Public Goods and Budget Deficit

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Abstract

We examine incentive-compatible mechanisms for fair financing and efficient selection of a public budget (or public good). A mechanism selects the level of the public budget and imposes taxes on individuals. Individuals' preferences are quasilinear. Fairness is expressed as weak monotonicity (called scale monotonicity) of the tax imposed on an individual as a function of his benefit from an increased level of the public budget. Efficiency is expressed as selection of a Pareto-optimal level of the public budget. The budget deficit is the difference between the public budget and the total amount of taxes collected from the individuals.

We show that any efficient scale-monotonic and incentive-compatible mechanism may generate a budget deficit. Moreover, it is impossible

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to collect taxes that always cover a fixed small fraction of the total cost.

1 Introduction

The Pareto-optimal social states depend on individuals' preferences, which are private information. Therefore, a collective choice that aims at selecting Pareto-optimal alternatives requires input from the individuals, and the input of the individuals should convey sufficient information regarding their preferences. For instance, if the input of each individual is his own preference over the alternatives, a collective choice mechanism can select an outcome that is Pareto optimal with respect to the reported preferences. However, a mechanism that selects the social state as a function of the individuals' input (e.g., the individuals' preferences) may be subject to manipulation. The individuals' selfish incentives may lead to input that does not convey sufficient information for the selection of an optimal outcome with respect to the individuals' true preferences. A common exception to this dilemma is a dictatorial collective choice that selects an alternative that is most preferred by the dictator.

In the most general setup, the result of Gibbard (1973) and Satterthwaite (1975) precluded the possibility of finding a non-dictatorial deterministic mechanism for choosing social states in which individuals do not have the possibility of manipulating the mechanism to their own advantage. In the more specialized context of quasilinear preferences, Groves (1973), Clarke

(1971), and Vickrey (1961) found a class of mechanisms, called VCG mechanisms, in which stating one's true preferences is a dominant strategy and a Pareto optimum is selected, i.e., a class of strategy-proof and efficient direct mechanisms.

We recall the classical model to which the VCG mechanisms applies. The model consists of a society of n individuals and a set S of public alternatives. A public alternative s is a vector $(k, t_1, \ldots, t_n)$, where k is an element of a set K of public projects, and t_i is a real number representing the monetary transfer from individual i. Individual i's preferences are described by a function $u_i : S \to \mathbb{R}$ that is of the form $u_i(k, t_1, \ldots, t_n) = v_i(k) - t_i$, where v_i, called agent i's *valuation functions*, is the private information of agent i. If the cost of project k is $c(k)$ then the budget deficit of the alternative $(k, t_1, \ldots, t_n)$ is $c(k) - \sum_i t_i$, and the project k is efficient with respect to $v_1, \ldots, v_n$ if and only if $\sum_i v_i(k) - c(k) \geq \sum_i v_i(k') - c(k')$ for all $k' \in K$.

A VCG mechanism maps a list of valuation functions $v = (v_1, \ldots, v_n)$ to the outcome $s(v) = (k^*, t_1(v), \ldots, t_n(v))$ where $k^* = k(v)$ is efficient with respect to $v_1, \ldots, v_n$ and

$$t_i(v) = c(k^*) - \sum_{j \neq i} v_j(k^*)) + c_i(v_{-i})$$

where v_{-i} is the vector of valuations with the i-th coordinate v_i omitted.

The VCG mechanisms are truthfully implementable in dominant strategies and select an efficient outcome. Green and Laffont (1977, 1979) and Holmstrom (1979) provide sets of conditions under which these are the only social choice functions that are truthfully implementable in dominant strategies and select an efficient outcome. Holmstrom's conditions hold in our model and therefore we focus in the Introduction on VCG mechanisms.

Another desirable property of a mechanism is balancing the budget, namely, $\sum_i t_i(v) = c(k(v))$. Unfortunately, in many cases it is impossible. Green and Laffont (1979) show that when every valuation function is possible, then there is no VCG mechanism that balances the budget.

However, there are VCG mechanisms that do not generate a deficit, for example, when there are finitely many projects. Another example is when the cost of all projects is bounded from below by a constant B, and the valuation functions are bounded from above by a constant C, in which case the VCG mechanisms with $c_i(v_{-i}) = (n-1)C - B(n-1)/n$ cover the cost of the selected public project. More generally, if there is a constant D (the previous example corresponds to $D = B - nC$) such that $c(k) - \sum_{j=1}^{n} v_j(k) \geq D$ for all projects k and all valuation functions $v_1, \ldots, v_n$, the VCG mechanisms with $c_i(v_{-i}) = -D(n-1)/n$ cover the cost of the selected public project.

In Section 7.4 we show that also in the model where $K = \mathbb{R}_+$, $c(k) = k$, and V_i consists of all nondecreasing concave functions v (normalized with $v(0) = 0$ and with $v'(x) \to 0$ as $x \to \infty$), there are VCG mechanisms that never generate a deficit. Note that in our model a valuation function need not be bounded. Even if we consider only bounded valuation functions, there is no single constant that bounds all valuation functions. Thus, even for a fixed k, the terms $c(k) - \sum_{j=1}^{n} v_j(k)$ are not bounded from below when $v_1, \ldots, v_n$ ranges over all valuation functions.

One possible interpretation of our model is that a point in $K = \mathbb{R}_+$

stands for the society's total budget. Once the budget is specified, a known mechanism or bargaining allocates (either deterministically or stochastically) the budget to public projects. Thus, an individual preference over the final bundle of projects transforms into a preference over the budget, represented by a valuation function $v : \mathbb{R}_+ \to \mathbb{R}$, with the normalization $v(0) = 0$. An alternative interpretation is that $K = \mathbb{R}_+$ stands for the possible levels of a public good, rescaled so that the cost of level x is x. In both interpretations, t_i stands for the tax imposed on individual i. We assume that the valuation functions v are nondecreasing concave functions with $v'(x) \to 0$ (where $v'(x)$ stands for the right-hand derivative of v at x) as $x \to \infty$.

A natural desirable fairness property of the taxation is monotonicity: if individual i's benefit from any increment in the public budget is never less than that of individual j, i.e., $v_i'(x) \geq v_j'(x)$ for all $x \geq 0$, then $t_i(v) \geq t_j(v)$.

Theorem 1 introduces a symmetric and monotonic VCG mechanism $\varphi^{\bar{x}}$ (that depends on the parameter $\bar{x}$) that never runs a deficit on the restricted domain $V^{\bar{x}}$ of all valuation functions v with $v'(x) \leq 1/n$ for every $x \geq \bar{x}$.

Theorem 2 asserts that there is no VCG mechanism that is monotonic and never runs a budget deficit. Moreover, we prove a stronger result by weakening the monotonicity requirement and allowing for some deficit. We formulate a much weaker fairness property, called *scale monotonicity* of a mechanism: if all valuation functions are multiples $\alpha_i w$ of a fixed valuation function w, then $t_i(\alpha_1 w, \ldots, \alpha_n w) \geq t_j(\alpha_1 w, \ldots, \alpha_n w)$ whenever $\alpha_i \geq \alpha_j$. Theorem 3 asserts that for any scale-monotonic VCG mechanism and every $\gamma > 0$ there is a valuation vector v for which the taxes do not cover even the γ fraction of the budget, and moreover such a budget deficit can arise for a valuation vector $v = (w, w, \dots, w)$ with identical valuations for all individuals.

