CHAPTER 15

# Values of smooth nonatomic games: the method of multilinear approximation

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## 1 Introduction

In their book *Values of Non-Atomic Games,* Aumann and Shapley [1] define the value for spaces of nonatomic games as a map from the space of games into bounded finitely additive games that satisfies a list of plausible axioms: linearity, symmetry, positivity, and efficiency. One of the themes of the theory of values is to demonstrate that on given spaces of games this list of plausible axioms determines the value uniquely. One of the spaces of games that have been extensively studied is $p$NA, which is the closure of the linear space generated by the polynomials of nonatomic measures. Theorem B of [1] asserts that a unique value $\phi$ exists on $p$NA and that $\|\phi\| = 1$. This chapter introduces a canonical way to approximate games in $p$NA by games in $p$NA that are "identified" with finite games. These are the multilinear nonatomic games – that is, games $v$ of the form $v = F \circ (\mu_1, \mu_2, \ldots, \mu_n)$, where $F$ is a multilinear function and $\mu_1, \mu_2, \ldots, \mu_n$ are mutually singular nonatomic measures.

The approximation theorem yields short proofs to classic results, such as the uniqueness of the Aumann – Shapley value on $p$NA and the existence of the asymptotic value on $p$NA (see [1, Theorem F]), as well as short proofs for some newer results such as the uniqueness of the $\mu$ value on $p$NA($\mu$) (see [4]). We also demonstrate the usefulness of our method by proving a generalization to $p$NA of Young's characterization [6 and Chapter 17 this volume] of the Shapley value without the linearity axiom, and by generalizing Young's characterization [7] of the Aumann-Shapley price mechanism. In the last chapter we use the ideas behind the multilinear approximation in order to supply an elementary proof to a classic result in analysis: the Weierstrass approximation theorem.

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## 2 Preliminaries

We follow basically the terminology and notations of Aumann and Shapley [1]. Let $(I,\mathscr{C})$ be a fixed standard measurable space (i.e., a measurable space isomorphic to $([0,1],\mathscr{B})$, where $\mathscr{B}$ denotes the Borel field in $[0,1]$). A *game* is a real-valued function $v$ on $\mathscr{C}$ with $v(\emptyset) = 0$. A game $v$ is *monotonic* if $v(S \cup T) \geq v(S)$ for all $S$ and $T$ in $\mathscr{C}$. A game $v$ has a *bounded variation* if it is the difference of two monotonic games. The variation $\|v\|$ of such game is $\|v\| = \inf\{w(I) + u(I)\}$, where the infimum ranges over all monotonic games $u$ and $w$ for which $v = w - u$. The space BV of all games with bounded variation is a Banach algebra (see [1, sect. 4]). The space of all finitely additive games in BV is denoted by FA, and the subspace of all nonatomic measures in FA is denoted by NA. The closed Banach algebra generated by NA is denoted by $p$NA. Equivalently, $p$NA is the closed linear subspace of BV generated by the powers of the nonatomic probability measures. Let $Q$ be a subset of BV. The set of monotonic games in $Q$ is denoted by $Q^+$. A map of $Q$ into BV is called *positive* if it maps $Q^+$ into $\mathrm{BV}^+$. An *automorphism* of $(I,\mathscr{C})$ is a 1-1 map $\theta$ of $I$ onto itself such that for every $S \subseteq I$, $S \in \mathscr{C}$ iff $\theta(S) \in \mathscr{C}$. The group of all automorphisms is denoted by $\mathscr{G}$. Each $\theta$ in $\mathscr{G}$ induces a linear map $\theta_*$ of BV onto itself, defined by $(\theta_* v)(S) = v(\theta S)$. A subset $Q$ of BV is called *symmetric* if $\theta_* Q \subseteq Q$ for all $\theta \in \mathscr{G}$. A map $\phi$ of a symmetric subset $Q$ of BV into BV is called *symmetric* if, for every $\theta \in \mathscr{G}$, $\theta_* \circ \phi = \phi \circ \theta_*$. A map $\phi$ of a subset $Q$ of BV into BV is called *efficient* if, for every $v \in Q$, $(\phi v)(I) = v(I)$. Let $Q$ be a symmetric subspace of BV. A *value* on $Q$ is a linear, positive, efficient, and symmetric map of $Q$ into FA. Let $Q$ be a subspace of BV. $Q$ is an *internal space* if for every $v \in Q$ and every $\epsilon > 0$ there exist $u$, $w \in Q^+$ with $v = w - u$ and $\|v\| \geq w(I) + u(I) - \epsilon$. The importance of internal spaces in the theory of values follows from [1, Propositions 4.7,4.12], where it was proven that every linear, positive, efficient map of an internal subspace of BV into FA is continuous and that the closure of an internal space is internal. These results provide a very efficient tool for deriving uniqueness theorems. One of the fundamental results of the theory of values of nonatomic games [1, Theorem B] is the existence of a unique value on $p$NA. The uniqueness of the value on a dense subspace of $p$NA follows from [1, Proposition 6.1], and the uniqueness on $p$NA is obtained by showing that $p$NA is an internal space.

Let $\Pi$ be a finite subfield of $\mathscr{C}$. The set of all atoms of $\Pi$ is denoted by $\pi$. The power set $2^\pi$ of $\pi$ is identified naturally with $\Pi$, and thus a finite game on the players' set $\pi$ is identified with a function $w:\Pi \to R$, with $w(\emptyset) = 0$. The restriction of $v \in \mathrm{BV}$ to $\Pi$ is denoted by $v_\Pi$. An *admissible* *sequence* of finite fields is an increasing sequence $(\Pi_n)_{n=1}^{\infty}$ of finite subfields of $\mathscr{C}$ such that $\cup_{n=1}^{\infty}\Pi_n$ generates $\mathscr{C}$. For every game $w$ of finitely many players, we denote by $\psi w$ the Shapley value of $w$ considered as a measure on the set of players. A game $v$ has an asymptotic value if there exists a game $\phi v$ such that, for every admissible sequence of finite fields $(\Pi_n)_{n=1}^{\infty}$ and every $S$ in $\Pi_1$, $\lim_{n\to\infty}\psi v_{\Pi_n}(S)$ exists and equals $\phi v(S)$. It follows that $\phi v$ is finitely additive, and it is called the *asymptotic value* of $v$. The set of all games in BV having an asymptotic value is denoted by ASYMP. Aumann and Shapley [1, Theorem F] have shown that ASYMP is a linear, symmetric, closed subspace of BV, and that the operator $\phi$ that associates to each $v$ its asymptotic value is a value on ASYMP with norm 1.

