Renewal Theory for Sampling Without Replacement

Abraham Neyman

The Annals of Probability, Vol. 10, No. 2 (May, 1982), 464-481.

Stable URL:
http://links.jstor.org/sici?sici=0091-1798%28198205%2910%3A2%3C464%3ARTFSWR%3E2.0.CO%3B2-2

The Annals of Probability is currently published by Institute of Mathematical Statistics.

The Annals of Probability
1982, Vol. 10, No. 2, 464-481

RENEWAL THEORY FOR SAMPLING WITHOUT REPLACEMENT[1]

BY ABRAHAM NEYMAN

University of California, Berkeley

Let π be a finite set, λ a probability measure on π, $0 < x < 1$ and $a \in \pi$. Let $P(a, x)$ denote the probability that in a random order of π, a is the first element (in the order) for which the λ-accumulated sum exceeds x. The main result of the paper is that for every $\varepsilon > 0$ there exist constants $\delta > 0$ and $K > 0$ such that if $\rho = \max_{a\in\pi} \lambda(a) < \delta$ and $K\rho < x < 1 - K\rho$ then $\sum_{a\in\pi} | P(a, x) - \lambda(a) | < \varepsilon$. This result implies a new variant of the classical renewal theorem, in which the convergence is uniform on classes of random variables.

1. Introduction. Let π be a finite set, λ a probability measure on π, $0 < x < 1$ and $a \in \pi$. Let $P(a, x)$ be the probability that in a random order of π, a is the first element (in the order) for which the λ-accumulated sum exceeds x. That is, if for every order $\mathscr{R}$ of π and $a \in \pi$, we denote $\mathscr{P}_a^{\mathscr{R}} = \{b : b \in \pi, b\mathscr{R}a\}$ the set of elements of π preceding a in the order $\mathscr{R}$, then $P(a, x) = |\{\mathscr{R} : \lambda(\mathscr{P}_a^{\mathscr{R}}) < x \leq \lambda(\mathscr{P}_a^{\mathscr{R}}) + \lambda(a)\}|/(|\pi|)!$ (where $|\pi|$ denotes the number of elements in π). The distance between the probability measures $P(\cdot, x)$ and $\lambda(\cdot)$ on π is defined by $\| P(\cdot, x) - \lambda(\cdot) \| = \sum_{a\in\pi} | P(a, x) - \lambda(a) |$. The main result of this paper is that $\| P(\cdot, x) - \lambda(\cdot) \|$ tends to 0 as $\max_{a\in\pi}\lambda(a)$ tends to 0. In fact we will prove the following stronger result.

MAIN THEOREM. *For every $\varepsilon > 0$ there exist constants $\delta > 0$ and $K > 0$ such that if $\rho = \max_{a\in\pi}\lambda(a) < \delta$, and $K\rho < x < 1 - K\rho$ then $\| P(\cdot, x) - \lambda(\cdot) \| < \varepsilon$.*

It will be shown that this result is an appropriate formulation of renewal theory for sampling without replacement. Moreover, it implies a new variant of the classical renewal theorem, in which the convergence is uniform on a class of random variables.

The origin of the problem. The problem arises in studying the asymptotic value for non-atomic games. In fact it is equivalent to a special case of the problem raised by Aumann and Shapley [1, page 10] whether $bv'NA \subset ASYMP$. In paper [6], the results obtained here are used to give an affirmative answer to this question, i.e., to prove that indeed $bv'NA \subset ASYMP$. A very brief summary of some basic related facts about non-atomic games will now be presented.

A *game* is a set-function $v : \mathscr{C} \to \mathbb{R}$, where $(I, \mathscr{C})$ is a measurable space (the *player space*) with $v(\emptyset) = 0$; it is called finite if $\mathscr{C}$ is finite. The Shapley value of a finite game v is the measure on $\mathscr{C}$ given by

$$(\psi v)(A) = \frac{1}{n!} \sum_{\mathscr{R}} (v(\mathscr{P}_A^{\mathscr{R}} \cup A) - v(\mathscr{P}_A^{\mathscr{R}})),$$

where the sum runs over all orders $\mathscr{R}$ on the players (atoms of $\mathscr{C}$) and $\mathscr{P}_A^{\mathscr{R}}$ is the set of players preceding A (an atom of $\mathscr{C}$) in the order $\mathscr{R}$.

To define a Shapley value for a game v that is not necessarily finite, one may approximate it by finite games. Specifically, if Π is a finite subfield of $\mathscr{C}$, define a finite game v_Π on Π by $v_\Pi = v | \Pi$. Given an S in $\mathscr{C}$ (a "coalition"), an increasing sequence $\{\Pi_1, \Pi_2, \cdots\}$ of finite subfield of $\mathscr{C}$ is called S-admissible if $S \in \Pi_1$ and $\cup_i\Pi_i$ generates $\mathscr{C}$. An

Received April 1979; revised January 1981.

[1] The work for this paper was partially supported by the Israel National Council for Research and Development and partially supported by National Science Foundation Grant MCS-79-06634.

AMS 1970 subject classifications. Primary 60K70, 60G30; secondary 90E70.

Key words and phrases. Renewal theory, sampling without replacement.

asymptotic value of v is a set function ϕv on $\mathscr{C}$ such that for all coalitions S and all S-admissible sequences, we have

$$\lim_{n\to\infty}\psi v_{\Pi_n}(S) = (\phi v)(S).$$

Not all games have asymptotic values; but when they exist, they are obviously unique.

Let μ be a non-atomic probability measure on $\mathscr{C}$, and let f be monotonic on [0, 1]. It has long been known [Kannai, Aumann-Shapley] that when f is absolutely continuous, $f\circ\mu$ has an asymptotic value. Our result implies immediately that when f is a step-function continuous at 0 and at 1, $f\circ\mu$ has an asymptotic value; the atoms of the finite fields Π appearing in the definition of the asymptotic value, correspond to the members of the sets π.

When f is singular continuous, we deduce in a separate paper from our result that $f\circ\mu$ has an asymptotic value. Putting all this together yields that $f\circ\mu$ has an asymptotic value whenever f is continuous at 0 and at 1. This is the essential content of the statement $bv'NA \subset ASYMP$.

The relation to classical renewal theory. Let X be a random variable having positive expectation $E(X)$ and let F_X denote its cumulative distribution function. Let X_i be a sequence of i.i.d. random variables with distribution dF_X. Let $S_n = \sum_{i=1}^{n} X_i$, and for every a, t, α *in* $\mathbb{R}$ define

$$V(a) = \textstyle\sum_{n=0}^{\infty} \operatorname{Prob}(S_n < a), \quad U(t, \alpha) = V(t) - V(t-\alpha), \quad g_X(t, \alpha) = U(t, \alpha) - \alpha/E(X).$$

Let $L(X)$ be the closed group generated by the support of dF_X. The classical renewal theorem is as follows.

Renewal Theorem. *If X is a random variable with $X \geq 0$, then for every $\alpha \in L(X)$,*

$$g_X(t, \alpha) \to_{t\to\infty} 0.$$

For nonnegative random variables X with finite expectation, we deduce from the Renewal Theorem by using Lebesgue dominated convergence theorem that $\int |g_X(t, \alpha)| \, dF_X(\alpha) \to_{t\to\infty} 0$ (for instance, bound $U(t, \alpha) \leq V(\alpha) \leq K(|\alpha| + 1)$). On the other hand, let X be a nonnegative random variable for which $\int |g_X(t, \alpha)| \, dF_X(\alpha) \to_{t\to\infty} 0$. Then $U(t, \alpha)$ converges to $\alpha/E(X)$ (as $t \to \infty$) in dF_X probability and by monotonicity in α, dF_X - a.s. Obviously $L = \{\alpha \mid g_X(t, \alpha) \to_{t\to\infty} 0\}$ is a group and by the same monotonicity it is also closed. All together it implies that $L \supset L(X)$. Therefore the following statement is equivalent to the Renewal Theorem.

Variant of Renewal Theorem. *If X is a random variable with $X \geq 0$ and $E(X) < \infty$ then*

$$G_X(t) = \int |g_X(t, \alpha)| \cdot dF_X(\alpha) \to_{t\to\infty} 0.$$

Our result implies this variant of the renewal theorem and actually proves that this limit holds uniformly on classes of random variables. For instance, it is possible to take the supremum over all bounded random variables $X \leq K$. The implication follows from the following simple observations:

(1) Define $N(t)$ by $S_{N(t)} \geq t > S_{N(t)-1}$ and $Y_t = X_{N(t)} = S_{N(t)} - S_{N(t)-1}$. Then for every measurable $B \subset \mathbb{R}$,

$$\operatorname{Prob}(Y_t \in B) = \int_B U([t-\alpha, t)) \, dF_X(\alpha).$$

(2) Let X be a nonnegative random variable with finite expectation $E(X)$, and let $M_t = 3(t/E(X))$. Then

$$\operatorname{Prob}(S_{M_t} > 2t) \to_{t\to\infty} 1$$

and for every $K > 0$

$$\text{Prob}(\exists k, 1 \le k \le M_t \quad \text{with} \quad X_k \ge t/K) \to 0.$$

(3) For every family of Borel subsets $B_t \subset \mathbb{R}$

$$\text{Prob}\left(\left|\frac{1}{M_t}\sum_{i=1}^{M_t} X_i I_{B_t}(X_i) - \int_{B_t} \alpha\, dF_X(\alpha)\right| > \varepsilon\right) \to_{t\to\infty} 0,$$

and finally, for $t > 0$.

$$\int (U[t-\alpha, t))\cdot dF_X(\alpha) = 1 = \int \alpha/E(X)\cdot dF_X(\alpha), \tag{4}$$

and therefore,

$$G_X(t) = 2 \sup_B \int_B (U([t-\alpha, t)) - \alpha/E(X))\, dF_X(\alpha)$$

$$= 2 \sup_B\left(\text{Prob}(Y_t \in B) - \left(\int_B \alpha\, dF_X(\alpha)\right)/E(X)\right).$$

The variant of the renewal theorem is then obtained from our main theorem by conditioning on the set $\{X_1, \cdots, X_{M_t}\}$.

2. Reformulation of the main result and basic concepts. We will present an equivalent formulation of the main theorem in terms of limits of sequences. We first start with some definitions.

Let π be a finite set, and let $(\pi, 2^\pi, \lambda)$ be a measure space, π is said to be a *partition* if λ is a probability measure on π. The *parameter* of a partition is the maximal measure of an atom in $(\pi, 2^\pi, \lambda)$ and is denoted by $\rho(\pi, \lambda)$ or $\rho(\pi)$, i.e., $\rho(\pi) = \max\{\lambda(a): a \in \pi\}$. A sequence $(\pi_k)_{k=1}^\infty$ of partitions is called *shrinking* if $\lim_{k\to\infty}\rho(\pi_k) = 0$. A sequence $(x_k)_{k=1}^\infty$ of numbers is called *null* if $\lim_{k\to\infty}x_k = 0$, and it is called (π_k)*-divergent* (where $(\pi_k)_{k=1}^\infty$ is a sequence of partitions) if $\lim_{k\to\infty}x_k/\rho(\pi_k) = \infty$.

