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SINGULAR GAMES HAVE ASYMPTOTIC VALUES*

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The asymptotic value of a game v with a continuum of players is defined whenever all the sequences of Shapley values of finite games that "approximate" v have the same limit. In this paper we prove that if v is defined by $v(S) = f(\mu(S))$, where μ is a nonatomic probability measure and f is a function of bounded variation on $[0, 1]$ that is continuous at 0 and at 1, then v has an asymptotic value. This had previously been known only when v is absolutely continuous. Thus, for example, our result implies that the nonatomic majority voting game, defined by $v(S) = 0$ or 1 according as $\mu(S) < 1/2$ or $\mu(S) > 1/2$, has an asymptotic value. We also apply our result to show that other games of interest in economics and political science have asymptotic values, and adduce an example to show that the result cannot be extended to functions f that are not of bounded variation.

Introduction. In their book *Values of Non Atomic Games* Aumann and Shapley extended the concept of value to certain classes of nonatomic games, i.e., infinite person games in which no individual has significance. One of the approaches, due to Kannai (1966), is the asymptotic one. Briefly, the asymptotic value is defined on each game v for which all the sequences of Shapley values, corresponding to sequences of finite games that 'approximate' v, have the same limit. The space of all games possessing an asymptotic value is denoted ASYMP.

The asymptotic value has been studied extensively, [2], [1], [6]. The basic theorem [1, Theorem F], asserts that ASYMP contains pNA. The space pNA is the subspace of BV spanned by powers of nonatomic measures, or alternatively, the subspace spanned by absolutely continuous scalar measure games. However, $bv'NA$, the maximal subspace of BV that is spanned by monotonic scalar measure games and on which there is a value (an axiomatic one), contains many other games of interest beside those of pNA. The space $bv'NA$ appears to be basic in the study of nonatomic games. Aumann and Shapley [1, Theorem A] proved the existence of a unique value on $bv'NA$, and they raised the question as to whether or not ASYMP contains $bv'NA$, or even whether the simplest single-jump functions are in ASYMP [1, p.10].

The main result of this paper answers the question in the affirmative, namely, that $bv'NA$ is contained in ASYMP. This is accomplished in §3. In order to prove this result, the author developed in [4] a renewal theorem for sampling without replacement, which is the main tool in proving our theorem. This renewal theorem is actually 'equivalent' to the existence of an asymptotic value on the simplest single jump function. The completion of the theorem is based on an integral representation for the Shapley value of the finite game approximating a scalar measure game, by means of those corresponding to jump functions (Lemma 3.4), and by known properties of the structure of $bv'NA$.

The existence of an asymptotic value on $bv'NA$ enables to prove the existence of the asymptotic value on many other spaces of interest. This is done in §4.

In §5, we show by means of a counter example that there is no hope to extend our result for scalar measure games with unbounded variation.

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2. Preliminaries. Most of the definitions and notations are according to [1]. Let $(I, \mathcal{C})$ be a measurable space isomorphic to $([0,1], \mathcal{B})$ where $\mathcal{B}$ is the σ-field of Borel sets in $[0,1]$. A *set function* is a real valued function v on $\mathcal{C}$ such that $v(\emptyset) = 0$. The members of I are called *players*, the members of $\mathcal{C}$ *coalitions*, and the set functions *games*. A game v is *monotonic* if for each $S, T \in \mathcal{C}$, $S \subset T \Rightarrow v(S) \leqslant v(T)$. If Q is a set of games, Q^+ denotes the subset of monotonic games in Q. A game v is of *bounded variation* if it is the difference between two monotonic games. The *variation norm* of v is defined by $\|v\| = \inf(u(I) + w(I))$, where the inf ranges over all monotonic set functions u and w such that $v = u - w$. The space of all games of bounded variation is called BV. The subspace of BV consisting of all bounded, finitely additive set functions is denoted FA, and that of all nonatomic measures (countably additive) is denoted NA. The subset of NA of all probability measures is denoted NA^1.

Let Q be any subspace of BV. A mapping of Q into BV is *positive* if it maps Q^+ into BV^+.

