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# Voting for Public Goods

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It is shown that when resources are privately owned, the institution of voting is irrelevant to the choice of non-exclusive public goods: the total bundle of such goods produced by Society is the same whether or not minority coalitions are permitted to produce them. This is in sharp contrast to the cases of redistribution and of exclusive public goods, where public decisions depend strongly on the vote. The analytic tool used is the Harsanyi–Shapley non-transferable utility value.

## 1. INTRODUCTION

Several years ago, a game theoretic model that took explicit account of power relationships was introduced to analyse taxation in a democratic society (Aumann and Kurz, 1977*a*, *b*). Those analyses were set in the context of private goods, so that only redistribution and exchange were at issue. In the current paper we apply a similar analysis to public goods.

The framework within which we work is that of a public goods economy, defined by a set of agents, a collection of public goods, a collection of non-consumable resources, and a technology (enabling public goods to be produced from resources); moreover, each agent has a utility function for public goods, an initial endowment of resources, and a voting weight. It will be assumed that the agents form a non-atomic continuum, i.e. that there are many agents, each of whom is individually insignificant.

We consider two games, the voting game and the non-voting game. In the voting game, any coalition (i.e. set of agents) with a majority of the vote may produce public goods, using its own resources only; once produced, the public goods may be enjoyed by all agents.[1] In the non-voting game, any coalition, irrespective of its size, may produce public goods, using its own resources only; public goods produced by different (disjoint) coalitions may be enjoyed by all. (For example, if we are discussing television, any programme produced by any coalition may be viewed by any agent.)

To these games we shall apply the solution notion known as the Harsanyi–Shapley Non-Transferable-Utility (NTU) Value;[2] the resulting outcomes (i.e. bundles of public goods produced) will be called value outcomes. We obtain the following:

**Theorem.** *The voting game has the same value outcomes as the non-voting game.*

In Aumann, Kurz and Neyman (1983) (henceforth AKN) we proved a related result, namely that the value outcomes in the voting game are independent of the voting weights. This follows from the current theorem, since obviously the voting weights cannot affect the outcome of the non-voting game. But the theorem of AKN was proved under weaker conditions, under which the voting and non-voting games can actually have different value outcomes (cf. Examples 6 and 7). Briefly, in AKN, it is the voting weights that turn out irrelevant; here, the whole institution of voting turns out to be irrelevant.

The paper is organized as follows. In Section 2 we formally describe public goods economies and set forth our assumptions. Section 3 contains the formal description of our games. Section 4 specifies the variant of the Harsanyi–Shapley NTU value used in this paper, thus completing the formal specification of all terms appearing in the above statements of the theorems. In Section 5 we demonstrate the theorem informally, stressing the intuitive background. Section 6 contains illustrations and counter-examples, and Section 7 the formal proof of the theorem. Section 8 is devoted to discussion.

The paper is constructed so that readers who are not interested in the formal treatment can avoid it entirely. Such readers, after completing the introduction, should go immediately to Section 5, then peruse the informal part of Section 6, and then read Section 8. Conversely, readers interested *only* in the formal proofs may omit Sections 5, 6, and 8.

## 2. PUBLIC GOODS ECONOMIES

The real line is represented by $\mathbb{R}$, the euclidean space of dimension $n$ by $E^n$, its non-negative orthant by $E^n_+$ (i.e. $E^n_+ = \{x \in E^n : x^j \geqq 0 \text{ for all } j\}$).

A non-atomic public goods economy consists of

(i) A measure space $(T, \mathscr{C}, \mu)$ ($T$ is the space of agents or players, $\mathscr{C}$ the family of coalitions, and $\mu$ the population measure); we assume that $\mu(T) = 1$ and that $\mu$ is $\sigma$-additive, non-atomic and non-negative.

(ii) Positive integers $l$ (the number of different kinds of resources) and $m$ (the number of different kinds of public goods).

(iii) A correspondence $G$ from $E^l_+$ to $E^m_+$ (the production correspondence).

(iv) For each $t$ in $T$, a member $e(t)$ of $E^l_+$ ($e(t)\mu(dt)$ is $dt$'s endowment of resources).