MAIN THEOREM.* *For every shrinking sequence of partitions $(\pi_k)_{k=1}^\infty$ and every sequence of subsets $T_k \subset \pi_k$, and every sequence $(x_k)_{k=1}^\infty$ such that $0 < x_k < 1$ and both $(x_k)_{k=1}^\infty$ and $(1 - x_k)_{k=1}^\infty$ are (π_k)-divergent*

$$\lim_{k\to\infty}\lambda(T_k) - P(T_k, x_k) = 0.$$

There are three concepts basic to the proof: the random walk, the continuous embedding, and the persistent numbers. We will present these concepts here.

The random walk generated by sampling without replacement. Let $(A, 2^A, \lambda)$ be a finite measure space. Let $|A| = n$, and let $\mathcal{R}$ be an order of A, $a_1\mathcal{R}a_2\mathcal{R}a_3 \cdots \mathcal{R}a_n$. The $(n+1)$-tuple$(\lambda(\emptyset), \lambda(\mathcal{P}_{a_2}^{\mathcal{R}}), \cdots, \lambda(\mathcal{P}_{a_n}^{\mathcal{R}}), \lambda(A))$ is called the *walk generated by A in the order $\mathcal{R}$*. The *random walk generated by A* is the walk generated by A in a random order $\mathcal{R}$ (i.e., every order has equal probability, namely $1/n!$). Let $I \subset \mathbb{R}$ be a subset of the reals. The number of visits of the walk generated by A in the order $\mathcal{R}$ to the set I is denoted by $N_A(\mathcal{R}, I)$:

$$N_A(\mathcal{R}, I) = \begin{cases} |\{a: a \in A \wedge \lambda(\mathcal{P}_a^{\mathcal{R}}) \in I\}| & \text{if } \lambda(A) \notin I \\ 1 + |\{a: a \in A \wedge \lambda(\mathcal{P}_a^{\mathcal{R}}) \in I\}| & \text{if } \lambda(A) \in I. \end{cases}$$

When $\mathcal{R}$ is an order on a set containing A, we define $N_A(\mathcal{R}, I)$ as $N_A(\bar{\mathcal{R}}, I)$ where $\bar{\mathcal{R}}$ is the order induced on A by $\mathcal{R}$. The expected number of visits of the random walk generated by A in the set I is denoted $U_A(I)$, i.e., $U_A(I) = E(N_A(\mathcal{R}, I)) \equiv (1/k!)\sum_{\mathcal{R}} N_A(\mathcal{R}, I)$ where $\mathcal{R}$ runs over all orders of a set having k elements and containing A.

The continuous embedding. Let π be a set. Consider the measure space $(2^\pi, \mathscr{C})$ where 2^π is the power set of π and $\mathscr{C}$ is the σ-field of subsets of 2^π generated by the sets that consist of all super sets of a given finite set, i.e., $\mathscr{C}$ is the minimal σ-field of subsets of 2^π, such that for every finite subset T of π, $\{A:A \subset \pi, A \supset T\} \in \mathscr{C}$. We associate to π a ("right continuous") stochastic process $Z_t:[0, 1] \to 2^\pi$, which is nondecreasing, has stationary increments, and the increments are sums (unions) of independent random variables. That means

(1) $o \le s \le t \le 1 \Rightarrow Z_t \supset Z_s$ (nondecreasing);
(2) $Z_t \backslash Z_{t'}$ has the same distribution as $Z_s \backslash Z_{s'}$ whenever $t - t' = s - s'$ (stationarity);
(3) for every t, Z_t is the union of independent random variables, $Z_1 = \pi$, and $Z_0 = \phi$.

This is done as follows: Let $(\Omega, \mathscr{B}, \mathbb{P})$ be a probability space such that for every $a \in \pi$ there is a real valued random variable X_a, defined on $(\Omega, \mathscr{B}, \mathbb{P})$, having uniform distribution on (0, 1], and such that the random variables X_a are mutually independent. The stochastic process $Z_t(\omega)$, which is defined by $Z_t(\omega) = \{a:a \in \pi, X_a(\omega) \le t\}$, has all these desired properties.

This continuous embedding induces for every $\omega \in \Omega$ a weak ordering $\mathscr{R}(\omega)$ on π by $a\mathscr{R}(\omega)b \Leftrightarrow X_a(\omega) \le X_b(\omega)$. When π is finite or countable then this is an order with probability one.

The continous embedding has many important merits[2]; part of them could be seen from the properties that are listed previously, and others will be understood along the proof. There is one basic hidden gain (probably the most important one) which we would like to stress here. The probability space of orders of a finite set is a finite one, and thus has a limited number of "equivalence classes" which restrict the "calculus of conditioning." By considering the orders as induced by our continuous probability space, the "calculus of conditioning" is enriched.

The persistent numbers. Our main result implies in particular the following (much) weaker property: Let (π_k) be a shrinking sequence of partition, $A_k \subset \pi_k$ with $\lambda(A_k) \ge \alpha > 0$ and let (x_k) be a null, (π_k)-divergent sequence. Then

$$\liminf_{k\to\infty} P(A_k, x_k) > 0.$$

We do not know how to prove this almost "obvious" fact without proving our result. Having that result would not shorten our proof, but would simplify it (as will be explained in the sequel). Actually, it would be helpful to have this property for a specific sequence $A_k = \pi_k(\alpha)$ which we now define. For every partition π, and every $0 < \varepsilon < 1$ we select a subset $\pi(\varepsilon)$ of π that satisfies the following properties:

(i) $a \in \pi(\epsilon),\ b \in \pi\backslash\pi(\epsilon) \Rightarrow \lambda(a) \le \lambda(b)$
(ii) $\epsilon \le \lambda(\pi(\epsilon)) < \epsilon + \max\{\lambda(a):a \in \pi(\epsilon)\}$.

DEFINITION. A number $0 < \epsilon < 1$ is called *persistent* if for every shrinking sequence of partitions (π_k), and every null, (π_k)-divergent sequence (x_k),

$$\liminf_{k\to\infty} P(\pi_k(\epsilon), x_k) > 0.$$

The set of all persistent numbers is denoted by Q.

REMARK. Indeed it follows from our result that every number $0 < \epsilon < 1$ is persistent. However, we could not prove it directly. We could prove that there are persistent numbers without much difficulty, but not that all numbers in (0, 1) are persistent. Having the result that all numbers in (0, 1) are persistent would save carrying the conditioning on the persistent numbers in Sections 5, 6 and 7.

[2] It can be used to prove many limit theorems for sampling without replacement. For instance, it provides an extremely short proof of the central limit theorem for sampling without replacement. Moreover, it might be used to unify the methods for proving limit theorems for sampling without replacement.

We close this section with some definitions and notations which will be used in the sequel. Let π be a finite measure space (We will often use the abbreviation π for the measure space $(\pi, 2^\pi, \lambda)$. No confusion will result.) Let $x > 0$ be a positive number. An element a of π is the *cover of x in the order $\mathscr{R}$* if $x - \lambda(a) \leq \lambda(P_a^{\mathscr{R}}) < x$. The cover of x in the order $\mathscr{R}$ is denoted $a(\mathscr{R}, x)$. Let π be a partition, and $\mathscr{R}$ an order of π, then $\mathscr{R}^-$ denotes the reverse order, i.e., $a\mathscr{R}^-b$ iff $b\mathscr{R}a$. For every partition π we select an order $\mathscr{R}_\pi$ satisfying $a\mathscr{R}_\pi b \Rightarrow \lambda(a) \leq \lambda(b)$, and we denote by $a(\epsilon)$ the cover of ϵ in the order $\mathscr{R}_\pi$, i.e., $a(\epsilon) = a(\mathscr{R}_\pi, \epsilon)$. If $\mathscr{R}$ is an order of π, $0 < x < 1$ then $\Gamma(\mathscr{R}, x)$, the *x-initial with respect to $\mathscr{R}$*, is defined by

$$\Gamma(\mathscr{R}, x) = \cap_{\{a:\lambda(\mathscr{P}_a^{\mathscr{R}}) \geq x\}} \mathscr{P}_a^{\mathscr{R}},$$

and $\Gamma^-(\mathscr{R}, x)$ – the *x-final with respect to $\mathscr{R}$* is defined by $\Gamma^-(\mathscr{R}, x) = \Gamma(\mathscr{R}^-, 1 - x)$. For every partition π and $\epsilon > 0$ denote by $\pi(\epsilon)$ the subset $\Gamma(\mathscr{R}_\pi, \epsilon)$ of π.

3. The structure of the proof. This section intends to guide the reader in the long and involved proof by describing the structure (not the ideas) of the proof. The main theorem regards the measure $P(\cdot, x)$ where x is far away from the "edges," i.e., when $x/\rho(\pi)$ and $(1 - x)/\rho(\pi)$ are large. However, we first derive a result concerning the measure $P(\cdot, x)$ for x "far but close" to the edges. We prove (Corollary 9.2) that if $(\pi_k)_{k=1}^\infty$ is a shrinking sequence of partitions and $(x_k)_{k=1}^\infty$ is null, (π_k)-divergent then $\|P(\cdot, x_k) - \lambda\| \to_{k\to\infty} 0$, or in other words for every $A_k \subset \pi_k$, $\lim_{k\to\infty} P(A_k, x_k) - \lambda(A_k) = 0$. The derivation of the main theorem from Corollary 9.2 is accomplished (in Section 9) by using "standard techniques" based on laws of large numbers for sampling without replacement. For proving Corollary 9.2 it is enough to prove that (Lemma 9.1) if $(\pi_k)_{k=1}^\infty$ is a shrinking sequence of partitions then for every $\eta > 0$ there exist sets $T_k \subset \pi_k$, with $\lambda(\pi_k) - \lambda(T_k) < \eta$ such that for every $\tilde{T}_k \subset T_k$ and for every null, (π_k)-divergent sequence (x_k),

$$\limsup_{k\to\infty} |P(\tilde{T}_k, x_k) - \lambda(\tilde{T}_k)| < \eta.$$

Now, fix a persistent number $\epsilon > 0$ which is not the smallest persistent number (alternatively, $\epsilon_1, \epsilon \in Q$ with $0 < \epsilon_1 < \epsilon < 1$). Every partition π will be decomposed into 3 parts. These parts will be denoted by π^1, π^2, π^3 and will depend on a number $K = K(\epsilon)$.