Let $\mathcal{G}$ be the group of automorphisms of $(I, \mathcal{C})$ (i.e., one to one mapping of I onto itself that are measurable in both directions). Each $\theta \in \mathcal{G}$ induces a linear mapping θ^* of BV onto itself, defined by $(\theta^* v)(S) = v(\theta S)$ for all $S \in \mathcal{C}$.

Let Q be a symmetric subspace of BV. A *value* on Q is a positive linear mapping ϕ from Q into FA that satisfies:

ϕ is symmetric, i.e., $\phi\theta^* = \theta^*\phi$ for all $\theta \in \mathcal{G}$,

ϕ is efficient, i.e., $\phi v(I) = v(I)$ for all $v \in Q$.

The space of all real valued functions f of bounded variation on $[0,1]$ that obey $f(0) = 0$ and are continuous at 0 and 1 is denoted bv'. The closed symmetric subspace of BV spanned by the set functions of the form $f \circ \mu$ where $f \in bv'$ and $\mu \in NA^1$ is called $bv'NA$. pNA is the closed subspace of $bv'NA$ spanned by all powers of NA^1 measures.

The Shapley value on finite games will be denoted by ψ. Let v be a finite game, N the set of players. For each player a and an order $\mathcal{R}$ on N, $\mathcal{P}_a^{\mathcal{R}}$ is the set of all players preceding a in $\mathcal{R}$. The Shapley value is given by

$$\psi v(a) = (1/|N|!) \sum_{\mathcal{R}} \left[v(\mathcal{P}_a^{\mathcal{R}} \cup \{a\}) - v(\mathcal{P}_a^{\mathcal{R}}) \right].$$

$\psi v(a)$ can be regarded as the expectation of the contributions of player a, where each order $\mathcal{R}$ has the same probability, namely $1/|N|!$. We will write $v(\mathcal{P}_a^{\mathcal{R}} \cup a)$ instead of $v(\mathcal{P}_a^{\mathcal{R}} \cup \{a\})$.

A *partition* Π of the underlying space $(I, \mathcal{C})$ is a finite family of disjoint measurable subsets whose union is I. A partition Π_2 is a *refinement* of another partition Π_1 if each member of Π_1 is a union of members of Π_2. In such a case we denote it by $\Pi_2 \succ \Pi_1$. A sequence $\{\Pi_n\}_{n=1}^{\infty}$ of partitions is called *admissible* if it satisfies

(1) it is *decreasing*, i.e., $\Pi_{m+1} \succ \Pi_m$ for each m;

(2) it is *separating*, i.e., for each $s, t \in I$ with $s \neq t$, there is m such that s and t are in different members of Π_m.

For each partition Π and set function v, let v_π be the finite game whose players are the members of Π, and for $A \subset \Pi$ $v_\pi(A)$ is defined by

$$v_\pi(A) = v\Big(\bigcup_{a \in A} a\Big).$$

A set function $\phi v \in FA$ is said to be the *asymptotic value* of v, if for every $T \in \mathcal{C}$ and every admissible sequence of partitions $\{\Pi_k\}_{k=1}^{\infty}$ with $\Pi_1 \succ \{T, I \setminus T\}$ the following limit and equality exists

$$\lim_{k \to \infty} \psi v_{\pi_k}(T_k) = \phi v(T)$$

where $T_k = \{a: a \in \Pi_k \text{ and } a \subset T\}$. The set of all games $v \in BV$ having an asymptotic value is denoted by ASYMP.

The main result of this paper is

MAIN THEOREM. $bv'NA \subset$ ASYMP.

We will use the following conventions; $\mathbf{R}$ stands for the real numbers, and if A is a finite set then $|A|$ denotes the number of elements in A.

3. The asymptotic value on $bv'NA$. In this section we will prove the main result of this paper, namely, that $bv'NA \subset$ ASYMP.

