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In honour of Martin Shubik

# Singular games in $bv'NA$☆

Abraham Neyman*

*Institute of Mathematics, and Center for the Study of Rationality, The Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel*

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ABSTRACT

Every simple monotonic game in $bv'NA$ is a weighted majority game. Every game $v \in bv'NA$ has a representation $v = u + \sum_{i=1}^{\infty} f_i \circ \mu_i$ where $u \in pNA$, $\mu_i \in NA^1$ and $f_i$ is a sequence of $bv'$ functions with $\sum_{i=1}^{\infty} \|f_i\| < \infty$. Moreover, the representation is unique if we require $f_i$ to be singular and that for every $i \neq j$, $\mu_i \neq \mu_j$.



## 1. Introduction

Simple games with finitely many players were introduced by von Neuman and Morgenstern (1944). These games are appropriate to the study of political structures in which power is the fundamental driving force. A special class consists of the weighted majority games $[q; w_1, \ldots, w_n]$, where $n$ is the number of players, $w_i$ is the number (weight) of votes of player $i$, and $q$ is the quota of votes required for a winning coalition.

The seminal paper of Shapley and Shubik (1954), which interpreted the Shapley value as a measure of voting power in the context of simple games, initiated a long line of research. Special attention has been given to weighted majority games with a large number of small voters (see Milnor and Shapley, 1978; Neyman, 1981, 1988; Shapiro and Shapley, 1978); such games arise naturally in stockholder voting in corporations with one vote per share held. In these games, the voters are individually insignificant and wield influence only through coalitions. It is fruitful to model them as games with a continuum of players (see the definitive model of Aumann and Shapley, 1974).

The underlying set of players is modeled by a measurable space $(I, \mathcal{C})$ that admits non-atomic (positive) measures. Here $I$ is the set of *players*, and the $\sigma$-algebra $\mathcal{C}$ is the set of *coalitions*. A *game* is a real-valued function $v$ on $\mathcal{C}$ with $v(\emptyset) = 0$.

Two spaces of games that play a central role in Aumann and Shapley (1974) – $pNA$ and the larger space $bv'NA$ – are generated by non-atomic scalar measure games, i.e., by games of the form $f \circ \mu$, where $\mu$ is in $NA^1$, the set of all non-atomic probability measures on $(I, \mathcal{C})$, and where $f$ is a real-valued function defined on $[0, 1]$ (the range of $\mu$). The space $pNA$ is obtained by considering all functions $f$ in $ac$, where $ac$ is the set of all absolutely continuous functions with $f(0) = 0$. The closure (in the bounded variation norm) of the linear span of all games $f \circ \mu$, for $f \in ac$ and $\mu \in NA^1$, is $pNA$. The space $bv'NA$ is obtained by considering all functions $f$ in $bv'$, where $bv'$ is the set of all functions of bounded variation on $[0, 1]$ that are continuous at $0 = \mu(\emptyset) = f(0)$ and at 1. The closure of the linear span of all games $f \circ \mu$, for $f \in bv'$ and $\mu \in NA^1$, is $bv'NA$.

For $f \in bv'$, let $\|f\|$ denote the variation of $f$ on $[0, 1]$. Whenever $\mu$ is a non-atomic probability measure, the bounded variation norm $\|f \circ \mu\|$ of $f \circ \mu$ equals $\|f\|$. Therefore, if $(f_i)_{i=1}^{\infty}$ is a sequence of functions in $bv'$ with $\sum_{i=1}^{\infty} \|f_i\| < \infty$ and $\mu_i$ is a

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* Corresponding author. Tel.: +972 544661731; fax: +972 2 6513681.
*E-mail address:* aneyman@math.huji.ac.il.

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sequence of non-atomic probability measures, the series $\sum_{i=1}^{\infty} f_i \circ \mu_i$ converges to a game $v \in bv'NA$. If the functions $f_i$ are in *ac* the series converges to a game in *pNA*.

A game of the form $f \circ \mu$ where $\mu$ is a non-atomic measure and $f$ is an absolutely continuous function on the range of the scalar measure $\mu$ is called an *absolutely continuous scalar measure game*. If $f$ is a polynomial, it is called a *polynomial scalar measure game*. Therefore any game in *pNA* is approximated in the bounded variation norm by a linear combination of absolutely continuous scalar measure games. Moreover, any absolutely continuous scalar measure game is approximated in the bounded variation norm by a polynomial scalar measure game and therefore any game in *pNA* is approximated by a linear combination of polynomial scalar measure games.