$$\pi^1 = \pi(\epsilon)$$

$$\pi^3 = \{a: a \in \pi, \lambda(a) \geq K\cdot\lambda(a(\epsilon))\}$$

and

$$\pi^2 = \pi\backslash\pi^1\backslash\pi^3.$$

For every $T \subset \pi$, we denote by T^1, T^2, T^3 the subsets of T, $T \cap \pi^1$, $T \cap \pi^2$, $T \cap \pi^3$ respectively. Section 5 proves that whenever π_k is a shrinking sequence of partitions, $b_k \in \pi_k^3$, and x_k a null, (π_k)-divergent sequence, then

$$(*) \qquad \limsup_{k\to\infty} \left| \frac{P(b_k, x_k)}{\lambda(b_k)} - 1 \right| \leq 2/K$$

which in particular implies that for every subset T_k of π_k,

$$(**) \qquad \limsup_{k\to\infty} |P(T_k^3, x_k) - \lambda(T_k^3)| \leq 2/K.$$

This is done in Section 5. Actually a more general statement is proved there so that it can be used later for the estimations regarding π_k^2. For "taking care" of π_k^2 we decompose π_k^2 into a large number of "thin sets" $\pi^{2,1}, \cdots, \pi^{2,m}$ where $m = m(K, h)$ such that for every $1 \leq i \leq m$

$$\sup \frac{\lambda(a) - \lambda(b)}{\lambda(a(\epsilon))} \leq h$$

where the supremem is taken over all pairs a, b in $\pi^{2,i}$. We prove (Proposition 6.1) that if (π_k) is a shrinking sequence of partitions, then for every $\eta > 0$ there exist $B_k^i \subset \pi^{2,i}$ with $\lambda(\pi^{2,i}) - \lambda(B_k^i) < \eta$ such that for every $\tilde{B}_k^i \subset B_k$ and every null, (π_k)-divergent sequence

$(x_k)_{k=1}^{\infty}$,

$$\limsup_{k\to\infty} |P(\tilde{B}_k^i, x_k) - \lambda(\tilde{B}_k^i)| - g(h)\lambda(B_k^i) \le 0$$

where g is a function satisfying $g(h) \to_{h\to 0} 0$. From this we deduce (part of Proposition 7.1) that if $(\pi_k)_{k=1}^{\infty}$ is a shrinking sequence of partitions, then for every $\eta > 0$ there exist subsets B_k of π_k^2 with $\lambda(\pi_k^2) - \lambda(B_k) < \eta$ such that for every $\tilde{B}_k \subset B_k$ and for every null, (π_k)-divergent sequence (x_k),

$$(***) \qquad \limsup |P(\tilde{B}_k, x_k) - \lambda(\tilde{B}_k)| < \eta.$$

Combining $(**)$ and $(***)$, we conclude (Proposition 7.1) that if $(\pi_k)_{k=1}^{\infty}$ is a shrinking sequence of partitions, then for every $\eta > 0$ there exist sets $T_k \subset \pi_k^2 \cup \pi_k^3$, $(T_k = B_k \cup \pi_k^3)$ with $\lambda(\pi_k^3 \cup \pi_k^2) - \lambda(T_k) < \eta$ such that for every $\tilde{T}_k \subset T_k$ and for every null, (π_k)-divergent sequence (x_k),

$$\limsup |P(\tilde{T}_k, x_k) - \lambda(\tilde{T}_k)| \le \eta.$$

Assuming that every number in (0, 1) is persistent, it would be easy to complete the proof as is done in Section 9. However, we do not know that every number is persistent and thus we prove in Section 8, by using Proposition 7.1 together with Lemma 4.1, that $Q = (0, 1)$.

4. Preliminary results. The stochastic process Z_t associated with the set π will be denoted by π^t. No confusion will result. The following lemma could be easily verified.

LEMMA 4.1. *Let π be a partition. Then for every $a \in \pi$ and $0 < y < 1$,*

$$P(a, y) = \frac{1}{|\pi|} U_{\pi\setminus\{a\}}([y - \lambda(a), y)) = \int_0^1 \operatorname{Prob}(\lambda(\pi^t \setminus \{a\}) \in [y - \lambda(a), y))\, dt.$$

LEMMA 4.2. (The reflection principle). *Let I, $J \subset \mathbb{R}$ be subsets of the reals, π a partition and $A \subset \pi$. Then if $I \supset \lambda(A) - J$,*

$$U_A(I) \ge U_A(J).$$

PROOF. This follows immediately from the equality and inequality

$$N_A(\mathcal{R}; J) = N_A(\mathcal{R}^-; \lambda(A) - J) \le N_A(\mathcal{R}^-; I).$$

LEMMA 4.3. *Let π be a partition, $A \subset \pi$ and $\rho = \rho(A) = \max\{\lambda(a): a \in A\}$. If I and J are intervals in $[0, \lambda(A)]$ satisfying $\mu(I) > \mu(J) + \rho$ (where μ stands for Lebesgue measure on the line) then*

$$U_A(I) \ge U_A(J).$$

PROOF. By the positivity of the measure U_A, it suffices to consider only a closed interval I. Hence by conditioning on final segments of the random walk generated by A and appealing to the reflection principle,

$$U_A(J) \le U_A([0, \mu(J)]) \le U_A([0, \mu(I) - \rho]) \le U_A(I). \qquad \square$$

COROLLARY 4.4. *Let π be a partition, $A \subset \pi$ and $\rho = \rho(A) = \max_{a\in A}\lambda(a)$. Then for every interval I, $I \subset [0, \lambda(A)]$ we have*

$$\text{(a)} \qquad U_A(I) \le \left[\frac{\lambda(A)}{\mu(I) + 2\rho}\right]^{-1} (|A| - 1)$$

and if $\mu(I) > 2\rho$

$$\text{(b)} \qquad U_A(I) \ge \left(\frac{\lambda(A)}{\mu(I) - 2\rho} + 1\right)^{-1} (|A| + 1).$$

PROOF. (a) There exists $\left[\frac{\lambda(A)}{\mu(I)+2\rho}\right]$ disjoint intervals which are contained in (0, $\lambda(A)$) such that the length of each of these intervals is $\mu(I)+2\rho$. In the union of these intervals there are at most $|A|-1$ visits. (The entire walk has $|A|+1$ visits, one of them at 0 and another at $\lambda(A)$.) But in each of these intervals the expected number of visits is at least $U_A(I)$ (Lemma 4.3) and therefore

$$U_A(I) \le \left[\frac{\lambda(A)}{\mu(I)+2\rho}\right]^{-1} (|A|-1).$$

(b) is similarly proved.

LEMMA 4.5. *Let π be a partition and $A \subset B \subset \pi$, with $a \in A$, $b \in B\backslash A \Rightarrow \lambda(a) \le \lambda(b)$. Then for each interval $I \subset [0, \lambda(A)]$ which contains 0 we have*

$$U_B(I) \le U_A(I).$$

PROOF. Each order $\mathscr{R}$ on B induces an order $\bar{\mathscr{R}}$ on A. ($\bar{\mathscr{R}}$ is the order $\mathscr{R}$ restricted to A). As $N_B(I, \mathscr{R}) \le N_A(I, \bar{\mathscr{R}})$, the lemma is proved.

The rest of this section will be used only in Section 6.

LEMMA 4.6. *For every shrinking sequence of partitions $(\pi_k)_{k=1}^{\infty}$ and for every null, (π_k)-divergent sequence $(t_k)_{k=1}^{\infty}$,*

(a) $\mathrm{Prob}(\lambda(\bar{\pi}_k) > 1-4t_k) \to_{k\to\infty} 1$, *where* $\bar{\pi}_k = \pi_k^{1-t_k}\backslash\pi_k^{t_k}$;

(b) $\mathrm{Prob}(|\bar{\pi}_k| > (1-4t_k)|\pi_k|) \to_{k\to\infty} 1$;

(c) *if* $A_k \subset \pi_k$ *and* $\liminf(|A_k|\rho(\pi_k)) > 0$, *then*

$$\mathrm{Prob}(|A_k\backslash\bar{\pi}_k| \ge t_k|A_k|) \to 1 \quad as \quad k\to\infty.$$

PROOF. $\lambda(\bar{\pi}_k)$ is a sum of $|\pi_k|$ independent random variables Y_a, $a \in \pi_k$ satisfying $\mathrm{Prob}(Y_a = \lambda(a)) = 1-2t_k$ and $\mathrm{Prob}(Y_a = 0) = 2t_k$. Therefore

$$E(\lambda(\bar{\pi}_k)) = \textstyle\sum_{a\in\pi_k} E(Y_a) = 1-2t_k$$

and $V(\lambda(\bar{\pi}_k)) = \sum_{a\in\pi} V(Y_a) = \sum_{a\in\pi} 2t_k(1-2t_k)\lambda^2(a) \le 2t_k\rho(\pi_k)$. Using Chebyshev's inequality and the fact that (t_k) is (π_k)-divergent, we have,

$$\mathrm{Prob}(\lambda(\bar{\pi}_k) \le 1-4t_k) \le (2t_k\rho(\pi_k))/(4t_k^2) \to_{k\to\infty} 0.$$

Similarly $|\bar{\pi}_k|$ is the sum of $|\pi_k|$ i.i.d. random variables Y_a, $a \in \pi_k$ satisfying $\mathrm{Prob}(Y_a = 1) = 1-2t_k$ and $\mathrm{Prob}(Y_a = 0) = 2t_k$. Thus $E(|\bar{\pi}_k|) = (1-2t_k)|\pi_k|$ and $V(\bar{\pi}_k) = 2t_k(1-2t_k)|\pi_k|$. Using Chebyshev's inequality we get that $\mathrm{Prob}(|\bar{\pi}_k| \le (1-4t_k)|\pi_k|) \le (2t_k|\pi_k|)^{-1}$. As $\rho(\pi_k) \ge |\pi_k|^{-1}$ and as t_k is (π_k)-divergent we conclude that $P(|\bar{\pi}_k| \le (1-4t_k)|\pi_k|) \to 0$ as $k\to\infty$, which proves (b). Similarly applying Chebyshev's inequality to the random variable $|A_k\backslash\bar{\pi}_k|$ we have

$$\mathrm{Prob}(|A_k\backslash\bar{\pi}_k| < t_k|A_k|) \le 2/(t_k|A_k|) \to 0 \cdot \text{ as } \quad k\to\infty$$

which completes the proof of Lemma 4.6.

LEMMA 4.7. *Let ℓ be a positive integer, G a finite set, $\lambda: G \to [z_1, z_2]$. Let $F: (-\infty, \infty) \to [0, 1]$ be defined by:*

$$F(\alpha) = 2^{-|G|}|\{A : A \subset G, \textstyle\sum_{a\in A}\lambda(a) < \alpha\}|.$$

Then there exists a probability space $(\Omega, \mathscr{B}, \mathbb{P})$ with random variables X, Y such that:

(i) $\mathbb{P}(\ell\cdot z_1 \le Y - X \le \ell\cdot z_2) \ge 1 - \ell/\sqrt{|G|}$

(ii) $\mathbb{P}(X \le \alpha) = F_X(\alpha) = F_Y(\alpha) = \mathbb{P}(Y \le \alpha) = F(\alpha)$.

PROOF. We start with some notations that will serve us along the proof. If Ω_1, Ω_2 are finite sets and $(\Omega_i, 2^{\Omega_i}, P_i)$ $i = 1, 2$ are measure spaces then

$$(\Omega_1, 2^{\Omega_1}, P_1) \oplus (\Omega_2, 2^{\Omega_2}, P_2)$$

is defined to be the finite measure space in which the "sample space" is a union of two disjoint copies of Ω_1 and Ω_2 (this "sample space" is denoted by $\Omega_1 \oplus \Omega_2$) and the measure of a subset is the sum of the P_i measures of the intersections with Ω_i. This new measure will be denoted by $P_1 \oplus P_2$. Let n, k, m be integers. Then $\binom{n}{k}$ will denote the binomial coefficients, and we define $(n; k; m) = \min\left(\binom{n}{k}, \binom{n}{k+m}\right)$, and $\overline{(n; k; m)} = (n; k; -m)$.