2 The Model

Let W be the set of all concave functions $w : \mathbb{R}_+ \to \mathbb{R}$, with $w(0) = 0$ and $\lim_{x\to\infty} w'(x) = 0$, where for a nondifferentiable (concave) w we denote by $w'(x)$ the right-hand derivative, i.e., the limit $\lim_{0<\varepsilon\to 0} \frac{w(x+\varepsilon)-w(x)}{\varepsilon}$. An element $w \in W$ is called a *valuation function*. The model $\langle N, V_1, \dots, V_n\rangle$ consists of a set of individuals, $N = \{1, \dots, n\}$, and a family of valuation functions, $V_1, \dots, V_n \subset W$. If $V_1 = V_2 = \dots = V_n = V \subset W$ we denote by $\langle N, V\rangle$ the model $\langle N, V_1, \dots, V_n\rangle$.

The interpretation is as follows. The set N is the set of individuals. The set V_i is the set of player i's possible valuation functions of a level/quantity of the public budget/good. Each individual has a valuation function $v_i \in V_i$. The n-member society chooses $x \in \mathbb{R}_+$, interpreted as the level/quantity of the public budget/good, and assigns taxes to each individual in the amount of t_i. The resulting outcome is then expressed as a vector $(x, t_1, \dots, t_n)$ in $\mathbb{R}_+ \times \mathbb{R}^n$. Thus the set of outcomes associated with this public budget/good model is

$$\Omega = \{(x, t_1, \dots, t_n) \mid x \in \mathbb{R}_+, t_i \in \mathbb{R}\}.$$

The preference of individual i over the set of outcomes is given by a utility function

$$u_i(x, t_1, \ldots, t_n) = v_i(x) - t_i$$

where $v_i \in V_i$. The valuation function v_i of individual i is private information.

A social choice mechanism for such a public goods economy is an N-person game form $(N; (S_i)_{i \in N}; \varphi)$ where

$$\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_n) : \times_{i \in N} S_i \to \Omega.$$

I.e., a mechanism is a collection of n strategy sets $S_1, \ldots, S_n$ and an outcome function $\varphi : S_1 \times \ldots \times S_n \to \Omega$. It can be viewed as a procedure for making the collective choice. The feasible actions of individual i are summarized by his strategy set S_i, and the rule for how agents' actions specify the collective choice is given by the outcome function φ. The selected level/quantity of the public budget/good, as a function of the profile of actions/strategies $(s_1, \ldots, s_n)$, is given by $\varphi_0(s_1, \ldots, s_n)$, and the tax imposed on individual i is given by $\varphi_i(s_1, \ldots, s_n)$. A mechanism is called a *direct revelation mechanism* if for every individual i in N, $S_i = V$.

The interpretation of the first coordinate x of a point in the outcome space Ω is the level of the public budget (or the expenditure on the public good, or the level/quantity of the public good when we assume unit cost). Thus, the budget deficit associated with the outcome $(x, t_1, \ldots, t_n)$ is $x - \sum_{i \in N} t_i$.

A VCG mechanism is a direct revelation mechanism where for a vector of valuation functions $v = (v_1, \ldots, v_n)$ we have

$$\sum_{i=1}^{n} v_i(\varphi_0(v)) - \varphi_0(v) \geq \sum_{i=1}^{n} v_i(x) - x \text{ for all } x \geq 0$$

and

$$\varphi_i(v) = \varphi_0(v) - \sum_{j \neq i} v_j(\varphi_0(v)) + c_i(v_{-i})$$

where v_{-i} is the vector of valuation v without its i-th coordinate v_i.

3 Desirable Properties of a Mechanism

In this section we introduce a list of properties that are used in the statements of our results.

3.1 Efficiency

An outcome $(x, t_1, \ldots, t_n)$ is called *efficient* with respect to a list of valuations $v = v_1, \ldots, v_n$ with $v_i \in V$, if there is no other outcome $(y, s_1, \ldots, s_n)$ such that for every $i \in N$

$$v_i(y) - s_i > v_i(x) - t_i$$

and

$$y - \sum_{i=1}^{n} s_i < x - \sum_{i=1}^{n} t_i.$$

The first list of inequalities implies that all individuals prefer the outcome $(y, s_1, \ldots, s_n)$ to the outcome $(x, t_1, \ldots, t_n)$ and the second one guarantees further that the budget deficit of the outcome $(x, t_1, \ldots, t_n)$ is greater than the budget deficit of the outcome $(y, s_1, \ldots, s_n)$.

The next (straightforward) result characterizes the efficient outcomes.

Proposition 1 *An outcome* $(x, t_1, \ldots, t_n)$ *is efficient w.r.t.* $v_1, \ldots, v_n$ *if and only if*

$$x \in \arg\max_x [\sum_{i=1}^n v_i(x) - x].$$

Proof. If $y \geq 0$ with $\sum_{i=1}^n v_i(y) - y > \sum_{i=1}^n v_i(x) - x$, for sufficiently small $\varepsilon > 0$ and $s_i = t_i + v_i(y) - v_i(x) - \varepsilon$, we have $v_i(y) - s_i > v_i(x) - t_i$ and $y - \sum_{i=1}^n s_i < x - \sum_{i=1}^n t_i$ and therefore $(x, t_1, \ldots, t_n)$ is not efficient. On the other hand, assume that $(x, t_1, \ldots, t_n)$ is an outcome such that for every $y \geq 0$, $\sum_{i=1}^n v_i(y) - y \leq \sum_{i=1}^n v_i(x) - x$. If $(y, s_1, \ldots, s_n)$ is an outcome with $y - \sum_{i=1}^n s_i < x - \sum_{i=1}^n t_i$ then $\sum_{i=1}^n v_i(y) - \sum_{i=1}^n s_i < \sum_{i=1}^n v_i(x) - \sum_{i=1}^n t_i$ and therefore there is an individual i with $v_i(y) - s_i < v_i(x) - t_i$. Therefore $(x, t_1, \ldots, t_n)$ is efficient. ∎

The result illustrates that the efficiency of an outcome $(x, t_1, \ldots, t_n)$ depends on its first coordinate only. Note that the proof does not use the concavity of the valuation functions $v \in W$. Therefore, the proposition holds also when the set of valuation functions W is replaced by the set U of all functions $v : \mathbb{R}_+ \to \mathbb{R}$. If $v_1, \ldots, v_n \in U$ are concave and differentiable then an outcome $(x, t_1, \ldots, t_n)$ with $x > 0$ is efficient if and only if $\sum_{i=1}^n v_i'(x) = 1$ (and $(0, t_1, \ldots, t_n)$ is efficient if and only if $\sum_{i=1}^n v_i'(0) \leq 1$). If $v_1, \ldots, v_n \in U$ are continuous and $v_i(x) = o(x)$ as $x \to \infty$ (which holds in particular when v_i is differentiable and $v_i'(x) \to_{x\to\infty} 0$ or $v_i(x+1) - v_i(x) \to_{x\to\infty} 0$) then an efficient outcome exists.

A mechanism is *efficient* if it is a direct revelation mechanism such that for every list of valuations $v = v_1, \ldots, v_n$ with $v_i \in V$, $\varphi(v)$ is an efficient outcome with respect to this list, i.e.,

$$\varphi_0(v) \in \arg\max_x [\sum_{i=1}^{n} v_i(x) - x].$$

In that case we also say that φ is efficient with respect to V.