## 3 Multilinear nonatomic games and finite games

For every finite field $\Pi \subset \mathscr{C}$ we denote by $\pi$ the set of atoms of $\Pi$. Let $G(\Pi)$ be the space of all finite games on the players' set $\pi$. The power set $2^{\pi}$ of $\pi$ is naturally identified with $\Pi$. Thus, a game $w$ in $G(\Pi)$ is identified with a function $w:\Pi \to R$ satisfying $w(\varnothing) = 0$. The space of all additive games in $G(\Pi)$ is denoted by $AG(\Pi)$. The set of all monotonic games in a subset $H$ of $G(\Pi)$ is denoted by $H^+$. Let $w \in G(\Pi)$. Define $w^+$ and $w^-$ in $G(\Pi)^+$ such that $w = w^+ - w^-$ in the following way:

$$w^+(S) = \max \sum_{i=1}^{n} \max\{w(S_0 \cup S_1 \cup \cdots \cup S_i) - w(S_0 \cup S_1 \cup \cdots \cup S_{i-1}),0\},$$

where $n$ is the number of elements in $\pi$, $S_0 = \varnothing$, and the outer maximum ranges over all possible orders $S_1, S_2, \ldots, S_n$ of $\{\mathrm{T} \in \mathrm{n}:\mathrm{T} \subseteq \mathrm{S}\}$. The variation norm of $w \in G(\Pi)$ is given by $\|w\| = \inf\{w_2(I) + w_1(I)\}$, where the infimum ranges over all $w_1$, $w_2 \in G(\Pi)^+$ for which $w = w_2 - w_1$. Actually, $\|w\| = w^+(I) + w^-(I)$. Let $T_\Pi$ be the map from the set of all games into $G(\Pi)$ given by $T_\Pi v = v_\Pi$, where $v_\Pi$ is the restriction to $\Pi$ of $v$. Note that $T_\Pi$ is linear, efficient, positive, and $\|T_\Pi v\| \le \|v\|$ for every $v \in \mathrm{BV}$.

A *carrier* of a game $v$ is a set $I'$ in $\mathscr{C}$ such that $v(S) = v(S \cap I')$ for every $S \in \mathscr{C}$. Let $\lambda$ be a probability measure in NA. The set of all games $v$ in $p$NA having the property that every carrier of $\lambda$ is a carrier of $v$ is denoted by $p\mathrm{NA}(\lambda)$. Equivalently, $p\mathrm{NA}(\lambda)$ is the closed algebra generated by $\mathrm{NA}(\lambda)$, where $\mathrm{NA}(\lambda)$ denotes the space of all measures in NA that are absolutely continuous with respect to $\lambda$.

Given a nonatomic probability measure $\lambda$ and a finite field $\Pi \subset \mathscr{C}$ with the set of atoms $\pi = \{S_1, S_2, \ldots, S_n\}$ $(1 \le i < j \le n \Rightarrow S_i \neq S_j)$, denote the restriction of $\lambda$ to $S_i$, $1 \le i \le n$, by $\lambda_{S_i} = \lambda_i$. Let ML($\lambda$,$\Pi$) be the set of all games $v$ having the form $v = F \circ (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $F$ is a multilinear function (see [3, and Owen Chapter 10 this volume]) on $\Pi_{i=1}^n [0, \lambda(S_i)]$ – the range of the vector measure $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Note that ML($\lambda$,$\Pi$) is a linear space of games and that every game $v \in$ ML($\lambda$,$\Pi$) is a polynomial in the $\lambda_i$'s; therefore ML($\lambda$,$\Pi$) $\subset p$NA($\lambda$) $\subset p$NA. Also note that, for every $u$, $v \in$ ML($\lambda$,$\Pi$), $u = v$ iff $T_\Pi u = T_\Pi v$. That is, $T_\Pi$ is 1-1 on ML($\lambda$,$\Pi$). Let $G_\lambda(\Pi)$ be the set of all games $w$ in $G(\Pi)$ having the property that $w(S) = w(T)$ whenever $\lambda(S \,\Delta\, T) = 0$, that is, every atom in $\pi$ that is a null set for $\lambda$ is a null player for $w$. The space of all additive games in $G_\lambda(\Pi)$ is denoted by $AG_\lambda(\Pi)$. For every game $w$ in $G_\lambda(\Pi)$ there exists a unique game $v \in$ ML($\lambda$,$\Pi$) with $T_\Pi v = w$ (just take $F$ to be the multilinear function for which $F(\lambda_1(S), \ldots, \lambda_n(S)) = w(S)$ for all $S \in \Pi$). Thus, the map $T_\Pi$ from ML($\lambda$,$\Pi$) onto $G_\lambda(\Pi)$ has an inverse $T^\lambda: G_\lambda(\Pi) \to$ ML($\lambda$,$\Pi$). Moreover, $v \in$ ML($\lambda$,$\Pi$)$^+$ iff $T_\Pi v \in G_\lambda(\Pi)^+$. Therefore $T^\lambda$ is also positive. Thus, both $T_\Pi$ and $T^\lambda$ are linear, positive, efficient operators. Let $v \in$ ML($\lambda$,$\Pi$) and set $w = T_\Pi v$. Because $w = w^+ - w^-$ with $\|w\| = w^+(I) + w^-(I)$, we have $v = T^\lambda(w^+) - T^\lambda(w^-)$, which implies that

$$\|v\| \le T^\lambda(w^+)(I) + T^\lambda(w^-)(I) = w^+(I) + w^-(I) = \|w\| = \|T_\Pi v\|.$$

Because we also have $\|T_\Pi v\| \le \|v\|$, we get that $T_\Pi$, and therefore, $T^\lambda$, is an isometry. Finally, because we have shown that $\|v\| = T^\lambda(w^+)(I) + T^\lambda(w^-)(I)$, ML($\lambda$,$\Pi$) is an internal space.

Altogether, the following proposition holds.

**Proposition 1.**

(i) ML($\lambda$,$\Pi$) $\subset p$NA($\lambda$).

(ii) $T_\Pi$: ML($\lambda$,$\Pi$) $\to G_\lambda(\Pi)$ is a linear, efficient, positive isometry.

(iii) $T^\lambda: G_\lambda(\Pi) \to$ ML($\lambda$,$\Pi$) is a linear, efficient, positive isometry.

(iv) $T_\Pi \circ T^\lambda$ is the identity map on $G_\lambda(\Pi)$, and $T^\lambda \circ T_\Pi$ is the identity map on ML($\lambda$,$\Pi$).

(v) ML($\lambda$,$\Pi$) is internal. Moreover, every $v \in$ ML($\lambda$,$\Pi$) is the difference of two monotonic games $v_1$ and $v_2$ in ML($\lambda$,$\Pi$) with $\|v\| = v_1(I) + v_2(I)$.

Observe that $T^\lambda$ maps $AG_\lambda(\Pi)$ into NA $\cap$ ML($\lambda$,$\Pi$). Therefore for every function $f$: ML($\lambda$,$\Pi$) $\to$ NA $\cap$ ML($\lambda$,$\Pi$) there exists a unique function $g: G_\lambda(\Pi) \to AG_\lambda(\Pi)$ that makes diagram (D1) commutative:

$$\begin{array}{ccc} \mathrm{ML}(\lambda,\Pi) & \xrightarrow{f} & \mathrm{NA} \cap \mathrm{ML}(\lambda,\Pi) \\ {\scriptstyle T^\lambda}\uparrow & & \downarrow{\scriptstyle T_\Pi} \\ G_\lambda(\Pi) & \xrightarrow{g} & AG_\lambda(\Pi) \end{array} \tag{D1}$$

That is, $g = T_\Pi \circ f \circ T^\lambda$.

