cdot\alpha=F(\alpha)\right\}.$$

Then ζ obeys (4.2)–(4.6) but violates (4.1).

Example 4.11 (consistency). We start by defining a price mechanism ψ for the class $\mathscr{F}$ of all cost problems. For every vector α in R_{++}^n and every subset S of $\{1,\dots,n\}$ we denote by α_S the vector in R_+^n that is given by

$$\begin{aligned}(\alpha_S)_i&=\alpha_i \quad\text{if}\quad i\in S,\\ &=0 \quad\text{if}\quad i\notin S.\end{aligned}$$

For every integer $n\geqq 1$ and every cost problem (F,α) in $\mathscr{F}_n\times R_{++}^n$, associate the finite game $v\equiv v^{(F,\alpha)}$, in coalitional form on the set $N=1,2,\dots,n$ of players that is given by

$$v(S)=F(\alpha_S).$$

Let ϕ denote the Shapley value for finite games and define the price[5] mechanism ψ by

$$\psi_i(F,\alpha)=\frac{(\phi v^{(F,\alpha)})(\{i\})}{\alpha_i}\quad\text{for all}\quad 1\leqq i\leqq n.$$

It is easy to verify that this price mechanism is additive, cost sharing, positive, continuous, and rescaling invariant, and obeys (4.4) on all of $\mathscr{F}$, and thus obeys these properties on every class of cost problems. In particular it satisfies these properties for $\mathscr{H}^c$, but it violates consistency. Moreover it obeys on $\mathscr{H}^c$ weak consistency: for all $(F,\alpha)\in\mathscr{H}_n^c$ if $F(x_1,\dots,x_n)=f(x_1+\dots+x_n)$ on $\{x\in R_+^n:0\leqq x\leqq\alpha\}$ then $\psi_i(F,\alpha)=\psi(f,\sum\alpha_i)$.

Example 4.12 (continuity). The price mechanism $\bar{\psi}$ on $\mathscr{H}^c$ given by: if $(F,\alpha)\in\mathscr{H}_n^c$,

[5]This price mechanism for the class of continuously differentiable cost problems was suggested by Andras Simonovitz in order to show that weak consistency is essential for the results of Mirman–Tauman (1982) and could not be replaced by symmetry, i.e., by permutational invariance.

$$\bar{\psi}_i(F,\alpha)=\frac{F(\alpha_{\{i\}})}{\alpha_i},$$

obeys properties (4.1), (4.2), (4.4), (4.5), and (4.7)–(4.9).

The price correspondence ψ for $\mathscr{H}^c$ given by: if $(F,\alpha)\in\mathscr{H}^c_n$,

$$\psi(F,\alpha)=\{P\in R^n_+ : P\cdot\alpha=F(\alpha) \quad \text{and} \quad P\leqq\bar{\psi}(F,\alpha)\}$$

obeys (4.1), (4.2), and (4.4)–(4.7) but violates continuity.

We will now show that both the homogeneity and the smoothness assumption on the domain $\mathscr{H}^c$ are essential for the theorem by showing that dropping any one of these does not yield a unique continuous C.A.P.C. We start by addressing differentiability. Using a recent development (due to Mertens) in the theory of values of nonatomic games it can be shown that there exists a continuous C.A.P.C. ψ for $\bar{\mathscr{H}}^c$ with $|\psi(F,\alpha)|=1$ for all (F,α) in $\bar{\mathscr{H}}^c$. The core price correspondence $C(\cdot,\cdot)$ for $\bar{\mathscr{H}}^c$ is also a continuous C.A.P.C. As there are cost problems (F,α) in $\bar{\mathscr{H}}^c$ with $|C(F,\alpha)|=\infty$, we conclude that there is more than one continuous C.A.P.C. for $\bar{\mathscr{H}}^c$. Next we address homogeneity. It can be shown that for any probability measure μ on $[0,1]$ that is absolutely continuous with respect to the Lebesque measure ν with $\mathrm{d}\mu/\mathrm{d}\nu$ bounded on $[0,1]$, the price mechanism ψ_μ on $\mathscr{D}$ that is given by

$$\psi_\mu(F,\alpha)=\int_0^1 F'(t\alpha)\,\mathrm{d}\mu(t)$$

is a continuous C.A.P.C. for $\mathscr{D}$. If $\mathscr{G}$ is either the class of all cost problems (F,α) in $\mathscr{D}$ with F subadditive or with F convex then two different such measures yield different continuous C.A.P.C.s for G.

Example 4.13. The price correspondence ψ on $\mathscr{H}^c$ given by: if $(F,\alpha)\in\mathscr{H}^c_n$,

$$\psi(F,\alpha)=\{P\in R^n_+ : P\cdot\alpha=F(\alpha)\}$$

obeys properties (4.1)–(4.3) and (4.5)–(4.8) but violates (4.4). However, any price correspondence $\psi:\mathscr{H}^c\to\bigcup_{n=1}^\infty R^n$ that satisfies (4.2), (4.3), (4.5), (4.6) and (4.9) also obeys (4.4).

We finally address rescaling invariance. It is possible to show that for any price correspondence $\psi:\mathscr{H}^c\to\bigcup_{n=1}^\infty R^n$ satisfying (4.1)–(4.4), if $(F,\alpha)\in\mathscr{H}^c_n$ and for all $1\leqq i\leqq j\leqq n$, α_i/α_j is rational,

$$\psi(F,\alpha)=\{F'(\alpha)\}.$$

But whether this conclusion could be obtained without the rationality requirement of α is not known to us.

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