3.2 Incentive Compatibility

An *incentive-compatible* mechanism is a direct mechanism in which stating one's true preference is a dominant strategy. It follows from [8, Theorem 2] that the VCG mechanisms are the only efficient incentive-compatible mechanisms whenever the domains of valuation functions $V_1, \ldots, V_n$ ($\subset W$ of $i = 1, \ldots, n$ respectively) are convex.

3.3 Feasibility (Budget Balance)

A mechanism is *feasible* (*fully funded*) if for every list of valuation functions $v = (v_1, \ldots, v_n)$ with $v_i \in V$,

$$\sum_{i=1}^{n} \varphi_i(v) \geq \varphi_0(v).$$

A mechanism *balances the budget* if for every list of valuation functions v with $v_i \in V$,

$$\sum_{i=1}^{n} \varphi_i(v) = \varphi_0(v).$$

A mechanism is γ*-funded*, $\gamma > 0$, if for every list of valuation functions v with $v_i \in V$,

$$\sum_{i=1}^{n} \varphi_i(v) \geq \gamma\varphi_0(v).$$

It is quite common to observe societies/countries whose budget is not balanced. Many such societies run a budget deficit. However, it is quite common to observe a budget deficit that is a small fraction of the budget. The concept of γ-fundedness enables us to quantify the maximal deficit as a fraction of the budget. For example a .95-funded mechanism translates to never running a budget deficit of more that 5% of the public expenditure.

3.4 Monotonicity

In this subsection we define various concepts of monotonicity that capture different equitable constraints by way of the distribution of the total tax among the individuals.

A mechanism is *monotonic* (w.r.t. V) if for every two individuals $i, j \in N$, $v_1, \ldots, v_n \in V$, with $v_i(b) - v_i(a) \geq v_j(b) - v_j(a)$ for all $b > a$,

$$\varphi_i(v_1, \ldots, v_n) \geq \varphi_j(v_1, \ldots, v_n).$$

The above monotonicity requires that whenever we are faced with valuation functions in which the marginal benefits to individual i from an increased level/quantity of the public budget/good are no less than those to individual j, the taxes levied on individual i be at least as much as those levied on individual j.

The next monotonicity property requires the above monotonicity of taxes only in cases where all valuations are a multiple of one fixed valuation function.

A mechanism is *scale-monotonic* (w.r.t. V) if for every two individuals $i, j \in N$, $w \in V$, positive numbers α_k, $k = 1, \ldots, n$, with $\alpha_k w \in V$ and $\alpha_i > \alpha_j$,

$$\varphi_i(\alpha_1 w, \ldots, \alpha_n w) \geq \varphi_j(\alpha_1 w, \ldots, \alpha_n w).$$

We introduce the following notation: given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+$ and $w \in W$, $\alpha * w$ denotes the list $\alpha_1 w, \ldots, \alpha_n w$ of valuation functions.

A mechanism is *strongly scale-monotonic* (w.r.t. V) if for every two individuals $i, j \in N$, $w \in V$, positive numbers α_k, $k = 1, \ldots, n$, with $\alpha_i > \alpha_j$,

$$\varphi_i(\alpha * w) - \alpha_i w(\varphi_0(\alpha * w)) \geq \varphi_j(\alpha * w) - \alpha_j w(\varphi_0(\alpha * w)).$$

The concept of a strongly scale-monotonic mechanism is related to progressive taxation. As $\alpha_i w(\varphi_0(\alpha * w))$ represents the utility of individual i from the level/quantity $\varphi_0(\alpha * w)$ of the public budget/good specified by the mechanism φ, the difference $\varphi_i(\alpha * w) - \alpha_i w(\varphi_0(\alpha * w)$ represents individual i's net taxation. A mechanism φ is strongly scale-monotonic if whenever all individuals possess a valuation function that is a multiple of a fixed valuation function w, those with higher valuation and thus also higher marginal valuation have larger net taxation.

3.5 Symmetry

The symmetric property defined below is a desirable requirement for an equitable mechanism. It is not, however, an assumption in our impossibility theorems. Nevertheless, in order to show that the assumptions of the theorems below are tight, we wish to demonstrate that each condition is necessary, even if we consider only symmetric mechanisms.

A social choice mechanism $\varphi : \times_{i\in N} S_i \to \Omega$ is called *symmetric* if $S_i = S_j$ for every $i, j \in N$ and for every $(\sigma_1, \ldots, \sigma_n) \in \times_{i\in N} S_i$ and every permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$

$$\varphi_0(\sigma_{\pi(1)}, \ldots, \sigma_{\pi(n)}) = \varphi_0(\sigma_1, \ldots, \sigma_n)$$

and

$$\varphi_{\pi^{-1}(i)}(\sigma_{\pi(1)}, \ldots, \sigma_{\pi(n)}) = \varphi_i(\sigma_1, \ldots, \sigma_n).$$

4 Monotonic VCG Mechanisms with No Deficit

In this section we demonstrate the existence of monotonic and symmetric VCG mechanisms that never generate a deficit on some domain of the valuation function. For example, let K be a sufficiently large constant, and let $V(K)$ be the set of all valuations $v \in W$ s.t. $v(x) \le K + x/n$. It follows that for every list of valuations $v_1, \ldots, v_n$ we have $\sum_{i=1}^n v_i(x) \le x + nK$. Then the VCG mechanism φ with $\varphi_i(v_1, \ldots, v_n) = \varphi_0(v_1, \ldots, v_n) - \sum_{j\neq i} v_j(\varphi_0(v_1, \ldots, v_n)) + (n-1)K$ is efficient, feasible (as $\sum_{i=1}^n \varphi_i(v_1, \ldots, v_n) \ge \varphi_0(v_1, \ldots, v_n)$), monotonic and strongly scale-monotonic (moreover, $v_i \ge v_j$, which follows from $v_i' \ge v_j'$ and $v_i(0) = v_j(0)$, implies $\varphi_i(v_1, \ldots, v_n) \ge \varphi_j(v_1, \ldots, v_n)$ and moreover $\varphi_i(v_1, \ldots, v_n) - v_i(\varphi_0(v_1, \ldots, v_n)) \ge \varphi_j(v_1, \ldots, v_n) -$

$v_j(\varphi_0(v_1, \ldots, v_n))$; this last inequality is in fact an equality), and strategy-proof. Other, more interesting examples follow.

Consider the following special class of valuation functions. Let the valuation function $w : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $w(x) = x$ if $x \leq 1$ and $w(x) = 1$ if $x \geq 1$, and set $V = V_w$. One interpretation is as follows. There are two public projects. The status quo project k_0 needs no further budget and thus the cost of k_0 is 0, and a transition (improvement or replacement) to another project k_1 costs one unit of money. For a cost of $0 < x < 1$ one can obtain an external contractor to build project k_1 with probability x, or alternatively, a budget $0 < x < 1$ can be placed in a risky asset yielding 1 (and thus covering the cost of project k_1) with probability x and yielding 0 with probability $1 - x$. A valuation function αw of an individual represents a valuation of α to the replacement of the status quo k_0 with the project k_1.