Let $|G| = n$, $\Omega = \{(k; \mathscr{R}) : 0 \le k \le n, \mathscr{R}$ an order of $G\}$, and let Ω_i, $i = 1, 2, 3, 4$ be disjoint copies of Ω. We define measures P_i and Ω_i by:

$$P_1(k; \mathscr{R}) = 2^{-n}(n; k; \ell)/n!$$

$$P_2(k; \mathscr{R}) = 2^{-n}\overline{(n; k; \ell)}n!$$

$$P_3(k; \mathscr{R}) = 2^{-n}\left(\binom{n}{k} - (n; k; \ell)\right)/n!$$

$$P_4(k; \mathscr{R}) = 2^{-n}\left(\binom{n}{k} - \overline{(n; k; \ell)}\right)/n!$$

Observe that $P_1 \oplus P_3$ ($P_2 \oplus P_4$) is a probability measure on $\Omega_1 \oplus \Omega_3$ ($\Omega_2 \oplus \Omega_4$). First, we show that $P_3(\Omega_3) \le \ell/\sqrt{n}$

$$P_3(\Omega_3) = \sum_{(k,\mathscr{R})\in\Omega_3} P_3(k; \mathscr{R}) = \sum_{0\le k\le n} \sum_{\mathscr{R}} P_3(k; \mathscr{R})$$

$$= \sum_{0\le k\le n} 2^{-n}\left(\binom{n}{k} - (n; k; \ell)\right).$$

Observe that if $k \le n/2 - \ell/2$ then $(n; k; \ell) = \binom{n}{k}$ and if $k > n/2 - \ell/2$ then $(n; k; \ell) = \binom{n}{k+\ell}$. Hence,

$$P_3(\Omega_3) = \sum_{n/2-\ell/2<k\le n} 2^{-n}\left(\binom{n}{k} - \binom{n}{k+\ell}\right) \le \ell\cdot 2^{-n}\binom{n}{[n/2]} \le \ell/\sqrt{n}.$$

Now, we will define a random variable $X_1 = X$ on $\Omega_1 \oplus \Omega_3$ and a random variable Y_1 on $\Omega_2 \oplus \Omega_4$ such that $F = F_X = F_Y$, and we will define a 1-1 measure preserving mapping T: $\Omega_1 \oplus \Omega_3 \to \Omega_2 \oplus \Omega_4$ such that

(1) $T\Omega_1 = \Omega_2$

(2) $\ell\cdot z_1 \le Y_1 \circ T|_{\Omega_1} - X_1|_{\Omega_1} \le \ell\cdot z_2$.

Thus, by defining the random variable Y on $\Omega_1 \oplus \Omega_3$ by $Y = Y_1 \circ T$, we conclude that $F_X = F_Y = F$ and $P_1 \oplus P_3(\ell\cdot z_1 \le Y - X \le \ell\cdot z_2) \ge 1 - P_3(\Omega_3)$. Therefore it is enough to prove the existence of such X_1, Y_1, and T. Define $X = X_1$ on $\Omega_1 \oplus \Omega_3$ by $X_1(k; \mathscr{R}) = \lambda(R_k(\mathscr{R}))$ where $R_k(\mathscr{R})$ is the initial k elements of G in the order $\mathscr{R}$ and for $A \subset G$, $\lambda(A) = \sum_{a\in A} \lambda(a)$. Similarly we define Y_1 on $\Omega_2 \oplus \Omega_4$ by $Y_1(k; \mathscr{R}) = \lambda(R_k(\mathscr{R}))$. Obviously $F = F_{X_1} = F_{Y_1}$. The mapping $T: \Omega_1 \oplus \Omega_3 \to \Omega_2 \oplus \Omega_4$, is defined by: if $(k; \mathscr{R}) \in \Omega_1$ then $T(k; \mathscr{R}) \in \Omega_2$ and $T(k; \mathscr{R}) = (k + \ell (\mathrm{mod}\ n); \mathscr{R})$, and if $(k; \mathscr{R}) \in \Omega_3$ then $T(k; \mathscr{R}) \in \Omega_4$ and $T(k; \mathscr{R}) = (n - k; \mathscr{R})$. It is straightforward to check that T is 1-1 and measure preserving and that (1) and (2) are satisfied. This completes the proof of Lemma 4.7.

5. The expected number of visits in small intervals. The purpose of this section is to prove the following.

PROPOSITION 5.1. *Let $(\pi_k)_{k=1}^{\infty}$ be a sequence of shrinking partitions, and let $(x_k)_{k=1}^{\infty}$ be a null (π_k)-divergent sequence of numbers. Then for every null sequence of numbers $(b_k)_{k=1}^{\infty}$ with*

$$K\cdot\lambda(a(\epsilon_2)) \le b_k \qquad (K > 2)$$

where ϵ_1, ϵ_2 *are persistent numbers* ($\epsilon_1, \epsilon_2 \in Q$) *with* $0 < \epsilon_1 < \epsilon_2 < 1$, *we have*

$$\limsup_{k\to\infty} \left| \frac{U_{\pi_k}([x_k, x_k + b_k))}{b_k\cdot|\pi_k|} - 1 \right| \le 2/K.$$

The proof of the proposition will make use of the following lemma.

LEMMA 5.2. *Let* $(\pi_k)_{k=1}^{\infty}$ *be a sequence of shrinking partitions, and* $(x_k)_{k=1}^{\infty}$ *be a null* (π_k)*-divergent sequence. Let* $(y_k)_{k=1}^{\infty}$ *be a sequence of numbers such that* $(1 - y_k)_{k=1}^{\infty}$ *is a null,* (π_k)*-divergent sequence. Then for every null sequence* $(b_k)_{k=1}^{\infty}$ *with*

$$2\lambda(a(\epsilon_2)) < b_k$$

where $\epsilon_1, \epsilon_2 \in Q$ *with* $0 < \epsilon_1 < \epsilon_2 < 1$, *we have*

$$\limsup \frac{U_{\pi_k}(I_k)}{b_k\cdot|\pi_k|} - \frac{U_{\pi_k}(J_k)}{b_k\cdot|\pi_k|} \le 0$$

where

$$I_k = [x_k, x_k + b_k)$$

$$J_k = (y_k - b_k - 2\lambda(a(\epsilon_2)), y_k]$$

and

$$\liminf \frac{U_{\pi_k}(I_k)}{b_k\cdot|\pi_k|} - \frac{U_{\pi_k}(J_k^-)}{b_k\cdot|\pi_k|} \ge 0$$

where $J_k^- = (y_k - b_k + 2\lambda(a(\epsilon_2)), y_k]$.

PROOF OF LEMMA 5.2. Define $L_k = \min(x_k/\rho(\pi_k), (1 - y_k)/\rho(\pi_k))$, and x_k^i, y_k^i $i = 1, \ldots, \ell_k = [\sqrt{L_k}/2]$ by:

$$x_k^i = x_k - i\sqrt{L_k}\cdot\rho(\pi_k) - \lambda(a(\epsilon_2)); \qquad y_k^i = y_k + i\sqrt{L_k}\rho(\pi_k).$$

We define a partition of the space of orders into two subsets; H, H_0 by: $\mathscr{R} \in H$ iff there exists $1 \le i \le \ell_k$ such that

$$(*) \qquad \lambda(a(\mathscr{R}, x_k^i)) \le \lambda(a(\epsilon_2))$$

and

$$\lambda(a(\mathscr{R}^-, 1 - y_k^i)) \le \lambda(a(\epsilon_2))$$

and $\mathscr{R} \in H_0$ iff $\mathscr{R} \notin H$. Now for each $1 \le i \le \ell_k$ we define H_i by: $\mathscr{R} \in H_i$ iff $\mathscr{R} \in H$ and i is the maximal index for which $(*)$ holds.

SUBLEMMA 5.2.1. *For every* $1 \le i \le \ell_k$

$$U_\pi(I_k \mid H_i) \le U_\pi(J_k \mid H_i).$$

The idea of the proof is to condition on the x_k^i-initial and on the y_k^i-final and appealing to the reflection principle.

PROOF. It is enough to prove that

$$U_\pi(I_k \mid H_i, \Gamma(\mathscr{R}, x_k^i), \Gamma^-(\mathscr{R}, y_k^i)) \le U_\pi(J_k \mid H_i, \Gamma(\mathscr{R}, x_k^i), \Gamma^-(\mathscr{R}, y_k^i)).$$

Let $\tilde{\pi} = \pi\backslash\Gamma(\mathscr{R}, x_k^i)\backslash\Gamma^-(\mathscr{R}, y_k^i)$. Then,

$$\begin{aligned} &U_\pi(I_k \mid H_i, \Gamma(\mathscr{R}, x_k^i), \Gamma^-(\mathscr{R}, y_k^i)) \\ (5.3) \qquad &= U_{\tilde{\pi}}(I_k - \lambda(\Gamma(\mathscr{R}, x_k^i))) = U_{\tilde{\pi}}(\lambda(\tilde{\pi}) - I_k + \lambda(\Gamma(\mathscr{R}, x_k^i))). \end{aligned}$$

As $0 \le \lambda(\Gamma(\mathscr{R}, x_k^i)) - x_k^i < \lambda(a(\epsilon_2))$ and $0 \le y_k^i - \lambda(\Gamma^-(\mathscr{R}, y_k^i)) < \lambda(a(\epsilon_2))$, we have

$$(5.4) \qquad \lambda(\tilde{\pi}) - I_k + \lambda(\Gamma(\mathcal{R}, x_k^i)) \subset J_k - \lambda(\Gamma(\mathcal{R}, x_k^i)).$$

From (5.3) and (5.4) we deduce that

$$U_\pi(I_k \mid H_i, \Gamma(\mathcal{R}, x_k^i), \Gamma^-(\mathcal{R}, y_k^i)) \leq U_{\tilde{\pi}}(J_k - \lambda(\Gamma(\mathcal{R}, x_k^i))) = U_\pi(J_k \mid H_i, \Gamma(\mathcal{R}, x_k^i), \Gamma^-(\mathcal{R}, y_k^i))$$

which completes the proof of Sublemma 5.2.1.

From Sublemma 5.2.1 we deduce that

$$(5.5) \qquad U(I_k \mid H) \leq U(J_k \mid H).$$

We proceed with the next sublemma.