In order to prove this result, the author developed the 'Renewal Theory for Sampling without Replacement' [4]. We start by introducing a result of that paper. Let Π be a partition and let $\rho(\Pi) = \max_{a\in\Pi}\lambda(a)$ where λ is in NA^1. For every element a of Π, and $0 < x < 1$, define $P(a,x)$ by

$$P(a,x) = (1/|\Pi|!)|\{\mathcal{R}: \lambda(\mathcal{P}_a^{\mathcal{R}}) < x \leqslant \lambda(\mathcal{P}_a^{\mathcal{R}}) + \lambda(a)\}|$$

(for $A \subset \Pi$ we mean by $\lambda(A)$ the sum $\sum_{a\in A}\lambda(a)$). Roughly speaking, $P(a,x)$ is the probability that in a random order of Π, a is the first element (in the order) for which the λ-accumulated sum exceeds x. For $A \subset \Pi$ let $P(A,x) = \sum_{a\in A}P(a,x)$. The distance between the probability measures $P(\cdot,x)$ and $\lambda(\cdot)$ (on$(\Pi, 2^\pi)$) is defined by

$$\|P(\cdot x) - \lambda(\cdot)\| = \sum_{a\in\Pi} |P(a,x) - \lambda(a)|.$$

Observe that as both $P(\cdot x)$ and $\lambda(\cdot)$ are probability measures,

$$\|P(\cdot,x) - \lambda(\cdot)\| = 2\max_{A\in\Pi}|P(A,x) - \lambda(A)|.$$

LEMMA 3.1. *For every $\epsilon > 0$ there exist constants $\delta > 0$ and $K > 0$ such that if $\rho = \max_{a\in\Pi}\lambda(a) < \delta$, and $K\rho < x < 1 - K\rho$ then $\|P(\cdot,x) - \lambda(\cdot)\| > \epsilon$.*

PROOF. This is theorem 9.8 of [4].

We proceed by proving that the "jump functions" are in ASYMP. We use the following terminology and notations. Let λ be in NA^1, and $\{\Pi_k\}_{k=1}^\infty$ a sequence of partitions; then $\{\Pi_k\}_{k=1}^\infty$ is said to be λ-*shrinking* if

$$\max_{a\in\Pi_k}\lambda(a) \underset{k\to\infty}{\to} 0.$$

If v is any set function on $(I, \mathcal{C})$, the dual v^* of v is defined by $v^*(S) = v(I) - v(I\setminus S)$. Observe that $v \in$ ASYMP iff $v^* \in$ ASYMP (easily shown by reversing order, see [1, p.140]).

LEMMA 3.2. *For every $0 < y < 1$, let $f_y, \bar{f}_y: [0,1] \to \mathbf{R}$ be defined by:*

$$f_y(x) = \begin{cases} 1 & \text{if } x \geqslant y, \\ 0 & \text{if } x < y, \end{cases} \qquad \bar{f}_y(x) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x \leqslant y. \end{cases}$$

Then for every $\lambda \in NA^1$, $f_y \circ \lambda$ and $\bar{f}_y \circ \lambda$ have asymptotic values.

PROOF. Let $0 < y < 1$, $T \in \mathcal{C}$ and $\lambda \in NA^1$ be given. Let $\{\Pi_k\}_{k=1}^\infty$ be an admissible sequence with $\Pi_1 \succ \{I\setminus T, T\}$. Then $\{\Pi_k\}_{k=1}^\infty$ is λ-shrinking (Lemma 18.6 of [1] and Halmos (1950) p. 172, Theorem A). Let $\eta > 0$, and let K and δ be given by Lemma 3.1 such that $\rho(\Pi) < \delta$ and $K\cdot\rho(\Pi) < x < 1 - K\cdot\rho(\Pi)$ (where $\rho(\Pi) = \max_{a\in\Pi}\lambda(a)$) implies that $\|P(\cdot,x) - \lambda(\cdot)\| < \eta$. As $\{\Pi_k\}_{k=1}^\infty$ is λ-shrinking, there exists k_0 such that for every $k > k_0$, $K\cdot\rho(\Pi_k) < y < 1 - K\cdot\rho(\Pi_k)$ and $\rho(\Pi_k) < \delta$.