A VCG mechanism here is of the form

$$\varphi_0(\alpha_1 w, \ldots, \alpha_n w) = x^*(\alpha_1, \ldots, \alpha_n)$$

where

$$x^*(\alpha_1, \ldots, \alpha_n) = \begin{cases} 1 & \text{if } \sum_i \alpha_i > 1 \\ 0 & \text{if } \sum_i \alpha_i < 1 \end{cases}$$

where $x^*(\alpha_1, \ldots, \alpha_n)$ is on $\sum_i \alpha_i = 1$ a function of $(\alpha_1, \ldots, \alpha_n)$ with $0 \leq x^*(\alpha_1, \ldots, \alpha_n) \leq 1$, and for $1 \leq i \leq n$

$$\varphi_i(\alpha_1 w, \ldots, \alpha_n w) = x^*(\alpha_1, \ldots, \alpha_n) - \sum_{j \neq i} \alpha_j x^*(\alpha_1 w, \ldots, \alpha_n w) + c_i(\alpha_{-i})$$

The question arises whether one can select the functions c_i so that the corresponding VCG mechanism is feasible and monotonic. The answer is yes. In fact, we prove a more general result.

Let $\bar{x} > 0$ and let $V^{\bar{x}}$ stand for all valuation functions $v \in W$ with $v'(x) \leq 1/n$ for $x > \bar{x}$. It follows that for every list of valuation functions $v : v_1, \ldots, v_n$ with $v_i \in V^{\bar{x}}$ there is an efficient budget $x^* \leq \bar{x}$.

Theorem 1 *Assume that $V_i = V^{\bar{x}}$. There is a symmetric, monotonic, and strongly scale-monotonic VCG mechanism that never runs a budget deficit for a list of valuation functions with $v_i \in V^{\bar{x}}$.*

Proof. Consider the following VCG mechanism where

$$\varphi_0(v) = \inf\{x : \sum_i v_i'(x) \leq 1\}$$

(equivalently, $\varphi_0(v)$ is the minimal efficient budget) and

$$\varphi_i(v) = \varphi_0(v) - \sum_{j \neq i} v_j(\varphi_0(v)) + \sum_{j \neq i} v_j(\varphi_0(v_{-i})) - \varphi_0(v_{-i}) + \bar{x}/n$$

where v_{-i} is the list of valuation functions v with its i-th coordinate v_i replaced by the constant function 0. Obviously, φ is symmetric.

Let $v = v_1, \ldots, v_n$ and assume that $v_i \in V^{\bar{x}}$. We first demonstrate that $\sum_{i=1}^n \varphi_i(v) \geq \varphi_0(v)$. Note that for every x we have $\sum_{j \neq i} v_j(x) - \sum_{j \neq i} v_j(\varphi_0(v_{-i})) \leq x - \varphi_0(v_{-i})$. Therefore, setting $x = \varphi_0(v)$ we have $\varphi_i(v) \geq \bar{x}/n$, and thus $\sum_i \varphi_i(v) \geq \bar{x} \geq \varphi_0(v)$.

Next we demonstrate monotonicity. By the symmetry of φ it suffices to prove that $\varphi_1(v) \geq \varphi_2(v)$ whenever $v_1'(x) \geq v_2'(x)$ for every x. Assume that

$v_1'(x) \geq v_2'(x)$ for every x. Set $x = \varphi_0(v)$, $x_1 = \varphi_0(v_{-1})$, and $x_2 = \varphi_0(v_{-2})$. Then, $x_1 \leq x_2 \leq x$. Note that for every z we have $\sum_{j \neq 1} v_j'(z) \leq \sum_{j \neq 2} v_j'(z)$. Therefore,

$$\sum_{j \neq 1} v_j(x) - \sum_{j \neq 1} v_j(x_2) \leq \sum_{j \neq 2} v_j(x) - \sum_{j \neq 2} v_j(x_2).$$

By the efficiency of x_1 with respect to v_{-1} we have $\sum_{j \neq 1} v_j(x_2) - \sum_{j \neq 1} v_j(x_1) \leq x_2 - x_1$. Therefore,

$$\begin{aligned}
\varphi_1(v) &= x - \sum_{j \neq 1} v_j(x) + \sum_{j \neq 1} v_j(x_1) - x_1 + \bar{x}/n \\
&\geq x - \sum_{j \neq 1} v_j(x_2) - \sum_{j \neq 2} v_j(x) + \sum_{j \neq 2} v_j(x_2) \\
&\quad + \sum_{j \neq 1} v_j(x_1) - x_1 + \bar{x}/n \\
&\geq x - \sum_{j \neq 1} v_j(x_2) - \sum_{j \neq 2} v_j(x) + \sum_{j \neq 2} v_j(x_2) \\
&\quad + \sum_{j \neq 1} v_j(x_2) - x_2 + x_1 - x_1 + \bar{x}/n \\
&= \varphi_2(v)
\end{aligned}$$

■

5 The Impossibility Results

The main result asserts that it is impossible to find an efficient, scale-monotonic, and strategy-proof mechanism for $\langle N, W \rangle$ that always collects taxes that cover at least a fixed fraction of the cost:

Theorem 2 *Assume that the society $N = \{1, \ldots, n\}$ has at least two members. For every efficient, scale-monotonic, and strategy-proof mechanism φ on $\langle N, W \rangle$ and every $\gamma > 0$, there exists a list of valuation functions $v_1, \ldots, v_n$ s.t. $v_1 = v_2 = \ldots = v_n$ and*

$$\sum_{i \in N} \varphi_i(v_1, \ldots, v_n) < \gamma \varphi_0(v_1, \ldots, v_n).$$

We will actually state and prove a stronger result. Note that if $V \subset W$, the restriction of any efficient, scale-monotonic, and strategy-proof mechanism for $\langle N, W \rangle$ to $\langle N, V \rangle$ is an efficient, scale-monotonic, and strategy-proof mechanism for $\langle N, V \rangle$. Given $w \in W$ we denote by V_w the set of valuation functions $\{aw \mid a \geq 0\}$. The next theorem asserts the existence of a differentiable and strictly concave valuation function $w \in W$ s.t. for every $n > 1$ and every efficient, scale-monotonic, and strategy-proof mechanism φ on $\langle N, V_w \rangle$ (where $N = \{1, \ldots, n\}$) and every $\gamma > 0$, there exists a list of valuation functions $v_1, \ldots, v_n$ in V_w s.t. $v_1 = v_2 = \ldots = v_n$ and

$$\sum_{i \in N} \varphi_i(v_1, \ldots, v_n) < \gamma \varphi_0(v_1, \ldots, v_n).$$

Theorem 3 *There exists a smooth strictly concave function $w \in W$ such that for every society N with at least two members, every positive constant γ, and every efficient, strategy-proof, and scale-monotonic mechanism φ on $\langle N, V_w \rangle$, there exists $\alpha \in \mathbb{R}_+$ s.t.*

$$\sum_{i \in N} \varphi_i(\alpha w, \ldots, \alpha w) < \gamma \varphi_0(\alpha w, \ldots, \alpha w).$$

In Section 7.5 we demonstrate a smooth strictly concave function $w \in W$, $w(x) = \sqrt{x}$, and a symmetric, efficient, feasible, scale-monotonic, and strategy-proof mechanism for $\langle N, V_w \rangle$. However, by replacing scale monotonicity by strong scale monotonicity in Theorem 2, we obtain an impossibility result for $\langle N, V_w \rangle$, where $w \in W$ is any valuation function:

Theorem 4 *Assume $n \geq 2$ and $w \in W$. Then there is no efficient, strategy-proof, feasible, and strongly scale-monotonic mechanism for $\langle N, V_w \rangle$.*

6 Proofs

Assume that $w \in W$. Let V_w be the set of all valuation functions $v \in W$ of the form $v = \alpha w$. Define the map $\psi : \mathbb{R}_+^n \to \mathbb{R}_+ \times \mathbb{R}^n$ by

$$\psi(\alpha_1, \dots, \alpha_n) = \varphi(\alpha_1 w, \dots, \alpha_n w).$$

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$, we denote by α_{-i} the $n-1$ dimensional vector $(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n)$. The first lemma states a property of a list of functions ψ_i $(0 \leq i \leq n)$ that corresponds to an efficient and strategy-proof mechanism φ for $\langle N, V_w \rangle$. The domain V_w is convex and therefore, as mentioned earlier, by applying [8, Theorem 2] to the restricted domain V_w we have

Lemma 1 *Assume that φ is an efficient and strategy-proof mechanism. Then*

$$\psi_i(\alpha) = \psi_0(\alpha) - \sum_{j \neq i} \alpha_j w((\psi_0(\alpha))) + c_i(\alpha_{-i})$$

where c_i is an arbitrary function.

Proof of Theorem 3. Assume that φ is an efficient, feasible, strategy-proof, and strongly scale-monotonic mechanism for $\langle N, V_w \rangle$. Set $\psi(\alpha_1, \ldots, \alpha_n) = \varphi(\alpha_1 w, \ldots, \alpha_n w)$.

As φ is feasible, $\sum_{i=1}^n \psi_i(\alpha) \geq \psi_0(\alpha)$, and therefore

$$\sum_{i=1}^{n} \psi_0(\alpha) - \sum_{i=1}^{n} \sum_{j \neq i} \alpha_j (w \circ \psi_0)(\alpha) + \sum_{i=1}^{n} c_i(\alpha_{-i}) \geq \psi_0(\alpha),$$

implying that for every $\alpha \in \mathbb{R}_+^n$,

$$\begin{aligned} \sum_{j \neq i} c_j(\alpha_{-j}) &\geq (n-1) \left[\sum_{j=1}^{n} \alpha_j (w \circ \psi_0)(\alpha) - \psi_0(\alpha) \right] - c_i(\alpha_{-i}) \\ &= (n-1) q(\alpha) - c_i(\alpha_{-i}) \end{aligned}$$

where q is the function defined by $q(\alpha) = \sum_{j=1}^n \alpha_j (w \circ \psi_0)(\alpha) - \psi_0(\alpha)$.

We next prove that $q(\alpha) \to \infty$ as $\alpha_i \to \infty$. We first provide a proof in the case that ψ_0 and w are differentiable. Differentiating the function q with respect to α_i we obtain

$$\frac{\partial q}{\partial \alpha_i}(\alpha) = (w \circ \psi_0)(\alpha) + \sum_{j=1}^{n} \alpha_j w'(\psi_0(\alpha)) \frac{\partial \psi_0}{\partial \alpha_i}(\alpha) - \frac{\partial \psi_0}{\partial \alpha_i}(\alpha).$$

As $\sum_{j=1}^n \alpha_j w'(\psi_0(\alpha)) = 1$ by the efficiency of φ,

$$\frac{\partial q}{\partial \alpha_i}(\alpha) = (w \circ \psi_0)(\alpha).$$

Therefore, $\frac{\partial q}{\partial \alpha_i}(\alpha)$ is positive and nondecreasing in α_i, implying that $q(\alpha) \to \infty$ as $\alpha_i \to \infty$. We now prove that $q(\alpha) \to \infty$ as $\alpha_i \to \infty$ without the differentiability assumption on ψ_0.

Fix $x_0 > 0$ with $w'(x_0) > 0$ and thus $w(x_0) > 0$. In the remainder of the present proof $w'(x)$ stands for the left-hand derivative w at x. By the concavity of w, $w'(x) \leq \frac{w(x)-w(x_0)}{x-x_0}$ for every $x > x_0$, and $\frac{w(x)-w(x_0)}{x-x_0} > 0$ for every $x \neq x_0$. Therefore, for every $x > x_0$,

$$\frac{w(x_0)}{\frac{w(x)-w(x_0)}{x-x_0}} - x_0 = \frac{w(x)}{\frac{w(x)-w(x_0)}{x-x_0}} - x \leq \frac{w(x)}{w'(x)} - x.$$

The assumptions on w – concavity and $w'(y) \to_{y\to\infty} 0$ – imply that $\frac{w(x)-w(x_0)}{x-x_0} \to 0$ as $x \to \infty$. Therefore

$$\frac{w(x)}{w'(x)} - x \geq \frac{w(x_0)}{\frac{w(x)-w(x_0)}{x-x_0}} - x_0 \to_{x\to\infty} \infty.$$

Assume $\psi_0(\alpha) > 0$, which holds for sufficiently large α_i. As $\sum_{j=1}^{n} \alpha_j w'(\psi_0(\alpha))$ is ≥ 1 by efficiency, it follows that $\sum_{j=1}^{n} \alpha_j w(\psi_0(\alpha)) \geq \frac{w(\psi_0(\alpha))}{w'(\psi_0(\alpha))}$. Therefore, either there is x with $w'(x) = 0$ and then $\psi_0(\alpha)$ is bounded, or $\psi_0(\alpha) \to_{\alpha_i\to\infty} \infty$ and then $q(\alpha) \geq \frac{w(\psi_0(\alpha))}{w'(\psi_0(\alpha))} - \psi_0(\alpha)$. In either case, $q(\alpha) \to \infty$ as $\alpha_i \to \infty$.

As $c_i(\alpha_{-i})$ is independent of α_i and given α_j, $j \neq i$, $q(\alpha) \to \infty$ as $\alpha_i \to \infty$, the inequality $\sum_{j\neq i} c_j(\alpha_{-j}) \geq (n-1)q(\alpha) - c_i(\alpha_{-i})$ implies that $\sum_{j\neq i} c_j(\alpha_{-j}) \to \infty$ as $\alpha_i \to \infty$. Therefore, for any given α_j, $j \neq i$, there exists a sufficiently large number α_i, such that $\sum_{j\neq i} c_j(\alpha_{-j}) > (n-1)c_i(\alpha_{-i})$ and for every $j \neq i$ $\alpha_i > \alpha_j$. In particular, it implies that there is $j \neq i$ with $\alpha_j < \alpha_i$ and $c_j(\alpha_{-j}) > c_i(\alpha_{-i})$. As $(\psi_i(\alpha) - \alpha_i w(\psi_0(\alpha)) - (\psi_j(\alpha) - \alpha_j w(\psi_0(\alpha)) = c_i(\alpha_{-i}) - c_j(\alpha_{-j})$, we deduce that φ is not strongly scale-monotonic. ∎

Proof of Theorem 2. Let w be smooth and strictly concave. Assume that φ is an efficient, scale-monotonic, and strategy-proof mechanism on $\langle N, V_w\rangle$. For $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ we set $\psi(\alpha_1, \ldots, \alpha_n) := \varphi(\alpha_1 w, \ldots, \alpha_n w)$ and $c(w) = \sum_{i=1}^{n} \psi_i(0, \ldots, 0)$. Note that the mechanism φ is defined on $\langle N, V_w\rangle$

and thus $c(w)$ depends on the domain V_w of φ and thus it depends indirectly on w. For every $\beta > 0$ let $x(\beta)$ be the unique point with $\beta w'(x(\beta)) = 1$. For every $s = 1, \ldots, n$ and $\alpha > 0$, set $x_s(\alpha) = x(s\alpha)$, i.e., $x_s(\alpha)$ is the unique point s.t.