(v) For each $t$ in $T$, a function $u_t : E^m_+ \to \mathbb{R}$ ($dt$'s von Neumann–Morgenstern utility).

(vi) A $\sigma$-additive, non-atomic, non-negative measure $\nu$ on $(T, \mathscr{C})$ (the voting measure); we assume $\nu(T) = 1$.

Note that the total endowment of a coalition $S$—its input into the production technology if it wishes to produce public goods by itself—is $\int_S e(t)\mu(dt)$; for simplicity, this vector is sometimes denoted $e(S)$. A public goods bundle is called jointly producible if it is in $G(e(T))$, i.e. can be produced by all of society.

We assume that the measurable space $(T, \mathscr{C})$ is isomorphic[3] to the unit interval [0, 1] with the Borel sets. This assumption is less restrictive than it sounds; any non-denumerable Borel subset of any euclidean space (or indeed, of any complete separable metric space) is isomorphic to [0, 1]. We also assume the following (as usual $x \leqq y$ means $x^j \leqq y^j$ for all $j$):

*Assumption* 1. $u_t(y)$ is Borel measurable simultaneously in $t$ and $y$, continuous in $y$ for each fixed $t$, and bounded uniformly in $t$ and $y$.

*Assumption* 2. $G$ has compact and non-empty values.

*Assumption* 3. If $x \leqq y$, then $G(x) \subset G(y)$ and $u_t(x) \leqq u_t(y)$ for all $t$.

*Assumption* 4. $0 \in G(0)$.

*Assumption* 5. *Either* (i) $u_t$ is $C^1$ (continuously differentiable) on[4] $E^m_+$ and the derivatives $\partial u_t/\partial y^j$ are strictly positive and uniformly bounded, *or* (ii) there are only finitely many different utility functions $u_t$.

Assumption 3 may be called "monotonicity of production and utility" or "free disposal of resources and of public goods". The theorem is actually false without this assumption; see Example 6. Assumption 4 says that the technology is capable of producing nothing from nothing. In Assumption 5, we assume that either the utility functions are smooth, or that there are only finitely many "utility types" (though perhaps a continuum of "endowment types"). The situation is reminiscent of that in Geometric Topology, where to avoid "wild imbeddings" one may assume either that all maps are piecewise linear, or that they are differentiable. We require Assumption 5 for the proof, but we do not know whether the theorem is actually false without it; see Section 8d.

The other assumptions are of a technical nature. Note that conceptually, uniform boundedness involves no loss of generality. Indeed, in each of the games we will consider, the set of feasible public goods bundles is contained in a compact set (see the end of Section 3); by changing the $u_t$ outside this set, we can make them bounded without really affecting anything. Uniform boundedness can then be obtained by applying (possibly different) positive linear transformations to each of the $u_t$. Omitting the assumption altogether might however cause technical difficulties, since transformations of this kind might affect the integrability of the utility functions $u_t(y)$ when weighted by the comparison function $\boldsymbol{\lambda}(t)$ (see Section 4). While the difficulties may be circumventable, it did not seem worthwhile to expend our energy—or the readers'—in removing this conceptually harmless assumption.

## 3. THE GAMES

For a verbal description of the games we shall define here, see Section 1.

Recall that a strategic game[5] with player space $(T, \mathscr{C}, \mu)$ is defined by specifying, for each coalition $S$, a set $X^S$ of strategies, and for each pair $(\sigma, \tau)$ of strategies belonging respectively to a coalition $S$ and its complement $T\backslash S$, a payoff function $\boldsymbol{h}^S_{\sigma\tau}$ from $T$ to $\mathbb{R}$.

In formally defining the games, we shall describe pure strategies only; but it is to be understood that arbitrary mixtures of pure strategies are also available to the players. The pure strategies we shall describe will have a natural Borel structure, and mixed strategies should be understood as random variables whose values are pure strategies.

In the non-voting game, a pure strategy for $S$ is simply a member $x$ of $G(e(S))$, i.e. a choice of a public goods bundle which can be produced from the total resource bundle $e(S)$. If $S$ has chosen $x \in G(e(S))$ and $T\backslash S$ has chosen $y \in G(e(T\backslash S))$, then the payoff to any $t$ is $u_t(x+y)$.