Thus by denoting $T_k = \{a : a \in \Pi_k, a \subset T\}$, and using the obvious identity

$$\psi(f_y \circ \lambda)_\pi(A) = P_\pi(A, y) \quad \text{for every partition} \quad \Pi, \quad \text{and} \quad A \subset Y,$$

we deduce that

$$|\psi(f_y \circ \lambda)_{\pi_k}(T_k) - \lambda(T_k)| \equiv |P(T_k, y) - \lambda(T_k)|$$
$$\leqslant \tfrac{1}{2}\|P_{\pi_k}(\cdot, y) - \lambda(\cdot)\| < \eta/2 \tag{3.3}$$

and hence

$$\lim_{k\to\infty} \psi(f_y \circ \lambda)_{\pi_k}(T_k) = \lambda(T)$$

which completes the proof that $f_y \circ \lambda \in$ ASYMP. Observe that $(\bar{f}_y \circ \lambda)(I) - (\bar{f}_y \circ \lambda)(I \setminus S) = 1 - (f_y \circ \lambda)(I \setminus S) = (f_{1-y} \circ \lambda)(S)$, and therefore $f_{1-y} \circ \lambda$ is the dual of $\bar{f}_y \circ \lambda$. In view of the remark preceding the lemma and the fact that $f_{1-y} \circ \lambda \in$ ASYMP it follows that $\bar{f}_y \circ \lambda \in$ ASYMP. This completes the proof of the lemma.

LEMMA 3.4. *Let $f \in bv'$ be right continuous and let Π be a given partition. Then for every subset A of Π and every λ in NA^1*

$$\psi(f \circ \lambda)_\pi(A) = \int_0^1 \psi(f_y \circ \lambda)_\pi(A) \cdot df(y).$$

PROOF. Let a be an element of Π. It is enough to prove that

$$\psi(f \circ \lambda)_\pi(a) = \int_0^1 \psi(f_y \circ \lambda)(a) \cdot df(y).$$

For every order $\mathfrak{R}$ of Π, we define a function $\chi(\mathfrak{R}, a) : [0, \infty) \to \mathbf{R}$ by:

$$\chi(\mathfrak{R}, a)(y) = \begin{cases} 1 & \text{if} \quad \lambda(\mathcal{P}_a^{\mathfrak{R}}) < y \leqslant \lambda(\mathcal{P}_a^{\mathfrak{R}} \cup \{a\}), \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$(f \circ \lambda)_\pi(\mathcal{P}_a^{\mathfrak{R}} \cup \{a\}) - (f \circ \lambda)_\pi(\mathcal{P}_a^{\mathfrak{R}}) = \int_0^1 \chi(\mathfrak{R}, a)(y) \cdot df(y)$$

and therefore

$$\psi(f \circ \lambda)_\pi(a) = (1/n!) \sum_{\mathfrak{R}} \int_0^1 \chi(\mathfrak{R}, a)(y) \cdot df(y)$$

where $n = |\Pi|$ and the sum is over all orders of Π. But, as

$$(1/n!) \sum_{\mathfrak{R}} \chi(\mathfrak{R}, a)(y) = \psi(f_y \circ \lambda)_\pi(a)$$

we conclude by changing the order of summation and integration that,

$$\psi(f \circ \lambda)_\pi(a) = \int_0^1 \left[(1/n!) \sum_{\mathfrak{R}} \chi(\mathfrak{R}, a)(y) \right] \cdot df(y)$$
$$= \int_0^1 \psi(f_y \circ \lambda)_\pi(a) \cdot df(y)$$

which completes the proof of the lemma.

PROOF OF THE MAIN THEOREM. $bv'NA$ is the closed linear symmetric space spanned by set functions of the form $f \circ \nu$ where $f \in bv'$ and ν is a probability measure in NA. Every $f \in bv'$ can be represented as a sum

$$f = g + \sum_{i=1}^{\infty} \alpha_i \cdot \bar{f}_{y_i}$$

where $g \in bv'$ is right continuous, $y_i \in (0,1)$ and $\{a_i\}$ is a sequence of numbers with $\sum |a_i| < \infty$. As

$$\left\| \sum_{i=n}^{\infty} a_i(\bar{f}_{y_i} \circ \lambda) \right\| \leqslant \sum_{i=n}^{\infty} |a_i| \underset{n\to\infty}{\to} 0$$