However, any polynomial of a vector of non-atomic measures that vanishes at the vector 0 is a linear combination of polynomial scalar measure games and therefore the space *pNA* contains many other games of interest. For example, if $\mu_1, \ldots, \mu_n$ are non-atomic probability measures and $f$ is a continuously differentiable function on the range of the vector measure $(\mu_1, \ldots, \mu_n)$ with $f(0) = 0$, then $f \circ (\mu_1, \ldots, \mu_n)$ is in *pNA*. Such a vector measure smooth game need not have a representation as a convergent series $\sum_{i=1}^{\infty} f_i \circ \mu_i$ where $f_i \in ac$ and $\mu_i \in NA^1$.

Any function $f \in bv'$ has a unique representation as a sum $f = f^{ac} + f^s$ where $f^{ac} \in ac$ and $f^s$ is a singular function, i.e., a function whose variation is on a set of measure zero, in $bv'$. The subspace of all singular functions in $bv'$ is denoted $s'$. The closed linear space generated by all games of the form $f \circ \mu$ where $f \in s'$ and $\mu \in NA^1$ is denoted $s'NA$.

Aumann and Shapley (1974) show that $bv'NA$ is the algebraic sum of the two spaces *pNA* and $s'NA$ and that for any two games, $u \in pNA$ and $v \in s'NA$, $\|u + v\| = \|u\| + \|v\|$.

The present paper demonstrates two results. The first asserts that the only simple monotonic games in $bv'NA$ are the weighted majority games. The second result shows that any game $v \in s'NA$ is the sum of a convergent series $\sum_{i=1}^{\infty} f_i \circ \mu_i$ where $f_i \in s'$ and $\sum_{i=1}^{\infty} \|f_i\| < \infty$, and thus every game $v \in bv'NA$ has a representation $v = u + \sum_{i=1}^{\infty} f_i \circ \mu_i$ where $u \in pNA$, $\mu_i \in NA^1$ and $f_i$ is a sequence of $bv'$ functions with $\sum_{i=1}^{\infty} \|f_i\| < \infty$. Moreover, the representation is unique if we require that $f_i$ be singular and that for every $i \neq j$, $\mu_i \neq \mu_j$.

## 2. Simple monotonic games in *bv'NA*

The set of all positive and finitely additive games is denoted $FA^+$. A weighted majority game is a game of the form

$$v(S) = \begin{cases} 1 & \text{if} \quad \mu(S) \geq q \\ 0 & \text{if} \quad \mu(S) < q \end{cases}$$

or

$$v(S) = \begin{cases} 1 & \text{if} \quad \mu(S) > q \\ 0 & \text{if} \quad \mu(S) \leq q \end{cases}$$

where $\mu \in FA^+$ and $0 < q < \mu(I)$. The finitely additive measure $\mu$ is called the *weight measure* and $q$ is called the *quota*. Every weighted majority game has a representation with a weighted measure in $FA^1$, the set of all $\mu \in FA^+$ with $\mu(I) = 1$, and thus we further assume the normalization $\mu \in FA^1$.

**Theorem 1.** *Every simple monotonic game in bv'NA is a weighted majority game, and the weighted measure is a non-atomic probability measure.*

**Proof.** Assume that $v \in bv'NA$ is a simple monotonic game. As $bv'NA$ is the closed linear span of all games of the form $f \circ \mu$ where $\mu \in NA^1$ and $f \in bv'$, $v$ can be approximated by linear combinations of the form $\sum_{i=1}^{n} f_i \circ \mu_i$ where $f_i \in bv'$ and $\mu_i \in NA^1$. As any function $f \in bv'$ is the sum of an absolutely continuous function $f^{ac} \in bv'$ and a singular function $f^s \in s'$, any such linear combination is the sum of a game $u \in pNA$ and a linear combination $\sum_{i=1}^{n} f_i \circ \mu_i$ where each one of the functions $f_i$ is singular, i.e., in $s'$.

Fix $\varepsilon > 0$, and let $u \in pNA$, $n$ a positive integer, $f_i \in s'$, and $\mu_i \in NA^1$ $(i = 1, \ldots, n)$, so that

$$\|v - u - \sum_{i=1}^{n} f_i \circ \mu_i\| < \varepsilon$$

Let $S_0 \subseteq \ldots \subseteq S_m$ be an increasing chain of coalitions so that the variation of $u$ over it, $\sum_{0 \leq j < m} |u(S_{j+1}) - u(S_j)|$, is $\geq \|u\| - \varepsilon$. Let $S(t)$, $0 \leq t \leq 1$, be the increasing path of ideal coalitions defined by $S(t) = S_j + (tm - j)(S_{j+1} - S_j)$ if $j/m \leq t \leq (j+1)/m$. The function $G : [0, 1] \to \mathbb{R}$, defined by $G(t) = (v - u - \sum_{i=1}^{n} f_i \circ \mu_i)(S(t))$, is a sum of an absolutely continuous function, $t \mapsto -u(S(t))$, and a singular function of bounded variation $t \mapsto (v - \sum_{i=1}^{n} f_i \circ \mu_i)(S(t))$. Therefore the variation of the game $v - u - \sum_{i=1}^{n} f_i \circ \mu_i$ over the increasing path $(S(t))_{0 \leq t \leq 1}$ is greater than or equal to the variation of $u$ over this increasing path, which is $\geq \|u\| - \varepsilon$. Therefore

