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Uniqueness of the Shapley Value

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It is shown that the Shapley value of any given game v is characterized by applying the value axioms—efficiency, symmetry, the null player axiom, and either additivity or strong positivity—to the additive group generated by the game v itself and its subgames.

It has often been remarked (e.g., Hart and Mas-Colell, 1988) that the standard axiomatizations (e.g., Dubey, 1975; Shapley, 1953; Young, 1985) of the Shapley value require the application of the value axioms to a large class of games (e.g., all games or all simple games). It is shown in this paper that the Shapley value of any given game v is characterized by applying the value axioms to the additive group generated by the game v itself and its subgames. This is important, particularly in applications where typically only one specific problem is considered.

Let N be a finite set. We refer to N as the set of *players* and to a subset S of N as a *coalition*. A (cooperative) *game* (with side payments) on the set of players N is a function v: $2^N \to R$ satisfying $v(\phi) = 0$. For a given game v and a coalition T, we refer to $v(T)$ as the *worth* of T. Given a game v on the set of players N, and a coalition S, we denote by v_S the real valued function v_S: $2^N \to R$ given by $v_S(T) = v(S \cap T)$. Thus, v_S is a game on the set of players N and is called the *subgame* of v, obtained by restricting v to subsets of S only.

The set of all games on the set of players N is denoted G^N, or G, for short. Let $v \in G$ and $i, j \in N$. The players i, j are substitutes in v if for every coalition $S \subset N \backslash \{i, j\}$, $v(S \cup i) = v(S \cup j)$; player i is a null player in the game v if $v(S \cup i) = v(S)$ for every coalition S.

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Let Q be a set of games on the set of players N. A map Ψ: $Q \to R^N$ is efficient if for every v in Q, $\Sigma_{i \in N} \Psi_i v = v(N)$; it is additive if for every v, w in Q with $v + w \in Q$, $\Psi v + \Psi w = \Psi(v + w)$; it is symmetric if for every v in Q and every two players i, j that are substitutes in v, $\Psi_i v = \Psi_j v$. A map Ψ: $Q \to R^N$ obeys the null player axiom if for every v in Q and every player i in N that is a null player of v, $\Psi_i v = 0$. For a given game v in G we denote by $G(v)$ the additive group generated by the game v and all of its subgames, i.e., $G(v) = \{w \in G | w = \Sigma\, k_i v_{S_i}$ where k_i are integers and S_i coalitions$\}$.

THEOREM A. *Let $v \in G$. Any map Ψ from $G(v)$ into R^N that is efficient, additive, and symmetric and obeys the null player axiom is the Shapley value.*

Proof. Let φ be the Shapley value. We must prove that $\varphi w = \Psi w$ for every game w in $G(v)$.

For any two games w, u and any two coalitions T and S, $(w \pm u)_S = w_S \pm u_S$ and $(w_S)_T = w_{S \cap T}$. Thus, any subgame of a game in $G(v)$ is in $G(v)$. For any game w in G we denote by $I(w) = \{S \subset N | \exists T \subset S$ with $w(T) \neq 0\}$. Note that for any minimal coalition S in $I(w)$, $w(S) \neq 0$. Also, $I(w \pm u) \subseteq I(w) \cup I(u)$ and $I(w_S) \subset I(w)$. We prove that $\Psi w = \varphi w$ for w in $Q(v)$ by induction on $|I(w)|$, the number of elements in $I(w)$. If $|I(w)| = 0$ then w is the 0 game and thus, by the null player axiom $\Psi_i w = 0 = \varphi_i w$. Assume that $\Psi w = \varphi w$ whenever $|I(w)| \leq k$ and let $w \in G(v)$ with $|I(w)| = k + 1$. Let S be a minimal element in $I(w)$. Then, w_S is a unanimity game and thus, by symmetry, efficiency, and the null player axiom it follows that $\Psi_i w_S = 0 = \varphi_i w_S$ for every $i \notin S$ and $\Psi_i w_S = w(S)/|S| = \varphi_i w_S$ for every $i \in S$. Note that $S \notin I(w - w_S) \subset I(w) \cup I(w_S) \subset I(w)$ and thus,

$$|I(w - w_S)| < |I(w)| \qquad \text{for every minimal coalition } S \in I(w). \tag{1}$$

Thus, by the induction hypothesis $\Psi(w - w_S) = \varphi(w - w_S)$ and applying additivity, $\Psi w = \Psi(w_S) + \Psi(w - w_S) = \varphi(w_S) + \varphi(w - w_S) = \varphi w$. ■

Our next result characterizes the Shapley value without the additivity axiom. The axiom that is added is strong positivity, which asserts that the value of each player is a monotonic function of his marginal contributions.

A map Ψ: $Q \to R^N$ is strongly positive if for any player $i \in N$ and any two games, w and u, in Q with $w(S \cup i) - w(S) \geq u(S \cup i) - u(S)$, $\Psi_i w \geq \Psi_i u$.

THEOREM B. *Let $v \in G^N$. Any map Ψ from $G(v)$ into R^N that is efficient, strongly positive, and symmetric is the Shapley value.*

Proof. We prove that $\Psi w = \varphi w$ for any w in $G(v)$ by induction on $|I(w)|$. If $|I(w)| = 0$ then w is the 0 game and thus, by the efficiency and symmetry of Ψ, $\Psi_i w = \varphi_i w = 0$. Assume that $\Psi w = \varphi w$ whenever $|I(w)| \leq k$ and let $w \in G(v)$ with $|I(w)| = k + 1$. Let $\partial(w)$ denote the set of minimal coalitions in $I(w)$. Note that for every S in $\partial(w)$ and every $i \in N \backslash S$, $w_S(T \cup i) - w_S(T) = 0$ and therefore by strong positivity of Ψ, $\Psi_i(w) = \Psi_i(w - w_S)$ which by (1) and the induction hypothesis equals $\varphi_i(w - w_S) = \varphi_i(w)$. Thus, $\Psi_i w = \varphi_i w$ for every $i \in N \backslash S$ where $S \in \partial(w)$. Letting S_0 denote the intersection of all coalitions S in $\partial(w)$, i.e., $S_0 = \bigcap_{S \in \partial(w)} S$, we deduce that

$$\Psi_i w = \varphi_i w \qquad \text{for every } i \in N \backslash S_0. \tag{2}$$

Note that $w(T) = 0$ whenever $T \not\supset S_0$. Thus, every two players i, j in S_0 are substitutes in w. Thus, using (1) and the efficiency of both Ψ and φ we deduce that $\Sigma_{i \in S_0} \Psi_i w = \Sigma_{i \in S_0} \phi_i w$, and by applying the symmetry of both Ψ and φ,

$$\Psi_i w = \varphi_i w \qquad \text{for every } i \in S_0. \tag{3}$$

Combining (2) and (3) we conclude that $\Psi w = \varphi w$. ■

Remark a. The above results show, in particular, that any additive subgroup of G, that is closed under the subgame operator, has a unique value. For instance, let w be a given monotonic simple game. Then, $Q = \{wv | v \in G\}$ is an example of such an additive subgroup. Such spaces of games arise in models in which political approval that is described by the simple game w is needed for various economic activities.

Remark b. It follows from the proof of Theorem B that strong positivity could be replaced by the axiom that $\Psi_i v$ depends only on the marginal contributions $v(S \cup i) - v(S)$, $S \subset N$, of player i in v.

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