By (iv) we also get that for every $g: G_\lambda(\Pi) \to AG_\lambda(\Pi)$ there exists a unique $f: \mathrm{ML}(\lambda,\Pi) \to \mathrm{NA} \cap \mathrm{ML}(\lambda,\Pi)$ that makes (D1) commutative. That is, $f = T^\lambda \circ g \circ T_\Pi$. Alternatively, $f$ is the unique function that makes diagram (D2) commutative:

$$\begin{array}{ccc} \mathrm{ML}(\lambda,\Pi) & \xrightarrow{f} & \mathrm{NA} \cap \mathrm{ML}(\lambda,\Pi) \\ {\scriptstyle T_\Pi}\downarrow & & \uparrow{\scriptstyle T^\lambda} \\ G_\lambda(\Pi) & \xrightarrow{g} & AG_\lambda(\Pi) \end{array} \tag{D2}$$

Let $\phi$ denote the Aumann–Shapley value on $p$NA, and let $\psi$ denote the Shapley value for finite games. It is easy to verify that $\phi$ maps $\mathrm{ML}(\lambda,\Pi)$ into $\mathrm{NA} \cap \mathrm{ML}(\lambda,\Pi)$, that $\psi$ maps $G_\lambda(\Pi)$ into $AG_\lambda(\Pi)$, and that the diagrams (D1) and (D2) are commutative with $f = \phi$ and $g = \psi$ (e.g., use [5] and [1, note 1 on p. 166]). This gives us a very efficient tool to derive uniqueness theorems.

**Proposition 2.** Let $\hat{\phi}: p\mathrm{NA} \to \mathrm{FA}$ be a function that maps $\mathrm{ML}(\lambda,\Pi)$ into $\mathrm{NA} \cap \mathrm{ML}(\lambda,\Pi)$, where $\lambda$ is a nonatomic probability measure and $\Pi \subset \mathscr{C}$ is a finite field. Define $\hat{\psi}: G_\lambda(\Pi) \to AG_\lambda(\Pi)$ such that (D1) will be commutative with $f = \hat{\phi}$ and $g = \hat{\psi}$. Then if $\hat{\psi}$ is the Shapley value on $G_\lambda(\Pi)$, $\hat{\phi}$ equals the Aumann–Shapley value on $\mathrm{ML}(\lambda,\Pi)$.

Given any finite field $\Pi \subset \mathscr{C}$ and a nonatomic probability measure $\lambda$, we denote by $E_\Pi^\lambda$ the map from $p\mathrm{NA}(\lambda)$ onto $\mathrm{ML}(\lambda,\Pi)$ given by $E_\Pi^\lambda v = T^\lambda(T_\Pi v)$. Note that $E_\Pi^\lambda$ is a linear, efficient, positive projection onto $\mathrm{ML}(\lambda,\Pi)$, and $\|E_\Pi^\lambda\| = 1$. In particular, diagram (D3) (in which $i$ denotes the identity map) is commutative:

$$\begin{array}{ccc} p\mathrm{NA}(\lambda) & \xrightarrow{T_\Pi} & G_\lambda(\Pi) \\ {\scriptstyle E_\Pi^\lambda}\downarrow & & \downarrow{\scriptstyle T^\lambda} \\ \mathrm{ML}(\lambda,\Pi) & \xrightarrow{i} & \mathrm{ML}(\lambda,\Pi) \end{array} \tag{D3}$$

## 4 The approximation theorem

We start with an inequality that holds in any normed algebra $(X, \|\ \|)$; that is, $\|xy\| \le \|x\| \, \|y\|$ for every $x, y, \in X$. Let $x_1, x_2, \ldots, x_n$ be fixed elements in $X$. For every $J = (j_1, j_2, \ldots, j_k) \in \{1,2, \ldots, n\}^k$ denote the product $\Pi_{i=1}^k x_{j_i} = x_{j_1} x_{j_2}, \ldots, x_{j_k}$ by $x(J)$. It follows that

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{J \in \{1,2, \ldots, n\}^k} x(J).$$

Let $D$ be the set of all $J \in \{1,2, \ldots, n\}^k$ such that for every $1 \le i < m \le k, j_i \ne j_m$, and let $B$ be the set of all $J \in \{1,2, \ldots, n\}^k$ that are not in $D$.

**Lemma 3.** For every $x_1, x_2, \ldots, x_n$ in a normed algebra and for every integer $k > 1$ the following holds:

$$\left\| \sum_{J \in B} x(J) \right\| = \left\| \left( \sum_{i=1}^n x_i \right)^k - \sum_{J \in D} x(J) \right\|$$
$$\le \binom{k}{2} \left( \max_{i=1, \ldots, n} \|x_i\| \right) \left( \sum_{i=1}^n \|x_i\| \right)^{k-1}.$$

*Proof:* For every $1 \le l \le n$ let

$$B_l = \{J \in B : \exists\ 1 \le i < m \le k \text{ s.t. } j_i = j_m = l\}.$$

Then $B = \cup_{l=1}^n B_l$. For every $1 \le i < m \le k$ let $B_l^{i,m} = \{J \in B_l : j_i = j_m = l\}$. Then

$$\left\| \sum_{J \in B_l^{i,m}} x(J) \right\| \le \|x_l\|^2 \left\| \sum_{i=1}^n x_i \right\|^{k-2}.$$

Therefore,

$$\left\| \sum_{J \in B_l} x(J) \right\| \le \binom{k}{2} \|x_l\|^2 \left\| \sum_{i=1}^n x_i \right\|^{k-2}.$$

Hence,

$$\left\| \sum_{J \in B} x(J) \right\| \le \sum_{l=1}^n \left\| \sum_{J \in B_l} x(J) \right\| \le \binom{k}{2} \left\| \sum_{i=1}^n x_i \right\|^{k-2} \sum_{l=1}^n \|x_l\|^2.$$

Because

$$\sum_{i=1}^{n} \|x_i\|^2 \leq (\max_{i=1,\ldots,n} \|x_i\|) \sum_{i=1}^{n} \|x_i\| \quad \text{and} \quad \left\|\sum_{i=1}^{n} x_i\right\|^{k-2} \leq \left(\sum_{i=1}^{n} \|x_i\|\right)^{k-2}$$

The result follows. ■

**Theorem 4.** Let $(\Pi_n)_{n=1}^{\infty}$ be an admissible sequence of finite fields. Then for every $v \in p\mathrm{NA}(\lambda)$, $\lim_{n\to\infty} E_{\Pi_n}^{\lambda} v = v$ (in the bounded variation norm).

*Proof:* Because all operators $E_{\Pi_n}^{\lambda}$ are linear with norm 1, the space $Q$ consisting of all games $v \in p\mathrm{NA}(\lambda)$ for which $E_{\Pi_n}^{\lambda} v \xrightarrow[n\to\infty]{} v$ is a closed subspace of $p\mathrm{NA}(\lambda)$. Let $\gamma$ be a probability measure in $\mathrm{ML}(\lambda,\Pi_m)$. We will show that for every $k \geq 1$, $\gamma^k \in Q$.