the space $bv'NA$ is the closed linear symmetric space spanned by jump functions $\bar{f}_y \circ \lambda$ and set functions of the form $f \circ \lambda$, where $f \in bv'$ is right continuous. As ASYMP is a closed symmetric subspace of BV [1, Propositions 18.4, 18.5], and as for every $y \in (0,1)$ $\bar{f}_y \circ \lambda \in$ ASYMP, it is enough to prove that for every nondecreasing $f \in bv'$ which is right continuous (every right continuous $f \in bv'$ is the difference of two functions $f_1, f_2 \in bv'$ which are nondecreasing and right continuous), $f \circ \lambda \in$ ASYMP. Let $f \in bv'$ be nondecreasing and right continuous, $T \in \mathfrak{B}$ and $\{\Pi_k\}_{k=1}^{\infty}$ an admissible sequence with $\Pi_1 \succ \{T, I\setminus T\}$. For every k denote by T_k the subset of Π_k of all elements contained in T, i.e.,

$$T_k = \{a : a \in \Pi_k, a \subset T\}.$$

By Lemma 3.4,

$$\psi(f\circ\lambda)_{\pi_k}(T_k) = \int_0^1 \psi(f_y \circ \lambda)_{\pi_k}(T_k)\,df(y),$$

and by Lemma 3.2

$$\psi(f_y \circ \lambda)_{\pi_k}(T_k) \underset{k\to\infty}{\to} \lambda(T).$$

As $|\psi(f_y \circ \lambda)_{\pi_k}(T_k)| \leqslant 1$, we could use Lebesgue's dominated convergence theorem to conclude that

$$\psi(f\circ\lambda)_{\pi_k}(T_k) \underset{k\to\infty}{\to} \int_0^1 \lambda(T)\,df(y) = f(1)\lambda(T)$$

which completes the proof of the theorem.

4. Other subspaces of ASYMP. We start with several notations. A is the closed algebra generated by all set functions of the form $f \circ \mu$ where f is a continuous function in bv', and μ is in NA^1. If Q_1 and Q_2 are subsets of BV then $Q_1 * Q_2$ is the closed linear symmetric subspace spanned by all games of the form $v_1 \cdot v_2$ where $v_1 \in Q_1$ and $v_2 \in Q_2$.

In [3] the partition value is introduced and it is proved that there is a partition value on each of the spaces A, $pNA*bv'NA$, $A*bv'NA$, $bv'NA*bv'NA$ and $A*bv'NA*bv'NA$. It was proved in [3] that $bv'NA*bv'NA \not\subset$ ASYMP and hence also $A*bv'NA*bv'NA \not\subset$ ASYMP. However it turns out that

THEOREM 4.1. *$A \subset$ ASYMP, $pNA*bv'NA \subset$ ASYMP and $A*bv'NA \subset$ ASYMP.*

PROOF. Observe that $A \subset A*bv'NA$ and that $pNA \subset A$ and therefore it is enough to prove that $A*bv'NA \subset$ ASYMP. But this follows along the same lines as the proof in [3] of the existence of a partition value on $A*bv'NA$, substituting the fact that $bv'NA \subset$ ASYMP for Lemmas 4.9 and 4.10 of [3].

5. Weakening of assumption? The existence of an asymptotic value on $bv'NA$ is equivalent to the existence of an asymptotic value on all scalar measure games $f \circ \lambda$

where λ is in NA^1 and f: $[0, 1] \to \mathbb{R}$ is continuous at 0 and 1, and of bounded variation. It can be easily proved that for f of bounded variation the limits

$$\lim_{\substack{x\to 0\\ x>0}} f(x) = f(0+) \qquad \text{and} \qquad \lim_{\substack{x\to 1\\ x<1}} f(x) = f(1-)$$

exist, and that further, $f \circ \lambda$ has an asymptotic value iff $f(0+) - f(0) = f(1-) - f(1)$. In view of this result, the theory of the asymptotic value is comprehensive in the class of scalar measure games with bounded variation.

The study of nonatomic games has been concentrated, so far, on games with bounded variation. However, the definitions of a value, and that of an asymptotic value are naturally extended to include all set functions, and existence of a value can be shown on spaces which include $bv'NA$ and many other scalar measure games of unbounded variation. However, as will be seen in the following example the bounded variation is essential for the existence of an asymptotic value. We will construct an example of a bounded function f defined on $[0, 1]$, which is continuous at 0 and 1, vanishes outside a countable set of points and $f \circ \lambda$ does not have an asymptotic value.