SUBLEMMA (5.2.2). *The sequence $(U_\pi(I_k \mid H_0))/(b_k \cdot |\pi_k|)$ is bounded.*

PROOF. It is enough to prove that for every $\Gamma(\mathcal{R}, x_k)$ and $\Gamma^-(\mathcal{R}, y_k)$, the sequence

$$\frac{U_\pi(I_k \mid H_0, \Gamma(\mathcal{R}, x_k), \Gamma^-(\mathcal{R}, y_k))}{b_k \cdot |\pi_k|}$$

is bounded. Denote $\tilde{\pi} = \pi \backslash \Gamma(\mathcal{R}, x_k) \backslash \Gamma^-(\mathcal{R}, y_k)$ and $\bar{\pi} = \tilde{\pi} \cap \pi(\epsilon_2)$. We have

$$(5.6) \qquad U_\pi(I_k \mid \Gamma(\mathcal{R}, x_k), \Gamma^-(\mathcal{R}, y_k), H_0) = U_{\tilde{\pi}}(I_k - \lambda(\Gamma(\mathcal{R}, x_k))).$$

By Lemma 4.5 and the nullity of (x_k), $(1 - y_k)$ and (b_k) it follows that

$$(5.7) \qquad U_{\tilde{\pi}}(I_k - \lambda(\Gamma(\mathcal{R}, x_k))) \leq U_{\tilde{\pi}}([0, b_k)) \leq U_{\bar{\pi}}([0, b_k)).$$

We proceed by estimating $U_{\bar{\pi}}([0, b_k))$. Observe that $\lambda(\tilde{\pi}) \to_{k\to\infty} 1$ and thus $\lambda(\bar{\pi}) \to_{k\to\infty} \epsilon_2$. Using Corollary (4.4), we conclude that for sufficiently large k

$$U_{\bar{\pi}}([0, b_k)) \leq \left[\frac{\lambda(\bar{\pi})}{b_k + 2\rho(\bar{\pi})}\right]^{-1} (|\bar{\pi}| - 1) \leq \left[\frac{\epsilon_1}{b_k + 2\lambda(a(\epsilon_2))}\right]^{-1} \cdot |\pi_k|$$

$$\leq \frac{b_k + 2\lambda(a(\epsilon_2))}{\epsilon_1} \cdot 2|\pi_k| \leq \frac{6b_k}{\epsilon_1} \cdot |\pi_k|.$$

Thus, for sufficiently large k, $(U_{\bar{\pi}}([0, b_k)))/(b_k \cdot |\pi_k|) \leq 6/\epsilon_1$, which completes (with (5.7) and (5.6)) the proof of Sublemma 5.2.2.

SUBLEMMA (5.2.3). $\mathrm{Prob}(H_0) \to_{k\to\infty} 0$.

PROOF. As $\epsilon_1 \in Q$ there exist constants $\delta > 0$, $M > 0$, $\alpha > 0$ such that if π is a partition with $\rho(\pi) < \alpha/M$ and $M \cdot \rho(\pi) < x < \alpha$ then

$$P(\pi(\epsilon_1), x) \geq \delta.$$

Otherwise there exist sequences, $0 < \delta_k \to 0$, $M_k \to \infty$, $0 < \alpha_k \to 0$, and a sequence of partitions (π_k) and a sequence of numbers (x_k) such that:

$$M_k \cdot \rho(\pi_k) < x_k < \alpha_k \quad \text{and} \quad P(\pi_k(\epsilon_1), x_k) < \delta_k.$$

The first condition asserts that $\{\pi_k\}$ is shrinking and that the sequence (x_k) is null and (π_k)-divergent while the second condition implies that $P(\pi_k(\epsilon_1), x_k) \to 0$ which contradicts the persistency of ϵ_1.

Consider the following facts:

(a) The nullity of the sequences (x_k), $(1 - y_k)$ implies that for sufficiently large k, for each $1 \leq i, j \leq \ell_k$

$$\lambda(\pi(\epsilon_2) \backslash \Gamma(\mathcal{R}, x_k^i) \backslash \Gamma^-(\mathcal{R}, y_k^i)) > \epsilon_1$$

which in particular implies that

$$\frac{\lambda(\pi(\epsilon_2) \backslash \Gamma(\mathcal{R}, x_k^i) \backslash \Gamma^-(\mathcal{R}, y_k^i))}{\lambda(\pi \backslash \Gamma(\mathcal{R}, x_k^i) \backslash \bar{\Gamma}(\mathcal{R}, y_k^j))} > \epsilon_1.$$

(b) For sufficiently large k, for every $1 \le i, j \le \ell_k$ and for every order $\mathscr{R}$ of π_k the following hold:

(b.1) $x_k < \alpha$

(b.2) $1 - y_k < \alpha$

(b.3) $x_k^{i-1} - \lambda(\Gamma(\mathscr{R}, x_k^i)) > 2M\cdot\rho(\pi)$

(b.4) $\lambda(\Gamma(\mathscr{R}, x_k^i) \cup \Gamma^-(\mathscr{R}, y_k^i)) < (\epsilon_2 - \epsilon_1)/2$.

Therefore for sufficiently large k

$$P(\pi(\epsilon_2), x_k^{i-1} \mid \Gamma(\mathscr{R}, x_k^i), \Gamma^-(\mathscr{R}, y_k^i)) \ge \delta$$

and

$$P(\pi(\epsilon_2), y_k^{i-1} \mid \Gamma(\mathscr{R}, x_k^{i-1}), \Gamma^-(\mathscr{R}, y_k^i)) \ge \delta.$$

Hence, by denoting $\bar{H}_i = \Omega \backslash \cup_{j>i} H_j$, we have

$$\operatorname{Prob}(H_i \mid \bar{H}_i, \Gamma(\mathscr{R}, x_k^{i+1}), \bar{\Gamma}(\mathscr{R}, y_k^{i+1})) \ge \delta^2$$

which yield that $\operatorname{Prob}(H_i \mid \bar{H}_i) \ge \delta^2$ and therefore $\operatorname{Prob}(H_0) \le (1 - \delta^2)^{\ell_k} \to_{k\to\infty} 0$ which completes the proof of Sublemma (5.2.3).

From Sublemmas (5.2.1), (5.2.2) and (5.2.3) we conclude that

$$\limsup \frac{U_\pi(I_k)}{b_k\cdot|\pi_k|} - \frac{U_\pi(J_k)}{b_k\cdot|\pi_k|}$$

$$\le \limsup\left(\left(\frac{U_\pi(I_k \mid H)}{b_k\cdot|\pi_k|} - \frac{U_\pi(J_k \mid H)}{b_k\cdot|\pi_k|}\right)\operatorname{Prob}(H) + \frac{U_\pi(I_k \mid H_0)}{b_k\cdot|\pi_k|}\operatorname{Prob}(H_0)\right)$$

$$\le \limsup \frac{U_\pi(I_k \mid H_0)}{b_k\cdot|\pi_k|}\operatorname{Prob}(H_0) = 0.$$

This completes the proof of the first part of Lemma 5.2. The second part is similarly proved.

Proof of Proposition 5.1. Define $\Delta_k = b_k + 2\lambda(a(\epsilon_2))$ and let $y_k^i = 1 - x_k - i\cdot\Delta_k$, $i = 0, \cdots, n_k$. Denote

$$I_k = [x_k, x_k + b_k)$$

$$J_k^i = (y_k^i, y_k^{i-1}] \quad i = 1, \cdots, n_k$$

and

$$J_k = \cup_{i=1}^{n_k} J_k^i = (1 - x_k - n_k\Delta_k, 1 - x_k].$$

Let $(n_k)_{k=1}^\infty$ be a sequence of integers such that:

(a) $n_k\Delta_k/\rho(\pi_k) \to_{k\to\infty} \infty$ and

(b) $(1 - y_k^{n_k})_{k=1}^\infty$ is a null sequence.

From Corollary 4.4 and (a) we have that

$$\lim \frac{U_{\pi_k}(J_k)}{n_k\cdot\Delta_k\cdot|\pi_k|} = 1.$$

On the other hand we deduce from Lemma 5.2 that for any selection of i_k, $1 \le i_k \le n_k$

$$\limsup \frac{U_{\pi_k}(I_k)}{b_k\cdot|\pi_k|} - \frac{U_{\pi_k}(J_k^i)}{b_k\cdot|\pi_k|} \le 0.$$

As $U_{\pi_k}(J_k) = \sum_{i=1}^{n_k} U_{\pi_k}(J_k^i)$ we conclude that

$$\limsup \frac{U_{\pi_k}(I_k)}{b_k\cdot|\pi_k|} - \frac{\Delta_k}{b_k} \le 0,$$

which yield that

$$\limsup \frac{U_{\pi_k}(I_k)}{b_k\cdot|\pi_k|} \le 1 + 2/K.$$

Similarly, by using the other part of Lemma 5.2 we can show that

$$\liminf \frac{U_{\pi_k}(I_k)}{b_k \cdot |\pi_k|} \geq 1 - 2/K$$

which completes the proof of Proposition 5.1.

From Proposition 5.1 we easily deduce the following corollary.

COROLLARY OF PROPOSITION 5.1. *Let $(\pi_k)_{k=1}^{\infty}$ be a shrinking sequence of partitions, and let $(x_k)_{k=1}^{\infty}$ be a null sequence (π_k)-divergent. Then for every $K > 2$ and $b_k \in \pi_k$ with $\lambda(b_k) \geq K \cdot \lambda(a(\epsilon_2))$ where $\epsilon_1, \epsilon_2 \in Q$, $0 < \epsilon_1 < \epsilon_2 < 1$,*

$$\limsup \left| \frac{U_{\pi_k \setminus \{b_k\}}([x_k - \lambda(b_k), x_k))}{\lambda(b_k) \cdot |\pi_k|} - 1 \right| \leq 2/K.$$

6. Coverage probabilities by "thin sets."

PROPOSITION 6.1. *Let $(\pi_k)_{k=1}^{\infty}$ be a shrinking sequence of partitions, and let $\epsilon_1, \epsilon_2 \in Q$ with $0 < \epsilon_1 < \epsilon_2 < 1$. If $A_k \subset \pi_k \setminus \pi_k(\epsilon_2)$ satisfies*

$$\max_{a, b \in A_k} \frac{\lambda(a) - \lambda(b)}{\lambda(a(\epsilon_2))} \leq h < 10^{-6}$$

then for every $\eta > 0$ there exist $B_k \subset A_k$ with $\lambda(A_k) - \lambda(B_k) \leq \eta$ such that for every $B'_k \subset B_k$ and every null, (π_k)-divergent sequence x_k,

$$\limsup |P(B'_k, x_k) - \lambda(B'_k)| - 5\lambda(B_k) h^{1/6} \leq 0.$$

PROOF. Let $\eta > 0$ be given. Without loss of generality we may assume that $\lambda(A_k) \geq \eta$. For every $0 < \xi < 1$, there exists a decomposition $A_k = A_k(\xi, -1) \cup A_k(\xi, 0) \cup A_k(\xi, 1)$ of A_k into disjoint subsets such that

(a) For every $a_i \in A_k(\xi, i)$, $i = -1, 0, 1$, $\lambda(a_1) \geq \lambda(a_0) \geq \lambda(a_{-1})$, and
(b) $|A_k(\xi, 1)| = |A_k(\xi, -1)|$ and $||A_k(\xi, 0)| - \xi |A_k|| \leq 1$.