Because $\mathrm{ML}(\lambda,\Pi_m) \subseteq \mathrm{ML}(\lambda,\Pi_n)$ for every $n > m$, $\gamma \in \mathrm{ML}(\lambda,\Pi_n)$ for every $n > m$, so $E_{\Pi_n}^{\lambda}\gamma = \gamma$ for every $n > m$, which proves our claim for $k = 1$. As for $k > 1$, for every $n > m$ and every $J = (A_1,A_2, \ldots ,A_k) \in \pi_n^k$ let $\gamma(J) = \Pi_{i=1}^{k}\gamma_{A_i}$, where $\gamma_A$ denotes the restriction of $\gamma$ to $A$. Obviously, $\gamma(J) \in \mathrm{ML}(\lambda,\Pi_n)$ whenever $A_i \neq A_j$ for every $1 \leq i < j \leq k$. Because $\gamma = \Sigma_{A\in\pi_n}\gamma_A$, we can use Lemma 3 to prove the existence of $u_n \in \mathrm{ML}(\lambda,\Pi_n)$ such that

$$\|\gamma^k - u_n\| \leq \binom{k}{2} (\max_{A\in\pi_n} \|\gamma_A\|) \left(\sum_{A\in\pi_n} \|\gamma_A\|\right)^{k-1} = \binom{k}{2} \max_{A\in\pi_n} \gamma(A) \xrightarrow[n\to\infty]{} 0,$$

because $\gamma$ is a nonatomic probability measure on a standard measurable space.

Obviously $u_n \in Q$ for every $n > m$ (because $E_{\Pi_p}^{\lambda} u_n = u_n$ for every $p \geq n$). Therefore $\gamma^k \in Q$ for every $k \geq 1$.

Finally, since $\cup_{n=1}^{\infty}\Pi_n$ is dense in $\mathscr{C}$, then $\mathrm{NA} \cap (\cup_{n=1}^{\infty}\mathrm{ML}(\lambda,\Pi_n))$ is dense in the space of all measures which are absolutely continuous w.r.t. $\lambda$, and since $p\mathrm{NA}(\lambda)$ is a Banach algebra it is the closed algebra generated by $\mathrm{NA} \cap (\cup_{n=1}^{\infty}\mathrm{ML}(\lambda,\Pi_n))$. This completes the proof. ■

## 5 Internality

Let $\lambda$ be a probability measure in NA, and let $(\Pi_n)_{n=1}^{\infty}$ be an admissible sequence of finite fields. By (v) $\mathrm{ML}(\lambda,\Pi_n)$ is internal for every $n \geq 1$. Because $\mathrm{ML}(\lambda,\Pi_n) \supseteq \mathrm{ML}(\lambda,\Pi_m)$ for every $n > m$, $\cup_{n=1}^{\infty}\mathrm{ML}(\lambda,\Pi)_n$ is a linear space; and because the union of internal spaces is an internal space (if it is linear), $\cup_{n=1}^{\infty}\mathrm{ML}(\lambda,\Pi_n)$ is an internal space [moreover, for every $v$ in $\cup_{n=1}^{\infty}\mathrm{ML}(\lambda,\Pi_n)$ there exist $u,\ w \in (\cup_{n=1}^{\infty}\mathrm{ML}(\lambda,\Pi_n))^+$ such that $v = w - u$ and $\|v\| = w(I) + u(I)$]. Theorem 4 implies that $p\mathrm{NA}(\lambda)$ is the closure of $\cup_{n=1}^{\infty}\mathrm{ML}(\lambda,\Pi_n)$, and therefore $p\mathrm{NA}(\lambda)$ is internal by [1, Proposition 4.12]. Alternatively, for every $v \in p\mathrm{NA}(\lambda)$ and every $\epsilon > 0$ construct a sequence of games $(v_k)_{k=1}^{\infty}$ in $\mathrm{ML}(\lambda,\Pi_{n_k})$ (where $(n_k)$ is an increasing sequence) with $\|v - v_k\| < \epsilon 2^{-k-1}$ for every $k \geq 1$. Let $v_0 = 0$. For every $k \geq 1$ let $v_k - v_{k-1} = w_k - u_k$, where $u_k$ and $w_k$ are monotonic games in $\mathrm{ML}(\lambda,\Pi_{n_k})$ with $\|v_k - v_{k-1}\| = w_k(I) + u_k(I)$. Let $w = \Sigma_{k=1}^{\infty} w_k$ and $u = \Sigma_{k=1}^{\infty} u_k$. Then $u,\ w \in p\mathrm{NA}(\lambda)^+$ (because for every $k > 1$, $u_k(I) + w_k(I) < \epsilon 2^{-k} + \epsilon 2^{-(k+1)}$), $v = w - u$, and

$$\begin{aligned} u(I) + w(I) &= \sum_{k=1}^{\infty} (u_k(I) + w_k(I)) \\ &= u_1(I) + w_1(I) + \sum_{k=2}^{\infty} (u_k(I) + w_k(I)) \leq \|v\| + \epsilon. \end{aligned}$$

Because $p\mathrm{NA} = \cup_{\lambda} p\mathrm{NA}(\lambda)$, it is also internal.

## 6 The uniqueness of the value

The uniqueness of the Aumann–Shapley value on $p\mathrm{NA}$ is mainly the consequence of the internality of $p\mathrm{NA}$ that was proved in the previous section. However, the uniqueness of the $\lambda$ value on $p\mathrm{NA}(\lambda)$, which cannot be based on internality observations alone, can also be derived by the methods developed here.

Let $\lambda$ be a nonatomic probability measure. A set $Q$ of games is $\lambda$-symmetric if $\theta_* Q \subseteq Q$ for every automorphism $\theta$ with $\theta_* \lambda = \lambda$. Let $Q$ be a $\lambda$-symmetric set of games. A map $\phi$ from $Q$ into BV is $\lambda$-symmetric if $\theta_* \circ \phi = \phi \circ \theta_*$ for every automorphism $\theta$ that preserves $\lambda$. It satisfies the *dummy axiom* if $\phi v(S^c) = 0$ whenever $S$ is a carrier of $v$.

**Theorem (Monderer [4]).** There exists a unique linear $\lambda$-symmetric positive efficient operator $\phi : p\mathrm{NA}(\lambda) \to \mathrm{FA}$ that satisfies the dummy axiom. It is the restriction to $p\mathrm{NA}(\lambda)$ of the Aumann–Shapley value $\phi$ on $p\mathrm{NA}$.

*Proof:* Let $\Pi \subset \mathscr{C}$ be a finite field with $\lambda(A) = \lambda(B) = 1/\#\pi$ for every two atoms $A,\ B \in \pi$. Note that for every $v \in \mathrm{ML}(\lambda,\Pi)$, $\theta_* v = v$ for every automorphism $\theta$ of $(I,\mathscr{C})$ that preserves the measure $\lambda$ and satisfies $\theta S = S$ for every $S \in \pi$. Therefore, by a slight change in the proof of [1, Proposition 6.1] (using the $\lambda$-symmetry of $\hat{\phi}$), we get that, for every $v \in \mathrm{ML}(\lambda,\Pi)$, $\hat{\phi}v$ is a linear combination of the measures $\lambda_A$, $A \in \pi$. Hence, $\hat{\phi}v \in \mathrm{NA} \cap \mathrm{ML}(\lambda,\Pi)$. Let $\hat{\psi}: G_\lambda(\Pi) \to AG_\lambda(\Pi)$ be the function that makes the diagram D4 commutative:

$$\begin{array}{ccc} \mathrm{ML}(\lambda,\Pi) & \xrightarrow{\hat{\phi}} & \mathrm{NA} \cap \mathrm{ML}(\lambda,\Pi) \\ {\scriptstyle T^\lambda}\uparrow & & \downarrow{\scriptstyle T_\Pi} \\ G_\lambda(\Pi) & \xrightarrow{\hat{\psi}} & AG_\lambda(\Pi) \end{array} \tag{D4}$$