THE EXAMPLE. For each N define $M = 2^N$ and define D_N to be all the triples (m, n, N) such that

$$m \text{ is odd}, \tag{5.1}$$

$$M/2 + 2\sqrt{M} < n + \sqrt{M} < m < M/2 + 3\sqrt{M}\,. \tag{5.2}$$

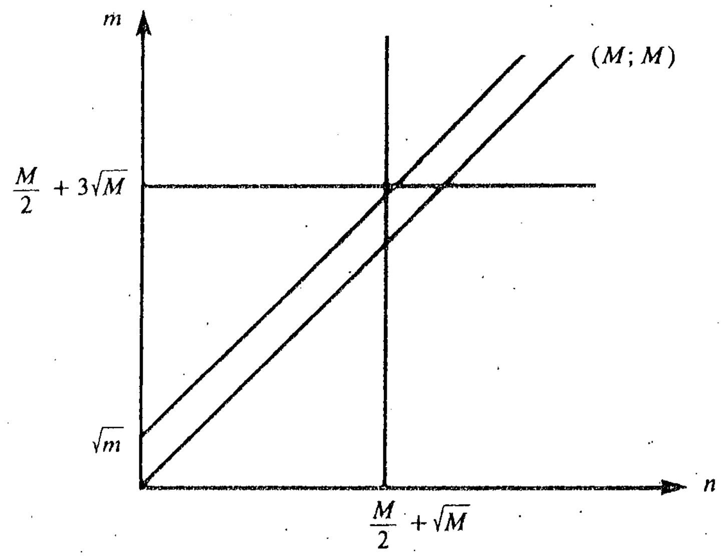

Figure 1

The set D_N consists of all the triples (m, n, N) such that m is odd and (m, n) is in the dashed area. It can be easily seen that

$$|D_N|/M \underset{N\to\infty}{\to} 1/4. \tag{5.3}$$

Let $t \in (0, 1)$ be an irrational. Define $D = \bigcup_{N=1}^{\infty} D_N$, and define $y: D \to (0, 1)$ by:

$$y(m, n, N) = tm/2^N + (1 - t)n/2^N. \tag{5.4}$$

y is a $1-1$ mapping, because $N' < N$ and $tm/2^N + (1-t)n/2^N = tm'/2^{N'} + (1-t)$ $n'/2^{N'}$ implies that both n and m are even which contradicts the assumption (5.1), and if $N = N'$ then $y(m,n,N) = y(m',n',N')$ implies that $m = m'$ and $n = n'$. Define Y_N to be the range of y restricted to D_N and $Y = \bigcup_{N=100}^{\infty} Y_N$, then $Y \subset (1/2, 3/4)$ and Y is countable. We define f to be the characteristic function of the set Y. We will prove that

$f \circ \lambda$ does not have an asymptotic value. If $f \circ \lambda$ has an asymptotic value then the asymptotic value must be 0, so in order to prove that $f \circ \lambda$ does not have an asymptotic value we will construct an admissible sequence of partitions $\{\Pi_N\}_{N=1}^{\infty}$ with $\Pi_1 \succ \{[0,t),[t,1]\}$ such that

$$\liminf_{N\to\infty} \psi(f\circ\lambda)_{\pi_N}(A_N) > 0$$

where $A_N = \{a: a \in \Pi_N, a \subset [0,t)\}$. Let Π_N be the partition that divides each of the intervals $[0,t)$ and $[t,1]$ into 2^N equal segments, and $A_N = \{a: a \in \Pi_N, a \subset [0,t)\}$. We shall show first that:

$$\psi(f\circ\lambda)_{\pi_N}(A_N) \geqslant \psi(\chi_{Y_N}\circ\lambda)_{\pi_N}(A_N) \tag{5.5}$$

and secondly we shall prove that there exists $\gamma > 0$ such that for every $y \in Y_N$

$$\psi(\chi_{\{y\}}\circ\lambda)_{\pi_N}(A_N) \geqslant \gamma \cdot 1/M. \tag{5.6}$$