$$\varepsilon > \|v - u - \sum_{i=1}^{n} f_i \circ \mu_i\| \geq \|u\| - \varepsilon$$

which implies that $\|u\| < 2\varepsilon$, and therefore, by the triangle inequality, we have

$$\|v - \sum_{i=1}^{n} f_i \circ \mu_i\| < 3\varepsilon$$

We can assume w.l.o.g. that $\mu_i \neq \mu_j$ for every $1 \le i < j \le n$.

Let $T \in \mathcal{C}$ with $\alpha_i := \mu_i(T) \neq \alpha_j := \mu_j(T)$ for every $i \neq j$. For each fixed $0 < r < 1$, the functions $f_i^r : [0,1] \to \mathbb{R}$ that are defined by $f_i^r(t) = f_i((1-r)t + r\alpha_i)$ are singular. By Corollary 8.10 of Aumann and Shapley (1974) there is $r$ sufficiently small such that for every $j \neq i$ the functions $f_i^r$ and $f_j^r$ are mutually singular when $i \neq j$. If $r$ is sufficiently small so that the variation of $f_i$ on each of the intervals $[0, r]$ and $[1-r, 1]$ is at most $\varepsilon/(2n)$, then the variation of $f_i^r$ is at least $\|f_i\| - \varepsilon/n$. Let $\chi(t) = (1-r)t + rT$, $0 \le t \le 1$, and consider the function $F : [0,1] \to \mathbb{R}$ defined by $F(t) = (v - \sum_{i=1}^{n} f_i \circ \mu_i)(\chi(t))$. Note that $F(t) = v(\chi(t)) - \sum_{i=1}^{n} f_i^r(t)$. As $\chi(t)$, $0 \le t \le 1$, is an increasing path of ideal coalitions, the variation of $F$ over $[0,1]$ is bounded by $\|v - \sum_{i=1}^{n} f_i \circ \mu_i\| < 3\varepsilon$. As $v$ is a simple monotonic game, the function $t \mapsto v(\chi(t))$ is singular to all but possibly one, say $f_1^r$, of the functions $f_i^r$. Therefore $\|F\| \ge \sum_{i=2}^{n} \|f_i^r\| \ge \sum_{i=2}^{n} \|f_i\| - \varepsilon$. Thus $\sum_{i=2}^{n} \|f_i\| < 4\varepsilon$. Therefore, using the triangle inequality, $\|v - f_1 \circ \mu_1\| < 7\varepsilon$. Therefore $v$ is a limit in the bounded variation norm of games of the form $f \circ \mu$ where $f \in s'$ and $\mu \in NA^1$. Consider the function $g(t) = v(tI)$. It follows that $\|f \circ \mu - g \circ \mu\| = \|f - g\| \le \|f \circ \mu - v\|$ and therefore $\|g \circ \mu - v\| \le 2\|f \circ \mu - v\|$ and thus $v$ is a limit in the bounded variation norm of weighted majority games. As the bounded variation distance of any two distinct weighted majority games is at least 1, the result follows. □

## 3. Singular games in *bv′NA*

Aumann and Shapley (1974) prove that the space *bv′NA* is the algebraic sum of two spaces: the space *pNA* and the space *s′NA*. The space *s′NA* is the closed linear space spanned by all games of the form $f \circ \mu$ where $f \in s'$ and $\mu$ is a non-atomic probability measure.

**Theorem 2.** *Every game $v \in s'NA$ is a countable (possibly finite) sum*

$$v = \sum_{i=1}^{\infty} f_i \circ \mu_i$$

*where $f_i \in s'$ with $\sum_{i=1}^{\infty} \|f_i\| < \infty$ and $\mu_i \in NA^1$.*

**Proof.** Let $(\mu_i)_{i=1}^{\infty}$ be a fixed sequence of distinct non-atomic probability measure. Consider the set $X = X(\mu_1, \mu_2, \ldots)$ of all games in $s'NA$ that are a countable sum $\sum_{i=1}^{\infty} f_i \circ \mu_i$ with $f_i \in s'$ and $\sum_{i=1}^{\infty} \|f_i\| < \infty$. Assume that $v = \sum_{i=1}^{\infty} f_i \circ \mu_i \in X$ with $\sum_{i=1}^{\infty} \|f_i\| < \infty$. Then $\sum_{i=1}^{n} f_i \circ \mu_i \to v$ as $n \to \infty$. By Proposition 8.17 of Aumann and Shapley (1974), $\|\sum_{i=1}^{n} f_i \circ \mu_i\| = \sum_{i=1}^{n} \|f_i\|$. Therefore $\|v\| = \sum_{i=1}^{\infty} \|f_i\|$.