Note that $\hat{\psi}$ satisfies the Shapley value axioms on $G_\lambda(\Pi)$, so, by Proposition 2, $\hat{\phi} = \phi$ on $\mathrm{ML}(\lambda,\Pi)$. Therefore, $\hat{\phi} = \phi$ on $\cup_{n=1}^{\infty}\mathrm{ML}(\lambda,\Pi_n)$, where $(\Pi_n)_{n=1}^{\infty}$ is any admissible sequence of finite fields with $\lambda(A) = 1/\#\pi_n$ for every atom $A$ of $\Pi_n$. That is, $\hat{\phi}$ and $\phi$ coincide on a dense subset of $p\mathrm{NA}(\lambda)$. Because $p\mathrm{NA}(\lambda)$ is internal and $\hat{\phi}$ is linear and positive, $\hat{\phi}$ is continuous and thus coincides with $\phi$ on all of $p\mathrm{NA}(\lambda)$. ■

## 7 The asymptotic value on *p*NA

In this section we will show that the asymptotic value exists in $p$NA. We will show that the asymptotic value of each $v \in p\mathrm{NA}$ equals its Aumann–Shapley value. Let $v \in p\mathrm{NA}$, let $(\Pi_n)_{n=1}^{\infty}$ be an admissible sequence of finite fields, and let $T \in \Pi_1$. We have to show that

$$\lim_{m\to\infty} (\psi v_{\Pi_m}(T) - \phi v(T)) = 0.$$

Note that there exists a probability measure $\lambda \in \mathrm{NA}$ for which $v \in p\mathrm{NA}(\lambda)$. Also note that for every $u \in \mathrm{ML}(\lambda,\Pi_n)$, $\psi u_{\Pi_n}(S) = \phi u(S)$ for every $S \in \Pi_n$. In particular, $\psi u_{\Pi_n}(T) = \phi u(T)$. Finally, recall that $E^{\lambda}_{\Pi_m}$ maps $p\mathrm{NA}(\lambda)$ into $\mathrm{ML}(\lambda,\Pi_m)$. Hence,

$$|\psi v_{\Pi_m}(T) - \phi v(T)| \le |\psi v_{\Pi_m}(T) - \psi(E^{\lambda}_{\Pi_m}v)_{\Pi_m}(T)| + |\phi(E^{\lambda}_{\Pi_m}v)(T) - \phi v(T)|.$$

Because for every $w \in \mathrm{BV}$, $\|w_{\Pi_m}\| \le \|w\|$ and $\|\psi\| = \|\phi\| = 1$, we deduce from the approximation theorem that

$$|\psi v_{\Pi_m}(T) - \phi v(T)| \le 2\|v - E^{\lambda}_{\Pi_m}v\| \xrightarrow[m\to\infty]{} 0.$$

## 8 Characterization of the Aumann–Shapley value without the linearity axiom

The Aumann–Shapley value $\phi$ on $p$NA is the unique linear, positive, efficient, and symmetric map from $p$NA to FA. It turns out that the value satisfies additional desirable properties like continuity, $\phi\mu = \mu$ for every $\mu \in$ NA, and stronger versions of positivity. One of the stronger versions of positivity will be called strong positivity: Let $Q \subseteq$ BV. A map $\bar{\phi}: Q \to$ FA is *strongly positive* if $\bar{\phi}v(S) \geq \bar{\phi}u(S)$ whenever $u, v \in Q$ and

$$v(T \cup S') - v(T) \geq u(T \cup S') - u(T)$$

for every $S' \subseteq S$ and $T$ in $\mathscr{C}$. Strong positivity is a desirable property for values, and it is satisfied by the value on $p$NA as well as by any other known nonpathological value. The following theorem asserts that strong positivity and continuity can replace positivity and linearity in the characterization of the value on $p$NA.

**Theorem 5.** Any symmetric, efficient, strongly positive, and continuous map from $p$NA to FA is the Aumann–Shapley value.

*Proof:* Let $\hat{\phi}: p\text{NA} \to$ FA be a map satisfying the conditions of the theorem. Denote by $\phi$ the Aumann–Shapley value on $p$NA. In order to prove that $\hat{\phi} = \phi$ it suffices to prove that they coincide on $p\text{NA}(\lambda)$ for every probability measure $\lambda \in$ NA. Let then $\lambda$ be a fixed arbitrary measure in NA. Because $\hat{\phi}$ and $\phi$ are continuous, it suffices to prove that $\hat{\phi} = \phi$ on $\text{ML}(\lambda,\Pi)$ for every finite field $\Pi \subset \mathscr{C}$. Because $\hat{\phi}$ is symmetric, it maps $\text{ML}(\lambda,\Pi)$ into $\text{NA} \cap \text{ML}(\lambda,\Pi)$ (we have already proved a similar claim in the proof of the theorem in Section 6). Let $\hat{\psi}: G_\lambda(\Pi) \to AG_\lambda(\Pi)$ be the function that makes diagram (D4) commutative. Obviously, $\hat{\psi}$ is a symmetric and efficient map. We will show that

$$\hat{\psi}u(A) \geq \hat{\psi}w(A) \tag{5.1}$$

for every atom $A$ of $\Pi$ for which

$$u(S \cup A) - u(S) \geq w(S \cup A) - w(S) \tag{5.2}$$

for every $S \in \Pi$. Therefore, $\hat{\psi}$ is strongly positive, and by Young's characterization [6, Theorem 2] it is the Shapley value $\psi$. By Theorem 2 we get that $\hat{\phi} = \phi$ on $\text{ML}(\lambda,\Pi)$.

Let then $u$, $w \in \mathrm{ML}(\lambda,\Pi)$ satisfy (5.2). Note that for every $v \in G_\lambda(\Pi)$ and every $R \in \mathscr{C}$,

$$T^\lambda v(R) = \sum_{S\in\Pi} v(S)\left[\prod_{B\in\pi;B\subseteq S} \lambda^B(R)\right]\left[\prod_{B\in\pi;B\subseteq S^c} (1-\lambda^B(R))\right],$$

where $\lambda^B(R) = \lambda(R \cap B)/\lambda(B)$ if $\lambda(B) > 0$, and $\lambda^B(R) = 0$ if $\lambda(B) = 0$. Therefore, for every $A' \in \mathscr{C}$, $A' \subseteq A$,

$$T^\lambda v(R \cup A') - T^\lambda v(R) = (\lambda^A(R \cup A') - \lambda^A(R)) \sum_{T\in\Pi;A\subseteq T^c}\left[\prod_{B\in\pi,B\subseteq T} \kappa^B(R)\right]$$
$$\cdot\left[\prod_{B\in\pi,\,B\subseteq T^c;B\neq A} (1-\lambda^B(R))\right](v(T\cup A) - v(T)).$$

Thus (2) implies that

$$T^\lambda u(R \cup A') - T^\lambda u(R) \geq T^\lambda w(R \cup A') - T^\lambda w(R)$$

for every $A' \subseteq A$ and $R$ in $\mathscr{C}$.