$$s\alpha w'(x_s(\alpha)) = 1.$$

The following lemma bounds from above the total taxes $\sum_{i \in N} \psi_i(\alpha, \ldots, \alpha)$ by a sum of three terms: the first term $c(w)$ which depends on w but does not depend on α, the second term $\theta = \theta(w, \alpha, n) := n\alpha \sum_{s=1}^{n-1} w(x_s(\alpha)) \leq n(n-1)w(x_{n-1}(\alpha))$ which is bounded from above by a function of the the restriction of w to the interval $[0, x_{n-1}(\alpha)]$, and the last term $d = d(w, n, \alpha)$ is given by $d(w, n, \alpha) := nx_n(\alpha) - n(n-1)\alpha w(x_n(\alpha))$. As $(n-1)\alpha w'(x_{n-1}(\alpha)) = 1$, the last term equals $n\left(x_n(\alpha) - \frac{w(x_n(\alpha))}{w'(x_{n-1}(\alpha))}\right)$.

Lemma 2 *For every* $\alpha \in \mathbb{R}_+$

$$\begin{aligned} \sum_{i \in N} \psi_i(\alpha, \ldots, \alpha) &\leq c(w) + \theta + n\left(x_n(\alpha) - \frac{w(x_n(\alpha))}{w'(x_{n-1}(\alpha))}\right) \\ &\leq c(w) + n(n-1)\alpha w(x_{n-1}(\alpha)) + n\left(x_n(\alpha) - \frac{w(x_n(\alpha))}{w'(x_{n-1}(\alpha))}\right) \end{aligned}$$

where $c(w) = \sum_{i \in N} \psi_i(0, \ldots, 0)$ *and* $\theta = n\alpha \sum_{s=1}^{n-1} w(x_s(\alpha))$.

Proof. Given a subset $S \subset N$ we define the vectors $e(S) \in \mathbb{R}^n$ by $e_i(S) = 1$ if $i \in S$ and 0 otherwise. For $i \in N$ and $S \subset N$ we denote by $e_{-i}(S)$ the vector $(e(S))_{-i}$, i.e., the vector obtained by eliminating the i-th coordinate of $e(S)$. Using scale monotonicity, $\psi_i(\alpha e(S)) - \psi_j(\alpha e(S)) \geq 0$ whenever $i \in S$, $j \notin S$, and $\alpha \in \mathbb{R}_+$. Assume that $i \in S$ and $j \notin S$. Then,

$$\psi_i(\alpha e(S)) = \psi_0(\alpha e(S)) - (|S| - 1)\alpha w(\psi_0(\alpha e(S))) + c_i(\alpha e_{-i}(S)) \quad (1)$$

and

$$\psi_j(\alpha e(S)) = \psi_0(\alpha e(S)) - |S|\alpha w(\psi_0(\alpha e(S))) + c_j(\alpha e_{-j}(S)).$$

Therefore, the inequality $\psi_i(\alpha e(S)) - \psi_j(\alpha e(S)) \geq 0$ implies that

$$c_j(\alpha e_{-j}(S)) \leq c_i(\alpha e_{-i}(S)) + \alpha w(\psi_0(\alpha e(S))). \tag{2}$$

For each fixed integer $s = 1, \ldots, n-1$, the average of $c_j(\alpha e_{-j}(S))$ over all pairs $j \notin S$ with $|S| = s$ is denoted $c_-(s)$. Similarly, the average of $c_i(\alpha e_{-i}(S))$ over all pairs $i \in S$ with $|S| = s$ is denoted $c^+(s)$. Note that as w is strictly concave, $\psi_0(\alpha e(S))$ depends only on the cardinality of S and α, and $x_s(\alpha) = \psi_0(\alpha e(S))$, the unique point x with $s\alpha w'(x) = 1$.

Averaging the inequalities (2) over all triples i, j, S with $|S| = s$, $i \in S$ and $j \neq S$ we obtain

$$c_-(s) \leq c^+(s) + \alpha w(x_s(\alpha)).$$

Note that $c^+(s) = c_-(s-1)$, and therefore $\sum_{s=2}^{n-1} c^+(s) = \sum_{s=1}^{n-2} c_-(s)$, and thus by summing the above inequalities over $s = 1, \ldots, n-1$,

$$c_-(n-1) \leq c^+(1) + \sum_{s=1}^{n-1} \alpha w(x_s(\alpha)). \tag{3}$$

By the definition of the functions c_- and c^+, $\sum_{i\in N} c_i(\alpha e_{-i}(N)) = nc^+(n) = nc_-(n-1)$; by the definition of the function x, $\psi_0(\alpha e(N)) = x_n(\alpha)$; and using (1),

$$\sum_{i=1}^{n} \psi_i(\alpha e(N)) = nx_n(\alpha) - n(n-1)\alpha w(x_n(\alpha)) + nc_-(n-1).$$

Therefore, together with (3), we deduce that

$$\sum_{i\in N}\psi_i(\alpha e(N)) \le nc^+(1) + n\alpha\sum_{s=1}^{n-1} w(x_s(\alpha)) + nx_n(\alpha) - n(n-1)\alpha w(x_n(\alpha)). \tag{4}$$

Note that $nc^+(1)$ depends on the domain V_w of φ, and thus is a function of n and w. The second summand on the right-hand side of the inequality, $n\alpha\sum_{s=1}^{n-1} w(x_s(\alpha))$, (equals θ by definition and) is $\le n(n-1)w(x_{n-1}(\alpha))$ by the monotonicity of w and $s \mapsto x_s(\alpha)$. Recall that $(n-1)\alpha w'(x_{n-1}(\alpha)) = 1$ and thus $(n-1)\alpha w(x_n(\alpha)) = \frac{w(x_n(\alpha))}{w'(x(n-1,\alpha))}$. This completes the proof of the lemma. ∎

Let $(n_k)_{k=1}^{\infty}$ be a sequence of positive integers with $n_k > 1$, $0 < \alpha(k) \uparrow \infty$ with $n_k\alpha(k) < \alpha(k+1)$, and $0 < \gamma_k \to_{k\to\infty} 0$.

Lemma 3 *There exists a strictly concave valuation function w in W and sequences $(x_k)_{k=0}^{\infty}$ and $(z_k)_{k=1}^{\infty}$ with $x_0 = 0$ and*

$$(n_k-1)\alpha(k)w'(z_k) = 1 \text{ and } n_k\alpha(k)w'(x_k) = 1 \quad \forall k \ge 1 \tag{5}$$

s.t. for every $k \ge 1$ we have

$$n_k(n_k-1)w(z_k) + n_k z_k \le \gamma_k x_k \tag{6}$$

and

$$w(x_k) = (x_k - z_k)w'(z_k) \tag{7}$$

(and thus $z_k < x_k < z_{k+1}$).