We prove next that $X = X(\mu_1, \mu_2, \ldots)$ is a closed subspace of $s'NA$. If $v_k = \sum_{i=1}^{\infty} f_{k,i} \circ \mu_i$ is a Cauchy sequence of games in $X$, $\|v_k - v_m\| \ge \|f_{k,i} - f_{m,i}\|$ for each fixed $1 \le i < \infty$, and thus $(f_{k,i})_{k=1}^{\infty}$ is a Cauchy sequence in $s'$, which is a Banach space, and therefore has a limit $f_i$ in $s'$. Moreover, for every $n$, $\sum_{i=1}^{n} \|f_i\| = \lim_{k\to\infty} \sum_{i=1}^{n} \|f_{k,i}\| \le \sup_{k\ge1} \|v_k\|$ and therefore $v := \sum_{i=1}^{\infty} f_i \circ \mu_i \in X$. As $\|v_k - v\| = \sum_{i=1}^{\infty} \|f_{k,i} - f_i\|$ and $\|v_k - v_m\| = \sum_{i=1}^{\infty} \|f_{k,i} - f_{m,i}\|$ we deduce that $\|v_k - v\| \le \sup_{m\ge k} \|v_k - v_m\| \to_{k\to\infty} 0$ and therefore $v_k \to v$ as $k \to \infty$.

As any sequence $v_k$ of finite linear combinations of singular scalar measure games is in $X(\mu_1, \mu_2, \ldots)$ where $\mu_1, \mu_2, \ldots$ is a sequence of pairwise different non-atomic probability measure containing all non-atomic probability measure appearing in the representations of the $v_k$ s, the result follows. □

## 4. Simple monotonic games in *′AN*

The space *′AN* is introduced in Mertens and Neyman (2003). It is the linear closed span of scalar measure games $f \circ \mu$ where $\mu$ is in $AN^1$, the set of finitely additive non-atomic positive measure, and $f$ (in $'$, namely, $f$ of bounded variation and) obeys a weakened continuity at $0 = \mu(\emptyset)$ and at $\mu(I)$.

**Theorem 3.** *Every simple monotonic game in ′AN is a weighted majority game, and the weighted measure is a non-atomic finitely additive probability measure.*

**Proof.** Assume that $v \in {}'AN$ is a simple monotonic game. As $'AN$ is the closed linear span of all games of the form $f \circ \mu$ where $\mu \in AN^1$ and $f \in {}'$, $v$ can be approximated by linear combinations of the form $\sum_{i=1}^{n} f_i \circ \mu_i$ where $f_i \in {}'$ and $\mu_i \in AN^1$ and w.l.o.g. for every $1 \le i < j \le n$, $\mu_i \neq \mu_j$. As $v$ has bounded variation, the game $\sum_{i=1}^{n} f_i \circ \mu_i$ has bounded variation. Lemma 3 of Mertens and Neyman (2003) implies that each function $f_i$ is a sum of a function $g_i \in s'$ and a function $h_i$ that is continuous on $[0,1]$ and absolutely continuous on its interior. Set $u = \sum_{i=1}^{n} h_i \circ \mu_i$. Let $\chi_0 \le \chi_2 \le \ldots \le \chi_m$ be an increasing chain of ideal coalitions so that the variation of $u$ over this chain is $\ge \|u\| - \varepsilon$. We can assume w.l.o.g. that for some $\delta > 0$ we have $\delta < \chi_0 \le \chi_m < 1 - \delta$. The proof proceeds as in the proof of Theorem 1; the only modification required is that of replacing the increasing path $S(t)$ with the increasing path $\chi(t)$ where $\chi(t) = \chi_j + (tm - j)(\chi_{j+1} - \chi_j)$ if $j/m \le t \le (j+1)/m$ and replacing the non-atomic probability measures $\mu_i$ with finitely additive non-atomic probability measures. This shows that $v$ can be approximated by the sum $\sum_{i=1}^{n} f_i \circ \mu_i$, where $f_i \in s'$ and $\mu_i \in AN^1$. □

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