In the voting game, a strategy for a coalition $S$ in the majority ($\nu(S) > \frac{1}{2}$) is again a member $x$ of $G(e(S))$. Minority coalitions ($\nu(S) < \frac{1}{2}$) have only one strategy (essentially "doing nothing"). If a majority coalition $S$ chooses $x \in G(e(S))$ and $T\backslash S$ chooses its single strategy (as it must), then the payoff to any $t$ is $u_t(x)$. The definition of strategies and payoffs for coalitions with exactly half the vote is not important, as these coalitions play practically no role in the analysis; the reader may define them in any way he considers appropriate.

Note that the set of feasible public goods bundles—those that can actually arise as outcomes of one of our games—is precisely the compact set $G(e(T))$.

## 4. VALUE OUTCOMES

We shall be working here with the asymptotic value,[6] an analogue of the finite-game Shapley value for games with a continuum of players, obtained by taking limits of finite approximants. Let $\Gamma$ be the voting or the non-voting game. A comparison function is a non-negative valued $\mu$-integrable function $\boldsymbol{\lambda}$ on $T$ that is positive on a set of agents of positive measure; the corresponding comparison measure $\lambda$ is defined by $\lambda(dt) = \boldsymbol{\lambda}(t)\mu(dt)$, i.e. $\lambda(S) = \int_S \boldsymbol{\lambda}(t)\mu(dt)$. A value outcome in $\Gamma$ is then a random bundle of public goods associated with the Harsanyi–Shapley NTU value based on $\phi$; i.e. a random variable $\underset{\sim}{y}$ with values in $G(e(T))$, for which there exists a comparison function $\boldsymbol{\lambda}$ such that the Harsanyi coalitional form[7] $v_\lambda^\Gamma$ of the game $\boldsymbol{\lambda}\Gamma$ is defined and has an asymptotic value, and

$$(\phi v_\lambda^\Gamma)(S) = \int_S Eu_t(\underset{\sim}{y})\lambda(dt) \quad \text{for all } S \in \mathscr{C}, \tag{1}$$

where $Eu_t(\underset{\sim}{y})$ is the expected utility of $\underset{\sim}{y}$.

## 5. AN INFORMAL DEMONSTRATION OF THE THEOREMS

We start by briefly reviewing Section 5 of AKN; the reader is referred to there for a more comprehensive treatment. Let us use the word outcome for a bundle of public goods.[8] Let $\lambda$ be a comparison measure, i.e. a non-negative measure on the agent space; $\lambda(dt)$ is interpreted as an infinitesimal "exchange rate" that enables comparison of agent $dt$'s utility $u_t$ with that of other agents. For each coalition $S$ and each outcome $y$, write

$$U^y(S) = \int_S u_t(y)\lambda(dt).$$

$U^y(S)$ represents the "total" payoff to $S$ when the exchange rates $\lambda(dt)$ are used and $y$ is the total bundle of public goods produced by all coalitions; this follows from the fact that all agents can enjoy all public goods produced by anybody.

Denote the non-voting and voting games by $A$ and $B$ respectively, and let $\Gamma$ be either $A$ or $B$. Define

$$w_\lambda^\Gamma(S) = \max\min\,[U^y(S) - U^y(T\backslash S)], \tag{2}$$

where the max and the min are over the strategies of $S$ and $T\backslash S$ respectively. Set

$$v_\lambda^\Gamma(S) = \frac{(w_\lambda^\Gamma(S) + w_\lambda^\Gamma(T))}{2}. \tag{3}$$

Briefly, $w_\lambda^\Gamma(S)$ measures $S$'s bargaining strength, or ability to threaten; $v_\lambda^\Gamma(S)$, the total utility that $S$ can expect from the resulting efficient compromise.

Recall that an outcome $y$ in $\Gamma$ is a value outcome iff

$$(\phi v_\lambda^\Gamma)(dt) = u_t(y)\lambda(dt), \tag{4}$$

where $\phi$ is the Shapley value; i.e, iff it is feasible, and its utility for each infinitesimal agent $dt$, in terms of the exchange rates $\lambda(dt)$, is precisely his value in the coalitional game $v_\lambda^\Gamma$.