Proof. We define inductively increasing sequences $(x_k)_{k\ge 0}$ and $(z_k)_{k\ge 1}$ with $z_k = x_{k-1} + n_k - 1 < x_k$, and smooth and strictly concave functions

$w_k : [0, x_k] \to \mathbb{R}_+$ and $v_k : [0, z_k] \to \mathbb{R}_+$ so that: $v_1(0) = 0$, w_k coincides with v_k on $[0, z_k]$, v_{k+1} coincides with w_k on $[0, x_k]$, $n_k\alpha(k)w'_k(x_k) = 1$, and $j\alpha(k)w'_k(x_{k-1} + j) = 1$ (for $1 \leq j \leq n_k - 1$).

Set $x_0 = 0$ and $z_k = x_{k-1}+n_k-1$. Let v_1 be a smooth and strictly concave function defined on $[0, z_1]$ so that $j\alpha(1)v'_1(j) = 1$ for every $1 \leq j \leq n_1 - 1$. Assume that w_k is a smooth and strictly concave function defined on the interval $[0, x_k]$ with $w_k(0) = 0$ and $w'(x_k) = \frac{1}{n_k\alpha(k)}$. As $\alpha(k + 1) > n_k\alpha(k)$ we can extend the function w_k to a smooth and strictly concave function v_{k+1} defined on the interval $[0, z_{k+1}]$ so that $v'_{k+1}(x_k + s) = \frac{1}{s\alpha(k+1)}$ for every $1 \leq s < n_{k+1}$.

Assume that v_k is a smooth and strictly concave function defined on $[0, z_k]$ with $(n_k - 1)\alpha(k)v'_k(z_k) = 1$. Let x_k be sufficiently large so that $\gamma_k x_k > n_k(n_k - 1)w(z_k) + n_k z_k$ and $(x_k - z_k)w'_k(z_k) > w_k(z_k) + (x_k - z_k)\frac{n_k-1}{n_k}w'_k(z_k)$. As $w_k(z_k)+(x_k-z_k)w'_k(z_k) > (x_k-z_k)w'_k(z_k) > w_k(z_k)+(x_k-z_k)\frac{n_k-1}{n_k}w'_k(z_k)$ there exists a smooth and strictly concave function w_k defined on $[0, x_k]$ so that the restriction of w_k to $[0, z_k]$ coincides with v_k, $w_k(x_k) = (y_k-z_k)w'_k(z_k)$, and $n_k\alpha(k)w'_k(x_k) = 1$.

The sequence x_k is increasing and the restriction of the smooth and strictly concave function w_k, which is defined on $[0, x_k]$, to the interval $[0, x_{k-1}]$ coincides with w_{k-1}. Therefore there is a smooth and strictly concave function $w : \mathbb{R}_+ \to \mathbb{R}_+$ that extends all functions w_k. This function w obeys condition (5). ∎

Let $(n_k)_k$ be a sequence so that for every n there are infinitely many values of k so that $n_k = n$, $0 < \alpha(k) \uparrow \infty$ with $n_k\alpha(k) < \alpha(k + 1)$, and

$0 < \gamma_k \downarrow 0$. Let $w \in W$ satisfy conditions (5), (6), and (7). Fix a society N with n members ($N = \{1, \ldots, n\}$), and let φ be an efficient, strategy-proof, and scale-monotonic direct mechanism on $\langle N, V_w \rangle$ and $\Psi : \mathbb{R}^n_+ \to \mathbb{R}_+ \times \mathbb{R}^n$ the associated function.

Fix $\gamma > 0$. Let k be sufficiently large so that $n_k = n$, $2\gamma_k < \gamma$, and $\gamma_k x_k > c(w)$. By Lemma 2 we have $\sum_{i=1}^n \Psi_i(\alpha(k)e(N)) \leq c(w) + n(n-1)w(z_k) + n(x_k - w(x_k)/w'(z_k))$. Using equation (7) we have $n(x_k - w(x_k)/w'(z_k) = nz_k$ and therefore by using condition (6) we have $\sum_{i=1}^n \Psi_i(\alpha(k)e(N)) \leq c(w) + \gamma_k x_k \leq 2\gamma_k \Psi_0(\alpha(k)e(N) \leq \gamma \Psi_0(\alpha(k)e(N))$. This completes the proof of Theorem 2. ∎

7 Tightness of the Assumptions

In this section we demonstrate that the conclusions of the theorems break down whenever we weaken our assumptions.

7.1 Efficiency

The mechanism that sets $\varphi_i(v) = 0$ for every $0 \leq i$ is strategy-proof, tax-monotonic, and balances the budget.

7.2 Strategy-Proofness

The mechanism that chooses an efficient outcome $\varphi_0(v)$ and shares the budget equally, i.e.,

$$\varphi_i(v) = \frac{v_i(\varphi_0(v))}{\sum_{i=1}^n v_i(\varphi_0(v))}\varphi_0(v),$$

is an efficient tax-monotonic mechanism that balances the budget.

7.3 Feasibility

The classical Clarke–Groves mechanism that chooses an efficient outcome $\varphi_0(v)$ and

$$\varphi_i(v) = \varphi_0(v) - \sum_{j \neq i} v_j(\varphi_0(v))$$

is symmetric, strongly scale-monotonic, strategy-proof, and efficient.

7.4 Scale Monotonicity

The efficient mechanism that selects

$$\varphi_i(v) = \varphi_0(v) - \sum_{j \neq i} v_j(\varphi_0(v) + 2n \max_{j,k \neq i} v_j(\varphi_0(\mathbf{v_k}))$$

where $\mathbf{v_k}$ stands for the vector of valuation functions $(v_k, \ldots, v_k)$ of length n, is strategy-proof but is not scale-monotonic. We now prove that it is also feasible, i.e., that $\sum_{i=1}^n \varphi_i(v) \geq \varphi_0(v)$. We distinguish two cases: $n \geq 3$ and $n = 2$.

First assume that $n \geq 3$. Then

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i(v) &\geq n\varphi_0(v) - (n-1)\sum_{i=1}^{n} v_i(\varphi_0(v)) + 2n\sum_{i=1}^{n} \max_{j,k\neq i} v_j(\varphi_0(\mathbf{v_k})) \\
&\geq n\varphi_0(v) - (n-1)\sum_{i=1}^{n} v_i(\varphi_0(v)) + 2n(n-2)\max_{j,k} v_j(\varphi_0(\mathbf{v_k})) \\
&\geq n\varphi_0(v) - (n-1)\sum_{i=1}^{n} v_i(\varphi_0(v)) + 2(n-2)\sum_{i=1}^{n} v_i(\varphi_0(v)) \\
&\geq n\varphi_0(v) \geq \varphi_0(v).
\end{aligned}$$

The second inequality uses the fact that for all but two possible values of i, $\max_{j,k\neq i} v_j(\varphi_0(\mathbf{v_k})) \geq \max_{j,k} v_j(\varphi_0(\mathbf{v_k}))$; the third inequality uses the inequality $\max_{j,k} v_j(\varphi_0(\mathbf{v_k})) \geq v_i(\varphi_0(v))$ which follows from the monotonicity of each v_i together with the inequality $\max_k \varphi_0(\mathbf{v_k}) \geq \varphi_0(v)$; the fourth inequality follows from $2(n-2) \geq n-1$ whenever $n \geq 3$.