Let ξ satisfy $2(1 - \xi) < \eta$ and let $B_k = A_k(\xi, 0)$. Then for sufficiently large k, $\lambda(A_k) - \lambda(B_k) \leq \eta$. Let (x_k) be a given null (π_k)-divergent sequence and let $t_k = x_k/8$. Let $\bar{\pi}_k(\omega) = \pi_k^{1-t_k}(\omega) \setminus \pi_k^{t_k}(\omega)$ and let H be the σ-field generated by the set-valued random variable $\bar{\pi}_k$. For every subset D_k of π_k we denote by $\bar{D}_k$ the set-valued random variable $D_k \cap \bar{\pi}_k$. Let f_k denote the indicator of the joint events $\lambda(\bar{\pi}_k) > 1 - 4t_k$, $|\bar{\pi}_k| > (1 - 4t_k)|\pi_k|$ and $(1 - 4t_k)|A_k(\xi, i)| \leq |\bar{A}_k(\xi, i)| \leq (1 - t_k)|A_k(\xi, i)|$, $i = -1, 0, 1$. Observe that f_k is H-measurable and that on $f_k = 1$, for every sequence $B'_k \subset B_k$, $\lambda(B'_k) - \lambda(\bar{B}_{k'}) < 4t_k$. Also by Lemma 4.6, $\mathrm{Prob}(f_k = 1) \to 1$ as $k \to \infty$. Therefore in order to prove Proposition 6.1, it is enough to show that for every $B'_k \subset B_k$ and for every sequence ω_k with $f_k(\omega_k) = 1$,

$$\limsup_{k \to \infty} |P(B'_k, x_k | H)(\omega_k) - \lambda(\bar{B}'_k(\omega_k))| - 5\lambda(B_k) h^{1/6} \leq 0.$$

Since $P(B'_k, x_k | H)(\omega_k) = P(\bar{B}'_k, x_k | H)(\omega_k)$ $(f_k(\omega_k) = 1)$, it is enough to show that for every sequence $b_k \in \bar{B}_k(\omega_k)$,

$$\limsup_{k \to \infty} \left| \frac{P(b_k, x_k | H)(\omega_k)}{\lambda(b_k)} - 1 \right| \leq 5h^{1/6}. \tag{6.2}$$

In all that follows in this section, U $(\bar{U})$ will stand for the expected number of visits of the random walk generated by $\pi_k \setminus b_k$ $(\bar{\pi}_k \setminus b_k)$.

LEMMA 6.3. *For every interval I_k with $I_k \subset [5t_k, 9t_k)$ and $K\lambda(a(\epsilon_2)) \geq \mu(I_k) \geq 2\lambda(a(\epsilon_2))$, we have for sufficiently large k,*

$$(\mu(I_k) - 3\lambda(a(\epsilon_2)))|\pi_k| \leq U(I_k | \bar{\pi}_k, \pi_k^{t_k})(\omega_k) \leq (3\lambda(a(\epsilon_2)) + \mu(I_k))|\pi_k|.$$

PROOF. As $f_k(\omega_k) = 1$, $U(I_k \mid \bar{\pi}_k, \pi_k^{t_k})(\omega_k) = \bar{U}(I_k - \lambda(\pi_k^{t_k}(\omega_k)) \mid \bar{\pi}_k)(\omega_k)$ and $I_k - \lambda(\pi_k^{t_k}(\omega_k)) \subset [t_k, 9t_k)$. Moreover, on $f_k = 1$, $|\bar{\pi}_k| \geq (1 - 4t_k)|\pi_k|$ and $1 \geq \lambda(\bar{\pi}_k) \geq 1 - 4t_k$ and thus in particular $\lambda(\bar{\pi}_k(\epsilon_2)) > ((\epsilon_1 + \epsilon_2)/2)\lambda(\bar{\pi}_k)$. Therefore we could apply Proposition 5.1 to show that

$$\limsup_{k\to\infty} \left| \frac{\bar{U}(I_k - \lambda(\pi_k^{t_k}(\omega_k)) \mid \bar{\pi}_k)(\omega_k)}{\mu(I_k)\,|\pi_k|} - 1 \right| - \frac{2\lambda(a(\epsilon_2))}{\mu(I_k)} \leq 0.$$

As $\lambda(a(\epsilon_2))\,|\pi_k|$ is bound from below (by ϵ_2), the result follows.

LEMMA 6.4. *On $f_k = 1$, for every $6t_k \leq y_k \leq 9t_k$,*

$$\limsup \frac{U([y_k - \delta_k, y_k) \mid H)}{\lambda(a(\epsilon_2))\,|\pi_k|} \leq 5\sqrt{h}$$

where $\delta_k = \max\{\lambda(a) - \lambda(b) : a, b \in B_k\}$.

PROOF. As $A_k(\xi, i)\backslash\bar{\pi}_k$ is H-measurable, the integer valued random variable $m(\omega) = \min\{|A_k(\xi, i)\backslash\bar{\pi}_k| : i = -1, 1\}$ is H-measurable and there are H-measurable selections of a subset D_k of $A_k(\xi, -1)\backslash\bar{\pi}_k$ with $|D_k| = m(\omega)$ and of a $1-1$ function $u: D_k \to A_k(\xi, 1)\backslash\bar{\pi}_k$. Define

$$S(\omega) = \{d \in D_k : 0 \leq X_d \wedge X_{u(d)} \leq t_k < 1 - t_k < X_d \vee X_{u(d)} \leq 1\}.$$

Observe that $|S(\omega)|$ is the sum of $m(\omega)$ i.i.d random variables $Y_a (a \in D_k)$ with $\text{Prob}(Y_a = 0) = 1/2$ and $\text{Prob}(Y_a = 1) = 1/2$. As on $f_k = 1$, $m(\omega) \geq t_k |A_k(\xi, -1)| \geq t_k(1 - \xi)|A_k|/4 \geq ((1-\xi)\eta/32)(x_k/\rho(\pi_k)) \to \infty$ as $k \to \infty$, it follows that on $f_k = 1$,

$$P(|S(\omega)| \geq n \mid H) \to 1 \text{ (uniformly)} \quad \text{as } k \to \infty. \tag{6.5}$$

Let H_1 be the σ-algebra generated by H and S. Let $n = \lceil 1/h \rceil$ (the smallest integer $\geq 1/h$) and let $S_1(\omega)$ be an H_1-measurable set valued random variable with $S_1(\omega) \subset S(\omega)$ and $|S_1(\omega)| = \min\{|S(\omega)|, n\}$ (e.g., fix a well ordering of $A_k(\xi, -1)$ and let $S_1(\omega)$ be the first n elements of $A_k(\xi, -1)$ in $S(\omega)$ if $|S(\omega)| \geq n$ and otherwise $S_1(\omega) = S(\omega)$). Let H_2 be the σ-field generated by H_1 and $\pi^{t_k}\backslash S_1\backslash u(S_1)$.Our next step is to show that on $f_k = 1$ and $|S_1| = n$,

$$\limsup \frac{U([y_k - \delta_k, y_k) \mid H_2)}{\lambda(a(\epsilon_2))\,|\pi_k|} \leq 5/\sqrt{n} \leq 5\sqrt{h}. \tag{6.6}$$

Denote

$$\gamma = \lambda(\pi^{t_k}\backslash S_1\backslash u(S_1))$$

$$T(\omega) = \{a \in S_1 : X_a(\omega) \leq t_k\}$$

$$y_k(T) = y_k - \gamma - \lambda(T \cup (u(S_1)\backslash u(T))) = y_k - \gamma - \lambda u(S_1) + \textstyle\sum_{a\in T} (\lambda(u(a)) - \lambda(a))$$

$$I_k(T) = [y_k(T) - \delta_k, y_k(T)).$$

Observe that for every $T \subset S_1$, $\text{Prob}(T(\omega) = T \mid H_2) = 2^{-|S_1|}$ and therefore on $f_k = 1$, $U([y_k - \delta_k, y_k) \mid H_2) = 2^{-|S_1|} \sum_{T\subset S_1} \bar{U}(I_k(T))$. Observe the following facts:

(i) $T_1 \subsetneq T_2 \subset S_1 \Rightarrow I_k(T_1) \cap I_k(T_2) = \phi$

(which follows from $y_k(T_2) - y_k(T_1) = \sum_{a\in T_2\backslash T_1} (\lambda(u(a)) - \lambda(a)) \geq \delta_k$).

(ii) For each $T \subset S_1$, $I_k(T) \subset I_k := [y_k - \delta_k - \lambda(u(S_1)), y_k - \lambda(S_1)) - \gamma$.

By (i) we have that for every x, the set $I(x) = \{T : T \subset S_1 \text{ and } x \in I_k(T)\}$ is a set of subsets of S_1 in which no two elements are comparable (in the inclusion partial order) and thus by a well known result

$$I(x) \leq \binom{|S_1|}{|S_1|/2} \leq \sqrt{\frac{2}{\pi}}\, 2^{|S_1|}/\sqrt{|S_1|}$$

which together with (ii) implies that on $f_k = 1$,

$$U(y_k - \delta_k, y_k \mid H_2) \leq \bar{U}(I_k \mid H_2)/\sqrt{|S_1|}. \tag{6.7}$$

As on $f_k = 1$, $I_k \subset [5t_k, 9t_k)$ and $\mu(I_k) \le \delta_k + |S_1| h\lambda(a(\epsilon_2)) \le (2 + 1/h)h\lambda(a(\epsilon_2)) < 2\lambda(a(\epsilon_2))$ we conclude by Lemma 6.3 that on $f_k = 1$, $\limsup_{k\to\infty} \dfrac{U(I_k | H_2)}{\lambda(a(\epsilon_2))|\pi_k|} \le 5$ which together with (6.7) and (6.5) proves (6.6) which completes the proof of Lemma 6.4.

LEMMA 6.8. *Let ℓ be a fixed positive integer. Then for every sequence $(y_k)_{k=1}^\infty$ with $7t_k < y_k \le 8t_k$ and for every sequence (b_k) and (ω_k) with $f_k(\omega_k) = 1$ and $b_k \in \bar{B}_k(\omega_k)$,*

$$\liminf \frac{U(\tilde{I}_k | H)(\omega_k)}{\lambda(b_k)|\pi_k|} - \frac{U(I_k | H)}{\lambda(b_k)|\pi_k|} \ge 0$$

where $I_k = [y_k - \lambda(b_k), y_k) - \ell\lambda(b_k)$ and $\tilde{I}_k = I_k + \ell\lambda(b_k) + [-\ell\delta_k, \ell\delta_k]$.

PROOF. Let H_3 be the σ-field generated by H and $\pi^{t_k}\backslash B_k$. Denote by F the cumulative distribution function of $\lambda(\pi_k^{t_k} \cap B_k)$ given $H_3(\omega_k)$, and by γ the H_3-measurable random variable $\lambda(\pi^{t_k}\backslash B_k)$. Then for every interval I contained in $(4t_k, 1 - 4t_k)$ we have,

$$U(I | H_3)(\omega_k) = \int \bar{U}(I - \gamma - x | H_3)(\omega_k)\, dF(x).$$

By Lemma 4.7 there exists a probability space $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{P})$ with random variables x, y, such that

(i) $F_x = F_y = F$ and

(ii) $\tilde{P}(\Omega_1) \ge 1 - \ell/\sqrt{|B_k\backslash\bar{\pi}_k|}$ where $\Omega_1 = \{\tilde{\omega} \in \tilde{\Omega} : \ell(\lambda(b_k) - \delta_k) \le y - x < \ell(\lambda(b_k) + \delta_k)\}$.