For proving (5.5) observe that $tm/2^N + (1-t)n/2^N \in Y$ implies because of the irrationality of t and (5.2) that $m > n$ which implies easily that $\psi(\chi_{\{y\}}\circ\lambda)_{\pi_N}(A_N) \geqslant 0$ for any $y \in Y$, which completes the proof of (5.5). For proving (5.6), observe that because of the irrationality of t, the equality $tk/M + (1-t)l/M = tm/M + (1-t)n/M$ implies that $k = m$ and $l = n$. Let $y \in Y_N$, $y = tm/2^N + (1-t)n/2^N$, then we have:

$$\begin{aligned}
\psi(\chi_{\{y\}}\circ\lambda)_{\pi_N}(A_N) &= \frac{\binom{M}{m}\binom{M}{n} m(m+n-1)!\,(2M-m-n)!}{(2M)!} \\
&\quad - \frac{\binom{M}{m}\binom{M}{n}(M-m)(m+n)!\,(2M-m-n-1)!}{(2M)!} \\
&= \frac{\binom{M}{m}\binom{M}{n}}{(2M)!}(m+n-1)!\,(2M-m-n-1)! \\
&\quad \left[m(2M-m-n) - (M-m)(m+n)\right] \\
&= \frac{\binom{M}{m}\binom{M}{n}}{\binom{2M-2}{m+n-1}} \cdot \frac{(m-n)M}{2M(2M-1)}
\end{aligned}$$

and by using the central limit theorem for Bernoulli trials and the properties of m, n we deduce that

$$\psi(\chi_{\{y\}}\circ\lambda)(A_N) \geqslant \frac{a\dfrac{2^M}{\sqrt{M}}\cdot\dfrac{2^M}{\sqrt{M}}\cdot M\cdot\sqrt{M}}{\beta\dfrac{2^{2(M-1)}}{\sqrt{2(M-1)}}\cdot 2M(2M-1)} \geqslant \gamma\cdot\frac{1}{M}$$

which completes the proof of (5.6). Combining (5.3), (5.5) and (5.6) we deduce that

$$\limsup \psi(f\circ\lambda)_{\pi_N}(A_N) > 0,$$

which completes the proof that $f \circ \lambda$ does not have an asymptotic example.

REMARK. (a) This example exhibits the fact that the bounded variation of f is essential for the existence of an asymptotic value for $f \circ \lambda$. (b) Denote the set of all set functions by SF. For any member v of SF we can assign the set function

$$\hat{v}(S) = \frac{v(S) - v(I \setminus S)}{2} + \frac{v(I)}{2}.$$

The set function $\bar{v} = v - \hat{v}$ is a 'symmetric' set function, i.e., $\bar{v}(S) = \bar{v}(I \setminus S)$. Hence $\bar{v}$ has an asymptotic value, which is the measure which vanishes identically. Therefore v has an asymptotic value iff $\hat{v}$ has an asymptotic value. Furthermore, the relation $\sim$ in SF defined by $v \sim \mu$ iff $\hat{u} = \hat{v}$ is an equivalence relationship. It is thus more convenient to investigate properties of the asymptotic value and state result in SF mod $\sim$.

Conjecture. Let $v = f \circ \mu$, where $\mu \in NA^1$ and $f: [0, 1] \to \mathbf{R}$ with $f(0) = 0$, continuous at 0 and 1. Then v has an asymptotic value iff $\hat{v}$ has bounded variation. (The if part is obvious from the above remark and the inclusion $bv'NA \subset$ ASYMP).

References

[1] Aumann, R. J. and Shapley, L. S. (1974). *Values of Non-Atomic Games*. Princeton Univ. Press, Princeton, N.J.

[2] Kannai, Yakar. (1966). Values of Games with a Continuum of Players. *Israel J. Math.* **4** 54–58.

[3] Neyman, Abraham and Tauman, Yair. (1979). The Partition Value, *Math. Oper. Res.* **4** 236–264.

[4] ———. (to appear). Renewal Theory for Sampling Without Replacement. *Ann. Probability*.

[5] Shapley, L. S. (1953). A Value for n-Person Games. In *Contribution to the Theory of Games, II*. Princeton Univ. Press, Princeton, N. J. 307–317.

[6] Hart, S. (1977). Asymptotic Values of Games with a Continuum of Players. *J. Math. Econon.* **4** 57–80.

[7] Halmos, P. R. (1950). *Measure Theory*. Van Nostrand, Princeton, N. J.

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