For each $\theta$ with $0 \leqq \theta \leqq 1$, let $\theta T$ denote a "diagonal coalition of size $\theta$". Intuitively, $\theta T$ may be thought of as a perfect sample of the population $T$ of all agents, containing a proportion $\theta$ of the agents. If $r$ is a non-atomic game and $dt$ an agent, then (AKN, (5.7)),

$$(\phi r)(dt) = \int_0^1 (r(\theta T \cup dt) - r(\theta T))d\theta. \tag{5}$$

Let $S$ be a perfect—or almost perfect—sample of the population $T$ with a clear majority; i.e. a coalition of the form $\alpha T$, $\alpha T \cup dt$, or $\alpha T\backslash dt$, where $\alpha$ is larger than $\frac{1}{2}$ by more than an infinitesimal. Then in the non-voting game, the optimal threat of the minority is not to produce any public goods. This is because any public goods produced by the minority will also be enjoyed by the majority. Both are perfect—or almost perfect—samples, so the per capita rise in utility from such production is about the same in the two coalitions; but the majority is larger than the minority, so its total utility rises by more. Thus in the difference $U^y(S) - U^y(T\backslash S)$ between the payoffs to the two coalitions—which is the criterion for defining the optimal threats (see (2))—any enjoyment by the minority is more than offset by the corresponding enjoyment of the majority. The upshot is that in the voting game, the minority *may* not produce; in the non-voting game, it chooses not to produce; in any case, it does not produce. Therefore the outcome is the same in the two cases, namely what the majority chooses to produce; thus suppressing the subscript $\lambda$, we conclude that

$$v^A(S) = v^B(S) \quad \text{and} \quad v^A(T \backslash S) = v^B(T \backslash S). \tag{6}$$

Set $r = v^A - v^B$; we wish to show that $\phi r = 0$. By (5), the relevant coalitions $U$ are those of the form $\theta T$ or $\theta T \cup dt$. If $\theta - (1/2)$ is non-infinitesimal, then each such coalition is either of the form $S$ considered in the previous paragraph, or is the complement of such a coalition. Hence by (6), $r$ vanishes on all such coalitions.

If $\theta - (1/2)$ is infinitesimal, then $U$ is "near"[9] $(1/2)T$, and so also near its complement $T \backslash U$; hence no matter what the outcome is, $U$ and $T \backslash U$ must have approximately the same utility, since all agents enjoy the same public goods. Hence $w^\Gamma(U)$ is infinitesimal, and since $w^A(T) = w^B(T)$, it follows from (3) that $r(U)$ is infinitesimal.

Summing up, the integrand in (5) vanishes when $\theta - (1/2)$ is not infinitesimal, and is infinitesimal when $\theta - (1/2)$ *is* infinitesimal. Hence $\phi r$ is the integral of an infinitesimal over an infinitesimal range, i.e. an infinitesimal of the second order, which may be ignored. Thus indeed $\phi r = 0$. Hence $\phi v_\lambda^A = \phi v_\lambda^B$ for all $\lambda$, and so by (4), $A$ and $B$ have the same value outcomes. This completes the demonstration of our theorem.

The second part of the argument ($\theta$ near $\frac{1}{2}$) breaks down in the case of redistribution of private goods (Aumann and Kurz 1977*a*, *b*), or when the majority may exclude the minority from enjoying the public goods. In both cases, the minority may be prevented from consuming anything, whereas the majority can at least use its own resources; so even when $S$ has only a slight $\nu$-majority, $w^B(S)$ will in general be very far from 0. The theorem is actually false in these cases (see Examples 1 and 2).

The first part of the argument ($\theta$ not near $\frac{1}{2}$) works only if utilities are monotonic (Assumption 3); for otherwise, the minority may threaten to lower the majority's total utility by producing public bads, more than it lowers its own (Example 6).

## 6. ILLUSTRATIONS AND COUNTEREXAMPLES

As above, $w(S)$ denotes the difference between total payoff to $S$ and that to $T \backslash S$ when they minimax the difference. We denote by $q(S)$ the total payoff to $S$ under the same circumstances (i.e. when the coalitions minimax the difference), so that

$$w(S) = q(S) - q(T \backslash S).$$

As above, $v(S)$ denotes $(w(S) + w(T))/2$ and $\lambda$ denotes a comparison measure. It may be verified that $\phi v = \phi q$.