Hence, by the strong positivity of $\hat{\phi}$, we have

$$\hat{\psi}u(A) = \hat{\phi}(T^\lambda u)(A) \geq \hat{\phi}(T^\lambda w)(A) = \hat{\psi}w(A). \quad \blacksquare$$

**Corollary 6.** Any $\lambda$-symmetric, efficient, strongly positive, continuous map from $p\mathrm{NA}(\lambda)$ to FA is the Aumann–Shapley value.

*Proof:* Let $\hat{\phi}: p\mathrm{NA}(\lambda) \to \mathrm{FA}$ be a map satisfying the conditions of the theorem. As in the proof Theorem 5, one can show that $\hat{\phi} = \phi$ on every $\mathrm{ML}(\lambda,\Pi)$ for which $\lambda(A) = 1/|\pi|$ for every atom $A \in \pi$.

Because for every probability measure $\lambda \in \mathrm{NA}^+$ there exists an admissible sequence of finite fields, each of them has the above property, the proof of the theorem follows from the continuity of $\hat{\phi}$. $\blacksquare$

## 9 Characterizations of the value on $p\mathrm{NA}\infty$

For every two games $u$ and $v$, we say that $v \succeq u$ if $v - u$ is a monotonic game. Note that $v \succeq u$ iff $v(T \cup S) - v(T) \geq u(T \cup S) - u(T)$ for every $S, T \in \mathscr{C}$. Also note that for any two additive games $\mu$ and $\gamma$, $\mu \succeq \gamma$ iff $\mu \geq \gamma$. The set of all games $v$ for which there exists $\mu \in \mathrm{NA}^+$ such that $-\mu \preceq v \preceq \mu$ is denoted by $\mathrm{AC}_\infty$. $\mathrm{AC}_\infty$ is a linear symmetric subspace of BV that contains NA. For every $v \in \mathrm{AC}_\infty$ let

$$\|v\|_\infty = \inf\{\mu(I): \mu \in \mathrm{NA}^+, -\mu \preceq v \preceq \mu\} + \max\{|v(S)|: S \in \mathscr{C}\}.$$

It can be easily verified that $(\mathrm{AC}_\infty, \| \|_\infty)$ is a Banach algebra and that $\| \| \leq \| \|_\infty$ on $\mathrm{AC}_\infty$. Also, for $\mu \in \mathrm{NA}$, $\|\mu\|_\infty \leq 2\|\mu\|$, which implies that the two norms are equivalent on NA.

**Lemma.** Let $Q \supseteq \mathrm{NA}$ be a subspace of $\mathrm{AC}_\infty$ and let $\overline{\phi}: Q \to \mathrm{FA}$ be a positive map satisfying $\overline{\phi}(v+\mu) = \overline{\phi}(v) + \mu$ for every $u \in Q$ and every $\mu \in \mathrm{NA}$. Then $\|\overline{\phi}v - \overline{\phi}u\| \leq \|v-u\|_\infty$ for every $u, v \in Q$.

*Proof:* For every $\mu \in \mathrm{NA}^+$ for which $-\mu \leq v - u \leq \mu$, we have $u - \mu \leq v \leq u + \mu$. Therefore, by the properties of $\overline{\phi}$, $\overline{\phi}u - \mu \leq \overline{\phi}v \leq \overline{\phi}u + \mu$. Hence, $-\mu \leq \overline{\phi}v - \overline{\phi}u \leq \mu$, which implies that $\|\overline{\phi}v - \overline{\phi}u\| \leq \mu(I)$. Because the last inequality holds for every $\mu$ for which $-\mu \leq v - u \leq \mu$, we get $\|\overline{\phi}v - \overline{\phi}u\| \leq \|v-u\|_\infty$. ∎

Let $p\mathrm{NA}_\infty$ be the $\| \|_\infty$-closed algebra generated by NA. $p\mathrm{NA}_\infty$ is a $\| \|$-dense subspace of $p\mathrm{NA}$ that contains NA as well as any game $v = F \circ (\mu_1, \ldots, \mu_n)$, where $\mu_i \in \mathrm{NA}$ and $F$ is continuously differentiable on the range of $(\mu_1, \ldots, \mu_n)$.

**Theorem 7.** There exists a unique value on $p\mathrm{NA}_\infty$. It is the restriction to $p\mathrm{NA}_\infty$ of the Aumann–Shapley value on $p\mathrm{NA}$.

*Proof:* Let $\hat{\phi}$ be a value on $p\mathrm{NA}_\infty$. Because $\hat{\phi}$ is symmetric and efficient, $\hat{\phi}\mu = \mu$ for every $\mu \in \mathrm{NA}$. Because $\hat{\phi}$ is linear, the previous lemma implies that it is $\| \|_\infty$-continuous. By [1, Note 2 on p. 54] $\hat{\phi}$ coincides with the Aumann–Shapley value on the algebra generated by NA. Because $\hat{\phi}$ and the Aumann–Shapley value are $\| \|_\infty$-continuous, they coincide on all of $p\mathrm{NA}_\infty$. ∎

We now turn to characterize the value on $p\mathrm{NA}_\infty$ without the linearity and positivity axioms. Both axioms will be replaced by the strong positivity axiom. We will not need any continuity assumption (compare with the analogous result in the previous chapter). Note that because Lemma 3 holds in any normed algebra, we can mimic the proof of Theorem 4 to get $\lim_{n\to\infty} E^{\lambda}_{\Pi_n} v = v$ (in the $\| \|_\infty$ norm) for every probability measure $\lambda \in \mathrm{NA}$, every admissible sequence of finite fields, and every $v \in p\mathrm{NA}_\infty(\lambda)$, where $p\mathrm{NA}_\infty(\lambda)$ is the $\| \|_\infty$-closed algebra generated by $\mathrm{NA}(\lambda)$. Therefore if $\hat{\phi}$: $p\mathrm{NA}_\infty \to \mathrm{FA}$ is a symmetric, efficient, strongly positive map that satisfies $\hat{\phi}(v+\mu) = \hat{\phi}(v) + \hat{\phi}(\mu)$ for every $v \in p\mathrm{NA}_\infty$ and for every $\mu \in \mathrm{NA}$, then it coincides with $\phi$ (the Aumann–Shapley value) on a $\| \|_\infty$-dense subspace of $p\mathrm{NA}_\infty$, and it is $\| \|_\infty$-continuous (by the lemma in this chapter); therefore, it coincides with $\phi$ on $p\mathrm{NA}_\infty$. As claimed, we have a stronger result.

**Theorem 8.** Any symmetric, efficient, strongly positive map from $p\mathrm{NA}_\infty$ to FA is the Aumann–Shapley value.

*Proof:* Let $\hat{\phi}: p\mathrm{NA}_\infty \to \mathrm{FA}$ be a symmetric, efficient, strongly positive map. As we have already mentioned, it suffices to prove that $\hat{\phi}(v+\mu) = \hat{\phi}(v) + \mu$ for all $v \in p\mathrm{NA}_\infty$ and for all $\mu \in \mathrm{NA}$.