Observe that on Ω_1, $\tilde{I}_k - \gamma - y(\tilde{\omega}) \supset I_k - \gamma - x(\tilde{\omega})$ and therefore

$$U(\tilde{I}_k | H_3)(\omega_k) \ge \int_{\Omega_1} \bar{U}(\tilde{I}_k - \gamma - y(\omega) | H_3)(\omega_k)\, d\tilde{P}(\omega)$$

$$\ge \int_{\Omega_1} \bar{U}(I_k - \gamma - x(\omega) | H_3)(\omega_k)\, d\tilde{P}(\omega).$$

By Lemma 6.3, $\bar{U}(I_k - \gamma - x(\omega) | H_3)(\omega_k) \le 4\lambda(b_k)|\pi_k|$ for k sufficiently large ($I_k \subset I'_k \subset [5t_k, 9t_k)$ where I'_k is some interval with $\mu(I'_k) = 2\lambda(b_k)$), and therefore $U(\tilde{I}_k | H_3)(\omega_k) \ge U(I_k | H_3)(\omega_k) - 4\lambda(b_k)|\pi_k| \ell\sqrt{|B_k\backslash\bar{\pi}_k(\omega_k)|}$. As $|B_k\backslash\bar{\pi}_k(\omega_k)| \ge t_k|B_k| \ge (\xi\eta/2)t_k/\rho(\pi_k) \to_{k\to\infty} \infty$, the lemma follows. ($H \subset H_3$).

PROOF OF (6.2). Observe that

$$P(b_k, x_k | H)(\omega_k) = U([x_k - \lambda(b_k), x_k) | H)(\omega_k)/|\bar{\pi}_k(\omega_k)|.$$

Let $\ell = [h^{-1/6}]$ and define

$$I_k = [x_k - \lambda(b_k), x_k), \qquad \tilde{I}_k = I_k + [-\ell\delta_k, \ell\delta_k]$$

$$J_k^i = I_k - i\lambda(b_k),\ i = 0, \cdots, \ell, \quad J_k = \cup_{i\le\ell} J_k^i.$$

By Lemma 6.8 we have for every $0 \le i \le \ell$

$$\liminf_{k\to\infty} \frac{U(\tilde{I}_k | H)(\omega_k)}{\lambda(b_k)|\pi_k|} - \frac{U(J_k^i | H)(\omega_k)}{\lambda(b_k)|\pi_k|} \ge 0.$$

Hence

$$\liminf_{k\to\infty} \frac{U(\tilde{I}_k | H)(\omega_k)}{\lambda(b_k)|\pi_k|} - \frac{U(J_k | H)(\omega_k)}{(\ell + 1)\lambda(b_k)|\pi_k|} \ge 0.$$

By Lemma 6.3, $\liminf(U(J_k | H)(\omega_k)/((\ell + 1)\lambda(b_k)|\pi_k|)) \ge 1 - 3/(\ell + 1)$ and therefore, $\liminf(U(\tilde{I}_k | H)(\omega_k)/(\lambda(b_k)|\pi_k|)) \ge 1 - 3/(\ell + 1)$. By Lemma 6.4,

$$\limsup((U(\tilde{I}_k | X)(\omega_k) - U(I_k | H)(\omega_k))/\lambda(b_k)|\pi_k|) \le 10\ell\sqrt{h}.$$

Thus we conclude that

$$\liminf_{k\to\infty} \frac{U(I_k \mid H)(\omega_k)}{\lambda(b_k) \mid \pi_k \mid} \geq 1 - \frac{3}{1+\ell} - 10\ell\sqrt{h} \geq 1 - 4h^{1/6}.$$

Similarly, setting $J_k^i = \tilde{I}_k - i(\lambda(b_k))$, $i = 0, \cdots, \ell$ we have by using Lemmas 6.3 and 6.4,

$$\limsup_{k\to\infty} \frac{U(I_k \mid H)(\omega_k)}{\lambda(b_k) \mid \pi_k \mid} \leq \limsup_{k\to\infty} \textstyle\sum_{i\leq\ell} U(J_k^i \mid H)(\omega_x)/(\ell + 1)$$

$$\leq \limsup_{k\to\infty} \frac{U(J_k \mid H)(\omega_k)}{\lambda(b_k) \mid \pi_k \mid (\ell + 1)} + 10\ell\sqrt{h}$$

$$\leq 1 + \frac{3}{\ell + 1} + 10h^{1/3} \leq 1 + 5h^{1/6}.$$

As $\lim |\bar{\pi}_k| / |\pi_k| \to 1$, we conclude that $\limsup \left| \frac{P(b_k, x_k)}{\lambda(b_k)} - 1 \right| \leq 5h^{1/6}$. This completes the proof of 6.2 and thus of Proposition 6.1.

7. An intermediate result.

PROPOSITION 7.1 *Let $(\pi_k)_{k=1}^{\infty}$ be a shrinking sequence of partitions and let ϵ_1, ϵ_2 be two persistent numbers with $0 < \epsilon_1 < \epsilon_2 < 1$. Then for every $\eta > 0$ there exist sets T_k, $T_k \subset \pi_k \backslash \pi_k(\epsilon_2)$, with $\lambda(\pi_k \backslash \pi_k(\epsilon_2)) - \lambda(T_k) < \eta$ such that for every $\tilde{T}_k \subset T_k$ and for every null (π_k)-divergent sequence (x_k),*

$$\limsup |P(\tilde{T}_k, x_k) - \lambda(\tilde{T}_k)| \leq \eta.$$

PROOF. There exists K_1 such that $2/K_1 < \eta$, and $K_1 > 2$. Define $T_k^1 = \{a : a \in \pi_k, \lambda(a) \geq K_1 \cdot \lambda(a(\epsilon_2))\}$. By the Corollary of Proposition 5.1, we have that for every sequence of subsets $\tilde{T}_k^1$, $\tilde{T}_k^1 \subset T_k^1$, and for every null (π_k)-divergent sequence (x_k),

$$\limsup_{k\to\infty} |P(\tilde{T}_k^1, x_k) - \lambda(\tilde{T}_k^1)| - (2/K_1)\lambda(T_k^1) \leq 0. \tag{7.2}$$

There exists h, $0 < h < 10^{-6}$ for which $5h^{1/6} < \eta$. For every k there exists a decomposition

$$\pi_k \backslash \pi_k(\epsilon_2) = T_k^1 \cup A_k^1 \cup \cdots \cup A_k^{\ell}$$

where $\ell \leq K_1/h$, such that for every $1 \leq i \leq \ell$,

$$\max_{a,b \in A_k^i} \frac{\lambda(a) - \lambda(b)}{\lambda(a(\epsilon_2))} \leq h.$$

By proposition 6.1 it follows that for every $1 \leq i \leq \ell$ there exist subsets B_k^i of A_k^i satisfying $\lambda(A_k^i) - \lambda(B_k^i) < 2^{-i}\eta$ such that every $\tilde{B}_k^i \subset B_k^i$ and for every null, (π_k)-divergent sequence (x_k),

$$\limsup_{k\to\infty} |P(\tilde{B}_k^i, x_k) - \lambda(\tilde{B}_k^i)| - \eta\lambda(B_k^i) \leq 0. \tag{7.3}$$

Define $T_k = T_k^1 \cup B_k^1 \cup \cdots \cup B_k^{\ell}$. By (7.2) and (7.3) it follows that for every $\tilde{T}_k \subset T_k$

$$\limsup_{k\to\infty} |P(\tilde{T}_k, x_k) - \lambda(\tilde{T}_k)| - (\lambda(\tilde{T}_k \cap T_k^1) + \textstyle\sum_{i=1}^{\ell} \lambda(B_k^i))\,\eta \leq 0$$

and thus

$$\limsup_{k\to\infty} |P(\tilde{T}_k, x_k) - \lambda(\tilde{T}_k)| \leq \eta$$

which completes the proof of Proposition 7.1.

8. Characterization of persistent numbers.

PROPOSITION 8.1. $Q = (0, 1)$.

LEMMA 8.2. *Let π be a partition and let $2\rho(\pi) < x < 1 - 2\rho(\pi)$. Then for every $0 < \epsilon$*

≤ 1 *and every* $a \in \pi\backslash\pi(\epsilon)$,

$$P(a, x) \leq 4\lambda(a)/\epsilon.$$

PROOF. Let $2\rho(\pi) < x < 1 - 2\rho(\pi)$, $b \in \pi(\epsilon)$ and $a \in \pi\backslash\pi(\epsilon)$. Then $\lambda(b) \leq \lambda(a)$ which together with the obvious implications $\lambda(\pi^t\backslash\{a\}) \in [x - \lambda(a), x) \Rightarrow \lambda(\pi^t\backslash\{b\}) \in [x - \lambda(a) - \lambda(b), x + \lambda(a)) \Rightarrow \mu\ \{y \in [x - 2\lambda(a), x + 2\lambda(a)]: \lambda(\pi^t\backslash\{b\}) \in [y - \lambda(b), y)\} \geq \lambda(b)$, imply that (using Fubini's theorem)

$$\begin{aligned}\mathrm{Prob}(\lambda(\pi^t\backslash\{a\}) \in [x - \lambda(a), x)) &\leq \mathrm{Prob}(\lambda(\pi^t\backslash\{b\}) \in [x - \lambda(a) - \lambda(b), x + \lambda(a)))\\ &\leq \frac{1}{\lambda(b)} \int_{x-2\lambda(a)}^{x+2\lambda(a)} \mathrm{Prob}(\lambda(\pi^t\backslash\{b\}) \in [y - \lambda(b), y))\, dy,\end{aligned}$$

and therefore (using Lemma 4.1 and Fubini's theorem)

$$\lambda(b)P(a, x) = \lambda(b) \int_0^1 \mathrm{Prob}(\lambda(\pi^t\backslash\{a\}) \in [x - \lambda(a), x))\, dt$$

$$\leq \int_0^1 \int_{x-2\lambda(a)}^{x+2\lambda(a)} \mathrm{Prob}(\lambda(\pi^t\backslash\{b\}) \in [y - \lambda(b), y))\, dy\, dt = \int_{x-2\lambda(a)}^{x+2\lambda(a)} P(b, y)\, dy.$$

Thus summing over all b in $\pi(\epsilon)$ and using the obvious inequality $\sum_{b\in\pi(\epsilon)} P(b, y) = P(\pi(\epsilon), y) \leq 1$, we conclude that $P(a, x) \leq 4\lambda(a)/\epsilon$.