Assume now that $n = 2$. In this case

$$\varphi_1(v) = \varphi_0(v) - v_2(\varphi_0(v) + 4v_2(\varphi_0(v_2, v_2))$$

$$\varphi_2(v) = \varphi_0(v) - v_1(\varphi_0(v) + 4v_1(\varphi_0(v_1, v_1)).$$

Assume without loss of generality that $v_1(\varphi_0(v)) \geq v_2(\varphi_0(v))$. We distinguish two possible cases.

Case 1: $v_1'(\varphi_0(v)) \geq \frac{1}{2} \geq v_2'(\varphi_0(v))$. Thus $\varphi_0(v_1, v_1) \geq \varphi_0(v)$, implying that $2v_1(\varphi_0(v_1, v_1)) \geq 2v_1(\varphi_0(v)) \geq v_1(\varphi_0(v)) + v_2(\varphi_0(v))$ and thus $\varphi_1(v)) + \varphi_2(v)) \geq 2\varphi_0(v) - v_1(\varphi_0(v)) - v_2(\varphi_0(v)) + 2v_1(\varphi_0(v_1, v_1)) \geq \varphi_0(v)$.

Case 2: $v_1'(\varphi_0(v)) < \frac{1}{2} < v_2'(\varphi_0(v))$. Thus, $\varphi_0(v_1, v_1) < \varphi_0(v) < \varphi_0(v_2, v_2)$.

Together with the monotonicity of v_2 this implies that

$$v_2(\varphi_0(v_2, v_2)) \geq v_2(\varphi_0(v)). \tag{8}$$

As for every x such that $\varphi_0(v_1, v_1) \leq x \leq \varphi_0(v)$, $v_1'(x) \leq \frac{1}{2} < v_2'(x)$,

$$v_2(\varphi_0(v)) - v_2(\varphi_0(v_1, v_1)) > v_1(\varphi_0(v)) - v_1(\varphi_0(v_1, v_1)).$$

I.e., by rearranging the terms,

$$v_1(\varphi_0(v_1, v_1)) > v_1(\varphi_0(v)) - v_2(\varphi_0(v)) + v_2(\varphi_0(v_1, v_1)). \tag{9}$$

Summing inequalities (8) and (9), we deduce that

$$v_1(\varphi_0(v_1, v_1)) + v_2(\varphi_0(v_2, v_2)) > v_1(\varphi_0(v)),$$

which, together with the assumption $v_1(\varphi_0(v)) \geq v_2(\varphi_0(v))$, implies that $\varphi_1(v) + \varphi_2(v) \geq 2\varphi_0(v)$.

7.5 Strong Scale Monotonicity

Theorem 3 shows that there is no efficient, feasible, and strategy-proof direct mechanism that is strongly scale-monotonic. We next show that it is impossible here to replace strong scale monotonicity with the weaker property of scale monotonicity.

We first illustrate an example with two members in the society, i.e., $N = \{1, 2\}$, and $V = \{aw \mid a > 0, w(x) = \sqrt{x}\}$. Simple calculations show that φ is efficient if and only if

$$\psi_0(\alpha_1, \alpha_2) = \frac{(\alpha_1 + \alpha_2)^2}{4}.$$

Set

$$\psi_i(\alpha) = \psi_0(\alpha) - \alpha_j (w \circ \psi_0)(\alpha) + \alpha_j^2/2,$$

where $j \neq i$. Then φ is strategy-proof. In addition, this mechanism is symmetric and scale-monotonic. Indeed, $\psi_1(\alpha) = \frac{(\alpha_1+\alpha_2)^2}{4} - \frac{\alpha_1\alpha_2}{2} = \psi_2(\alpha)$. In addition, $\psi_1(\alpha)+\psi_2(\alpha) = \frac{\alpha_1^2+\alpha_2^2}{2} \geq \frac{(\alpha_1+\alpha_2)^2}{4} = \psi_0(\alpha)$, and thus φ is feasible.

Next we consider the n-member society, i.e., $N = \{1, \ldots, n\}$, and $V = \{aw \mid a > 0, w(x) = \sqrt{x}\}$. Simple calculations show that φ is efficient if and only if

$$\psi_0(\alpha_1, \ldots, \alpha_n) = \frac{(\sum_{i=1}^n \alpha_i)^2}{4}.$$

Let f_i be the function $f_i : \mathbb{R}^n \to \mathbb{R}$ given by

$$f_i(x_1, \ldots, x_n) = \frac{\sum_{k \neq i} x_k^2}{2} + \frac{\sum_{k \neq i} \sum_{i \neq j > k} x_k x_j}{2}.$$

Note that

$$f_i(x_1, \ldots, x_n) - f_j(x_1, \ldots, x_n) = \frac{(x_j - x_i)}{2} \sum_{k=1}^n x_k$$

and

$$\sum_{i=1}^n f_i(x) = \frac{(n-1)}{2} \sum_{k=1}^n x_k^2 + \frac{n-2}{2} \sum_{j>k} x_j x_k.$$

Set

$$\psi_i(\alpha) = \psi_0(\alpha) - \sum_{j \neq i} \alpha_j (w \circ \psi_0)(\alpha) + f_i(\alpha).$$

Then φ is strategy-proof and symmetric. We next show that it is scale-monotonic. Indeed,

$$\psi_i(\alpha) = \frac{(\sum_{i=1}^n \alpha_i)^2}{4} - \sum_{k=1}^n \alpha_k (w \circ \psi_0)(\alpha) + \alpha_i \frac{\sum_{k=1}^n \alpha_k}{2} + f_i(\alpha).$$

Therefore,

$$\psi_i(\alpha) - \psi_j(\alpha) = (\alpha_i - \alpha_j)\frac{\sum_{k=1}^n \alpha_k}{2} + f_i(\alpha) - f_j(\alpha).$$

As $f_i(x) - f_j(x) = \frac{(x_j - x_i)}{2}\sum_{k=1}^n x_k$, we deduce that $\psi_i(\alpha) - \psi_j(\alpha) = 0$, which proves in particular that $\psi_i(\alpha) \geq \psi_j(\alpha)$ whenever $\alpha_i \geq \alpha_j$, thus φ is scale-monotonic. In addition, one can verify that φ is feasible. Indeed,

$$\begin{aligned}
\sum_{i=1}^n \psi_i(\alpha) - \psi_0(\alpha) &= (n-1)[\psi_0(\alpha) - \sum_{k=1}^n \alpha_k (w \circ \psi_0)(\alpha)] + \sum_{k=1}^n f_k(\alpha) \\
&= -(n-1)\frac{(\sum_{k=1}^n \alpha_k)^2}{4} + \sum_{k=1}^n f_k(\alpha) \\
&= -(n-1)\left[\frac{\sum_{k=1}^n \alpha_k^2}{4} + \frac{\sum_{j>k} \alpha_j \alpha_k}{2}\right] + \sum_{k=1}^n f_k(\alpha) \\
&= \frac{n-1}{4}\sum_{k=1}^n \alpha_k^2 - \frac{1}{2}\sum_{j>k} \alpha_j \alpha_k \\
&\geq \frac{n-1}{4}\sum_{k=1}^n \alpha_k^2 - \frac{1}{4}\sum_{j>k} (\alpha_j^2 + \alpha_k^2) \\
&= \frac{n-1}{4}\sum_{k=1}^n \alpha_k^2 - \frac{n-1}{4}\sum_{k=1}^n \alpha_k^2 \\
&= 0.
\end{aligned}$$

Therefore

$$\sum_{i=1}^n \psi_i(\alpha) \geq \psi_0(\alpha).$$

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