Let $v \in p\mathrm{NA}_\infty$. The map $G(\mu) = \hat{\phi}(v+\mu) - \hat{\phi}(v)$ is a strongly positive map of NA into FA. Therefore, $G(\mu)(S) = G(\mu_S)(S)$ for all $S \in \mathscr{C}$. As $(\mu_S)_{S^c} = 0_{S^c}$ and $G(0) = 0$, $G(\mu_S)(S^c) = G(0)(S^c) = 0$. Hence, by the efficiency axiom $G(\mu_S(S)) = \mu(S)$. Therefore $\hat{\phi}(v+\mu) = \hat{\phi}(v) + \mu$. ■

## 10 Application to cost allocation

A *cost problem* is a pair $(f,a)$, where $a \in R^n_{++}$ and $f$ is a real valued function on $D_a = \{x \in R^n_+ : 0 \le x \le a\}$ with $f(0) = 0$. For each $x \in D_a$ $f(x)$ is interpreted as the cost of producing the bundle $x = (x_1, \ldots, x_n)$ of commodities or services and $a$ is interpreted as the vector of quantities actually produced.

Let $n \ge 1$. The set of all cost problems $(f,a)$ for which $a \in R^n_{++}$ and $f$ is continuously differentiable on $D_a$ is denoted by $\mathscr{F}_n$. Let $\mathscr{F} = \cup_{n=1}^\infty \mathscr{F}_n$. A *price mechanism* is a function $\psi: \mathscr{F} \to \cup_{n=1}^\infty R^n$ such that $\psi(f,a) \in R^n$ for every $(f,a) \in \mathscr{F}_n$. The $i$-th coordinate of $\psi(f,a)$ will be denoted by $\psi_i(f,a)$.

A price mechanism $\psi$ is *cost sharing* if

$$\sum_{i=1}^n \psi_i(f,a)a_i = f(a). \tag{1}$$

for every $n \ge 1$ and for every $(f,a) \in \mathscr{F}_n$.

Let $m \ge n \ge 1$ and let $\pi = (S_1, S_2, \ldots, S_n)$ be an ordered partition of $\{1,2, \ldots, m\}$. We define $\pi^*: R^m \to R^n$ by $\pi^*_i(x) = \Sigma_{j\in S_i} x_j$ for every $x \in R^m$ and for every $1 \le i \le n$.

A price mechanism $\psi$ is *consistent* if for every $m \ge n \ge 1$, for every $b \in R^m_{++}$, for every ordered partition $\pi = (S_1, S_2, \ldots, S_n)$ of $\{1,2, \ldots, m\}$ and for every $(f, \pi^*(b)) \in \mathscr{F}_n$

$$\psi_i(f \circ \pi^*, b) = \psi_j(f, \pi^*(b)) \tag{2}$$

for every $1 \le j \le n$ and for every $i \in S_j$.

For each $x$, $y \in R^n$ denote $x * y = (x_1y_1, \ldots, x_ny_n)$ and for each $\lambda \in R^n_{++}$ denote $\lambda^{-1} = (1/\lambda_1, \ldots, 1/\lambda_n)$. Also, for each function $f$ on $R^n$ define $(\lambda * f)(x) = f(\lambda * x)$ for all $x \in R^n$.

A price mechanism $\psi$ is *rescaling invariant* if

$$\psi(\lambda * f, \lambda^{-1} * a) = \lambda * \psi(f,a) \tag{3}$$

for all $\lambda$, $a \in R^n_{++}$ and $(f,a) \in \mathscr{F}_n$.

A price mechanism $\psi$ is *strongly monotone* if

$$\frac{\partial f}{\partial x_i}(x) \geq \frac{\partial g}{\partial x_i}(x) \;\; \forall x \in D_a \Rightarrow \psi_i(f,a) \geq \psi_i(g,a) \tag{4}$$

for every $n \geq 1$, for every $(f,a)$, $(g,a) \in \mathscr{F}_n$ and for every $1 \leq i \leq n$.

The Aumann-Shapley price mechanism $\phi$ was defined in [2] and [3] by:

$$\phi_i(f,a) = \int_0^1 \frac{\partial f}{\partial x_i}(ta)dt$$

for every $n \geq 1$, for every $(f,a) \in \mathscr{F}_n$ and for every $1 \leq i \leq n$.

**Theorem 9.** There exists a unique price mechanism on $\mathscr{F}$ which is cost sharing, consistent, rescaling invariant and strongly monotone. It is the Aumann–Shapley price mechanism. ■

Theorem 9 is a generalization of a result of Young [7], who proved that the Aumann–Shapley price mechanism is the unique price mechanism which is cost sharing, strongly monotone and aggregation invariant, where aggregation invariance means that the price mechanism is covariant under linear transformations. The following example shows that consistency and rescaling invariance are weaker than aggregation invariance even in the presence of the cost sharing axiom.

**Example 10.** For every $n \geq 1$, for every $a \in R^n_{++}$, for every $(f,a) \in \mathscr{F}_n$, and for every $1 \leq j \leq n$ denote $s^a_j(f) = \max_{x \in D_a} \partial f/\partial x_j(x)$. Define

$$\psi_i(f,a) = s^a_i(f)f(a) \Big/ \sum_{j=1}^n a_j s^a_j(f)$$

whenever $\Sigma^n_{j=1} a_j s^a_j(f) \neq 0$, and

$$\psi_i(f,a) = \phi_i(f,a)$$

otherwise.

It is easily verified that $\psi$ is cost sharing, consistent, rescaling invariant and not aggregation invariant.

*Proof of Theorem 9:* We start with some definitions. Let $n \geq 1$. The space of all games on $\{1, \ldots, n\}$ will be denoted by $G(n)$. Let $a \in R^n_{++}$. The set of all $(f,a)$ in $\mathscr{F}$ will be denoted by $\mathscr{F}(a)$. The set of all $(f,a) \in \mathscr{F}(a)$ for which $f$ is a multilinear function on $D_a$ will be denoted by $\mathrm{ML}(a)$. Let $T^a: \mathscr{F}(a) \to G(n)$ be defined as follows: For every $S \subseteq \{1, \ldots, n\}$.

$$T^a(f,a)(S) = f(1_S * a),$$

where $(1_S)_i = 1\ \forall i \in S$ and $(1_S)_i = 0\ \forall i \notin S$.

It is easily verified that the restriction to $\mathrm{ML}(a)$ of $T^a$ is a $1-1$ function onto $G(n)$. Let $T^{-a}: G(n) \to \mathrm{ML}(a)$ be the inverse function of $T^a_{|\mathrm{ML}(a)}$. We now turn to the proof.

Let $\psi$ be a cost sharing, consistent, rescaling invariant, strongly monotone price mechanism on $\mathscr{F}$. For each $n \geq 1$ and for each $a \in R^n_{++}$ define $\psi^a: G(n) \to R^n$ such that the following diagram will be commutative:

$$\begin{array}{ccc} \mathrm{ML}(a) & \xrightarrow{a * \psi} & R^n \\ {\scriptstyle T^{-a}}\uparrow & & \downarrow{\scriptstyle i} \\ G(n) & \xrightarrow{\psi^a} & R^n, \end{array}$$

where $i$ is the identity map of $R^n$, and $(a * \psi)(f,a) = a * (\psi(f,a))$.