PROOF OF PROPOSITION 8.1. Summing the inequalities of Lemma 8.2 for all a in $\pi\backslash\pi(5/6)$ we have

$$P(\pi\backslash\pi(5/6), x) \leq (4/6)/(5/6) \leq 4/5$$

and therefore $P(\pi(5/6), x) \geq 1/5$ which proves that $5/6 \in Q$. Observe that if $x \in Q$ then $[x, 1) \subset Q$. Let $\epsilon \in Q$. Let (π_k) be a shrinking sequence of partitions, and let (x_k) be a null (π_k)-divergent sequence. Proposition 7.1 asserts the existence of sets $T_k \subset \pi_k\backslash\pi_k(\epsilon)$ with $\lambda(T_k) > 1 - \epsilon - \epsilon^2/17$ such that $\limsup_{k\to\infty} P(T_k, x_k) \leq 1 - \epsilon + \epsilon/4$. Let $\bar{\epsilon} = \epsilon - \epsilon^2/17$ and let $A_k = \pi_k\backslash\pi_k(\bar{\epsilon})\backslash T_k$. Then $\lambda(A_k) \leq \epsilon - \bar{\epsilon} + \epsilon^2/17$ and by Lemma 8.2 $\limsup_{k\to\infty} P(A_k, x_k) \leq 4(\epsilon - \bar{\epsilon} + \epsilon^2/17)/\bar{\epsilon} \leq \epsilon/2$. As $P(\pi_k(\bar{\epsilon}), x_k) + P(A_k, x_k) + P(T_k, x_k) = 1$ we conclude that $\liminf_{k\to\infty} P(\pi_k(\bar{\epsilon}), x_k) \geq 1 - (1 - \epsilon + \epsilon/4) - \epsilon/2 = \epsilon/4$. Therefore if $\epsilon \in Q$ then $\bar{\epsilon} = \epsilon - \epsilon^2/17 \in Q$ which shows that $Q = (x, 1)$ for some $0 \leq x \leq 1$. As for any $0 < x < 1$ there is $\epsilon > x$ such that $\bar{\epsilon} = \epsilon - \epsilon^2/17 < x$, $Q = (0, 1)$.

9. Completing the proof of the main result. First, we state an immediate generalization of Proposition 7.1 by applying to it the fact that $Q = (0, 1)$.

LEMMA 9.1. *Let* $(\pi_k)_{k=1}^\infty$ *be a shrinking sequence of partitions. For every* $\eta > 0$ *there exist sets* $T_k \subset \pi_k$, *with* $\lambda(T_k) > 1 - \eta$ *such that for every* $\tilde{T}_k \subset T_k$ *and for every null,* (π_k)*-divergent sequence* (x_k),

$$\limsup_{k\to\infty} |P(\tilde{T}_k, x_k) - \lambda(\tilde{T}_k)| < \eta..$$

COROLLARY 9.2. *Let* $(\pi_k)_{k=1}^\infty$ *be a shrinking sequence of partitions. Then, for every null,* (π_k)*-divergent sequence* (x_k) *and for every* $A_k \subset \pi_k$ *the following limit and equality hold:*

$$\lim_{k\to\infty} (P(A_k, x_k) - \lambda(A_k)) = 0.$$

PROOF. Otherwise, there exist a shrinking sequence of partitions $(\pi_k)_{k=1}^\infty$, a null, (π_k)-divergent sequence (x_k) and sets $A_k \subset \pi_k$ such that

$$\lim_{k\to\infty} |P(A_k, x_k) - \lambda(A_k)| = \alpha > 0.$$

Take $\eta = \alpha/5$. According to Lemma 9.1 there exist $T_k \subset \pi_k$ with $\lambda(T_k) > 1 - \eta$ such that

$$\limsup_{k\to\infty} |P(T_k \cap A_k, x_k) - \lambda(T_k \cap A_k)| < \eta$$

and
$$\limsup_{k\to\infty} |P(T_k, x_k) - \lambda(T_k)| < \eta.$$

Thus,

$$\begin{aligned}\limsup |P(A_k, x_k) - \lambda(A_k)| &\le \limsup |P(T_k \cap A_k), x_k) - \lambda(T_k \cap A_k)| \\ &\quad + \limsup |P(A_k \backslash T_k, x_k) + \limsup |\lambda(A_k \backslash T_k)| \\ &\le \eta + \limsup |1 - P(T_k, x_k)| + \eta \le 4\eta < \alpha.\end{aligned}$$

which contradicts the assumption. This completes the proof of Corollary 9.2.

We reformulate now the corollary in the "ϵ, δ language."

COROLLARY 9.2*. *For every $\eta > 0$ there exist $\delta > 0$, $K > 0$, $\alpha > 0$ such that if π is a partition and $0 < x < 1$ with $\rho(\pi) < \delta$ and $K\rho(\pi) < x < \alpha$ then for every $T \subset \pi$*

$$|P(T, x) - \lambda(T)| < \eta.$$

THEOREM 9.3. *Let $0 < \alpha < 1$ be given. For every $\eta > 0$ there exist $\delta > 0$, $K > 0$ such that if π is a partition with $\rho(\pi) < \delta$ and $0 < x < 1$ with $K\rho(\pi) < x < 1 - \alpha$ then for every $T \subset \pi$,*

$$|P(T, x) - \lambda(T)| < \eta.$$

PROOF. Let $0 < \eta < 1$ and $0 < \alpha < 1$ be given. Let K_1, δ_1, α_1 be the constants associated to $\eta/5$ by Corollary 9.2*. Without loss of generality $K_1 \ge 1 \ge \alpha_1 \ge \alpha$. Denote $\beta = \eta\alpha_1\alpha/2$, $t = \max(0, x - \beta)$. By Chebyshev's inequality,

$$\mathrm{Prob}(|\lambda(\pi^t(\omega)) - t| > \beta/4) \le \frac{V(\lambda(\pi^t(\omega)))}{(\beta/4)^2} \le \frac{16}{\beta^2}\rho(\pi)$$

and for every $T \subset \pi$

$$\mathrm{Prob}\left(|\lambda((\pi\backslash\pi^t(\omega)) \cap T) - (1-t)\lambda(T)| > \frac{\alpha\eta}{8}\right) \le \frac{64}{\alpha^2\eta^2}\rho(\pi).$$

Let δ_2 be a constant satisfying,

$$(64/\alpha^2\eta^2 + 16/\beta^2)\delta_2 < \eta/8$$

and take

$$\delta = \min\left(\frac{\eta\alpha^2\alpha_1}{4K_1}, \delta_2\right), \qquad K = K_1/\alpha.$$

Let π be a partition and $0 < x < 1$ with $\rho(\pi) < \delta$ and $K\cdot\rho(\pi) < x < 1 - \alpha$ and let $T \subset \pi$. Define $\bar\Omega \subset \Omega$ by:

$$\bar\Omega = \Omega_1 \cap \Omega_2 \text{ where } \Omega_1 = \{\omega: |\lambda(\pi^t(\omega)) - t| \le \beta/4\},$$
$$\Omega_2 = \left\{\omega: |\lambda((\pi\backslash\pi^t(\omega)) \cap T) - (1-t)\lambda(T)| \le \frac{\alpha\eta}{8}\right\}.$$

As $\delta \le \delta_2$ we have

$$\mathrm{Prob}(\bar\Omega) > 1 - \eta/8. \tag{9.4}$$

From the definition of $\bar\Omega$ it follows that for every $\omega \in \bar\Omega$,

$$\begin{aligned}\left|\frac{\lambda((\pi\backslash\pi^t(\omega)) \cap T)}{\lambda(\pi\backslash\pi^t(\omega))} - \lambda(T)\right| &\le \left|\frac{\alpha\eta/8}{\alpha}\right| + \left|\frac{(1-t)\lambda(T)}{\lambda(\pi\backslash\pi^t(\omega))} - \lambda(T)\right| \\ &\le \eta/8 + \eta\alpha\alpha_1/8\alpha \le \eta/4.\end{aligned} \tag{9.5}$$

For $x > \beta$ (i.e., $t > 0$) it follows from the definition of $\bar{\Omega}$ that for every $\omega \in \bar{\Omega}$

$$K_1 \frac{\rho(\pi)}{\lambda(\pi \backslash \pi^t(\omega))} \geq K_1 \frac{\delta}{\alpha} \leq \frac{\eta\alpha\alpha_1}{4} < \frac{x - \lambda(\pi^t(\omega))}{1 - \lambda(\pi^t(\omega))} < \frac{\eta\alpha\alpha_1}{\alpha} < \alpha_1 . \tag{9.6}$$

and for $x \leq \beta$, (i.e., $t = 0$) $\lambda(\pi^t(\omega)) = 0$, and therefore

$$K_1\rho(\pi) \leq K\rho(\pi) \leq x \leq \alpha_1. \tag{9.7}$$

By (9.4) it follows that

$$\begin{aligned} |P(T, x) - \lambda(T)| &\leq |P(T, x \mid \bar{\Omega}) - \lambda(T)| \\ &\quad + (1 - \text{Prob}(\bar{\Omega}))P(T, x \mid \bar{\Omega}) + |P(T, x \mid \Omega\backslash\bar{\Omega}) \cdot \text{Prob}(\Omega\backslash\bar{\Omega})| \\ &\leq |P(T, x \mid \bar{\Omega}) - \lambda(T)| + \eta/8 + \eta/8. \end{aligned}$$

But from Corollary 9.2*, (9.5), (9.6) and (9.7) it follows that

$$|P(T, x \mid \bar{\Omega}) - \lambda(T)| \leq \eta/4 + \eta/4;$$

thus,

$$|P(T, x) - \lambda(T)| < \eta$$

which completes the proof of Theorem 9.3.

THEOREM 9.8. *For every $\eta > 0$ there exists $\delta > 0$, $K > 0$ such that if π is a partition with $\rho(\pi) < \delta$ and $0 < x < 1$ with $K\rho(\pi) < x < 1 - K\rho(\pi)$, then*

$$\textstyle\sum_{a \in \pi} |P(a, x) - \lambda(a)| < \eta.$$

PROOF. By defining $P'(a, x) = \text{Prob}(x - \lambda(a) < \lambda(\mathscr{P}_a^{(\mathscr{H})}) \leq x)$, and thus $P'(a, y) = \frac{1}{|\pi|} U_{\pi\backslash\{a\}}((y - \lambda(a), y])$, we obtain (as in the derivation of Theorem 9.3) the following:

THEOREM 9.3*. *Let $0 < \alpha < 1$ be given. For every $\eta > 0$ there exist $\delta > 0$, $K > 0$ such that if π is a partition with $\rho(\pi) < \delta$ and $0 < x < 1$ with $K\rho(\pi) < x < 1 - \alpha$ then for every $T \subset \pi$*

$$|P'(T, x) - \lambda(T)| < \eta.$$

Now, observe that $P'(T, x) = P(T, 1 - x)$ and therefore by Theorems 9.3 and 9.3* and by the following obvious identity

$$\sup\nolimits_{T \subset \pi} |P(T, x) - \lambda(T)| = \frac{1}{2} \textstyle\sum_{a \in \pi} |P(a, x) - \lambda(a)|,$$

Theorem 9.8 follows.

REFERENCES

[1] AUMANN, R. J. and SHAPLEY, L. S. (1974). *Values of Non-Atomic Games.* Princeton University Press.

[2] BLACKWELL, D. (1948). A renewal theorem. *Duke Math. J.* **15** 145–150.

[3] BLACKWELL, D. (1953). Extension of a renewal theorem. *Pacific J. Math.* **3** 315–320.

[4] KANNAI, Y. (1966). Values of games with a continuum of players. *Israel J. of Math.* **4** 54–58.

[5] LINDVALL, T. (1977). A probabilistic proof of Blackwell's renewal theorem. *Ann. Probability* **5** 482–485.

[6] NEYMAN, A. (1981). Singular games have asymptotic values. *Math. Oper. Res.* **6** 205–212.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720