$\psi^a$ is efficient, symmetric, and strongly positive since $\psi$ is cost sharing, consistent and rescaling invariant, and strongly monotone respectively. Therefore, by [6], $\psi^a$ is the Shapley value. Hence by [5] $\psi$ coincides with the Aumann–Shapley price mechanism $\phi$ on $\mathrm{ML}(a)$. Thus we have proven that $\psi$ and $\phi$ coincide on $\cup_{n \geq 1, a \in R^n_{++}} \mathrm{ML}(a)$.

Let then $n \geq 1$ and let $a \in R^n_{++}$. We show that $\psi = \phi$ on $\mathscr{F}(a)$. For every $k \geq 1$ let $\pi(k) = (S_1, \ldots, S_n)$ be the ordered partition of $\{1,2, \ldots, kn\}$, where for each $1 \leq p \leq n$

$$S_p = \{(p-1)k + 1, (p-1)k + 2, \ldots, pk\}.$$

Also, denote $a(k) = (a_1(k), a_2(k), \ldots, a_{kn}(k))$, where $a_j(k) = a_p/k$ for every $j \in S_p$, and let $f(k) = f \circ \pi(k)$.

As $\psi$ is consistent, for every $1 \leq p \leq n$ and every $j \in S_p$,

$$\psi_j(f(k), a(k)) = \psi_p(f,a). \tag{9.1}$$

For each fixed $m \geq 1$ and for each $b \in R^m_{++}$, the space $C^1_0(D_b)$ of all continuously differentiable functions $g$ on $D_b$ with $g(0) = 0$ is a normed algebra with the norm

$$\|g\| = \max_{x \in D_b} |g(x)| + \sum_{l=1}^{m} \max_{x \in D_b} \left| \frac{\partial g}{\partial x_l}(x) \right| b_l,$$

and the polynomials that vanish at 0 are dense in $C_0^1(D_b)$. Note that the mapping $f \to f(k)$ is a linear isometry from $C_0^1(D_a)$ into $C_0^1(D_{a(k)})$.

Let $f \in C_0^1(D_a)$ and $\epsilon > 0$. Let $g$ be a polynomial on $D_a$ with $g(0) = 0$ such that $\|f - g\| < \epsilon$. Applying Lemma 3 to $C_0^1(D_{a(k)})$ we deduce that for a sufficiently large $k$, there exists a multilinear function $\gamma$ on $D_{a(k)}$ such that $\|\gamma - g(k)\| < \epsilon$. As $\|g(k) - f(k)\| < \|g - f\|$, it follows that

$$\|\gamma - f(k)\| < 2\epsilon. \tag{9.2}$$

Let $L$ be the linear function in $C_0^1(D_{a(k)})$ that is given by,

$$L(z) = \sum_{l=1}^{kn} \max_{x \in D_{a(k)}} \left| \frac{\partial(\gamma - f(k))}{\partial x_l}(x) \right| z_l. \tag{9.3}$$

As $\gamma + L$ and $\gamma - L$ are multilinear functions on $D_{a(k)}$,

$$\psi(\gamma \pm L, a(k)) = \phi(\gamma \pm L, a(k)). \tag{9.4}$$

It follows from (9.3) that for every $1 \le l \le kn$,

$$\frac{\partial(\gamma - L)}{\partial x_l} \le \frac{\partial f(k)}{\partial x_l} \le \frac{\partial(\gamma + L)}{\partial x_l}$$

on $D_{a(k)}$. Therefore, by (9.4), the consistency of both $\psi$ and $\phi$, and the linearity of $\phi$, we have:

$$a_p|\psi_p(f,a) - \phi_p(f,a)| = \left| \sum_{l \in S_p} a_l(k)(\psi_l(f(k),a(k)) - \phi_l(f(k),a(k))) \right|$$
$$\le \sum_{l \in S_p} \phi_l(2L, a(k)) a_l(k) \le 4\epsilon.$$

Thus for every $1 \le p \le n$ and for every $\epsilon > 0$

$$a_p|\psi_p(f,a) - \phi_p(f,a)| < 4\epsilon.$$

As $a_p > 0$ it follows that $\psi(f,a) = \phi(f,a)$. ■

## 11 Bernstein's polynomials

Let $f$ be a continuous function on $[0,1]$. Bernstein's theorem asserts that the sequence of polynomials $(B_n(t))_{n \ge 1}$ converges to $f(t)$ uniformly on $[0,1]$, where

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} f(k/n).$$

It is easily verified that it suffices to prove the theorem for continuously differentiable functions $f$. Let then $f$ be a continuously differentiable function on [0,1]. For every $n > 1$ define $f_n$ on $D_n = \{x \in R^n : 0 \le x_i \le 1 \; \forall 1 \le i \le n\}$ by $f_n(x) = f((x_1 + x_2 + \cdots + x_n)/n)$. Let $g_n$ be the multilinear function on $D_n$ whose values on the vertices $1_S$, $S \subseteq \{1,2, \ldots ,n\}$ of $D_n$ are $f_n(1_S)$.

Let $\epsilon > 0$ and let $n$ be an integer large enough such that $|f'(t_2) - f'(t_1)| < \epsilon$ whenever $|t_2 - t_1| \le 1/n$. We will show that

$$|g_n(x) - f_n(x)| \le \epsilon \; \forall x \in D_n. \tag{1}$$

The theorem will follow by substituting $x = (t,t, \ldots ,t)$ in the last inequality.

We now prove (1). For each $x \in D_n$ let $N(x)$ be the number of the indices $i$ for which $0 < x_i < 1$. Obviously $0 \le N(x) \le n$. We will prove by induction on $k$ that

$$|g_n(x) - f_n(x)| \le k\epsilon/n$$

whenever $N(x) = k$.

For $x \in D_n$ with $N(x) = 0$, $x = 1_S$ for some $S \subseteq \{1,2, \ldots ,n\}$ and therefore $g_n(x) = f_n(x)$. Assume the claim has been proven for $0 \le k < n$. We now prove it for $k + 1$. Let $x \in D_n$ with $N(x) = k + 1$. Let $l \in \{1,2, \ldots ,n\}$ with $0 < x_l < 1$. Since $g_n$ is a multilinear function,

$$g_n(x) = x_l g_n(z^1) + (1 - x_l)g_n(z^0),$$

where $z^0 = (x_1, \ldots ,x_{l-1},0,x_{l+1}, \ldots ,x_n)$ and $z^1 = (x_1, \ldots ,x_{l-1},1,x_{l+1}, \ldots ,x_n)$.

Observe that for $i = 0$, 1, $N(z^i) = k$, and therefore by the induction hypothesis

$$|g_n(z^i) - f_n(z^i)| \le k\epsilon/n.$$

Hence,

$$|g_n(x) - f_n(x)| \le k\epsilon/n + |x_l(f_n(z^1) - f_n(x)) - (1 - x_l)(f_n(x) - f_n(z^0))].$$

By the intermediate value theorem:

$$(f_n(z^1) - f_n(x)) = f'(a)((1 - x_l)/n)$$

and

$$(f_n(x) - f_n(z^0)) = f'(b)(x_l/n),$$

where $|a - b| \le 1/n$. Hence,

$$|g_n(x) - f_n(x)| \le k\epsilon/n + (x_l(1 - x_l))\epsilon/n \le (k + 1)\epsilon/n$$

since $x_l(1 - x_l) \le 1$. ■

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