In the examples of this section, we will describe the utilities only in the "relevant range"—the compact set of feasible public goods bundles (see the end of Section 3). Outside of the relevant range they can be chosen arbitrarily, as long as they satisfy our assumptions. For example, the phrase "linear utilities" means "utilities that are linear in the relevant range"; the utilities cannot, of course, be linear throughout $E_+^m$, since that would violate the boundedness condition.

In the first four examples of this section, the comparison measure $\lambda$ and the population measure $\mu$ coincide. The subscript $\lambda$ is omitted in these examples.

To provide contrast and perspective, we first consider two games *different* from those forming the main subject of this paper.

*Example* 1: A Redistribution Game. This is a variant of one of the games discussed by Aumann and Kurz (1977*a*).[10] There is one kind of commodity, serving both as resource and consumption good; it is private, in the sense that any amount consumed by one agent cannot also be consumed by another agent. Each agent $dt$ is endowed with an amount $e(dt) = \boldsymbol{e}(t)\mu(dt)$ of this commodity, $\mu$ being the population measure. Utilities are linear; specifically $u_t(x) = x$ for all $t$ and $x$. The vote measure $\nu$ may be different from $\mu$. A coalition with a majority of the vote may redistribute its own resources in any way it pleases among its own members; a minority coalition may consume nothing.[11] We assume $e(T) = \nu(T) = 1$.

It may be seen that $q(S) = e(S)$ or 0 according as $\nu(S) > \frac{1}{2}$ or $< \frac{1}{2}$. In particular, $q$ is a function of the vector measure $(e, \nu)$. For diagonal coalitions, we get $q(\theta T) = 0$ or $\theta$ according as $\theta < \frac{1}{2}$ or $\theta > \frac{1}{2}$ (see Figure 1). For the value, (5) yields

$$(\phi q)(dt) = \left[\int_0^{1/2-\nu(dt)} + \int_{1/2-\nu(dt)}^{1/2} + \int_{1/2}^{1}\right](q(\theta T \cup dt) - q(\theta T))d\theta. \tag{7}$$

In the first integral, both $\theta T \cup dt$ and $\theta T$ are minorities, so the integrand vanishes identically. In the second integral, $\theta T \cup dt$ is a majority whereas $\theta T$ is a minority; hence the integrand $\approx ((1/2)T) = \frac{1}{2}$, and so the integral is $\approx (1/2)\nu(dt)$. In the last integral, both are majorities; hence the integrand is $e(\theta T \cup dt) - e(\theta T) = e(dt)$, and the integral is $(1/2)e(dt)$. Summing up, we get $\phi q = (\nu + e)/2$. Thus the vote measure $\nu$ is an important component of the value, so that our theorem does not hold here.

*Example* 2: Public Goods With Exclusion. Here we are back in a public goods context, but we allow the coalition producing the goods to exclude others from using it. To keep the example simple, we assume one kind of resource ($l = 1$), which is non-consumable; one public good ($m = 1$); one unit of the public good may be produced from each unit of the resource ($G(x) = [0, x]$); agent $dt$ is endowed with $e(dt) = \boldsymbol{e}(t)\mu(dt)$ units of the resource; and $u_t(x) = x$. Only coalitions with a majority of the vote measure $\nu$ may produce public goods, using their own resources only; they may (and therefore in the optimal threat do) prevent non-members from using the output. We assume $\mu(T) = e(T) = \nu(T) = 1$.

If $S$ is a majority coalition, it will produce $e(S)$, and hence its payoff $q(S)$ is

$$\int_S u_t(e(S))\mu(dS) = \mu(S)e(S).$$

If $S$ is a minority, then $q(S) = 0$. For diagonal coalitions, we get $q(\theta T) = 0$ or $\theta^2$ according as $\theta < (1/2)$ or $\theta > (1/2)$ (see Figure 1). Applying (7), we find that the first integral vanishes and the second one yields $\nu(dt)/4$. The integrand in the third integral is

$$\mu(\theta T \cup dt)e(\theta T \cup dt) - \mu(\theta T)e(\theta T),$$

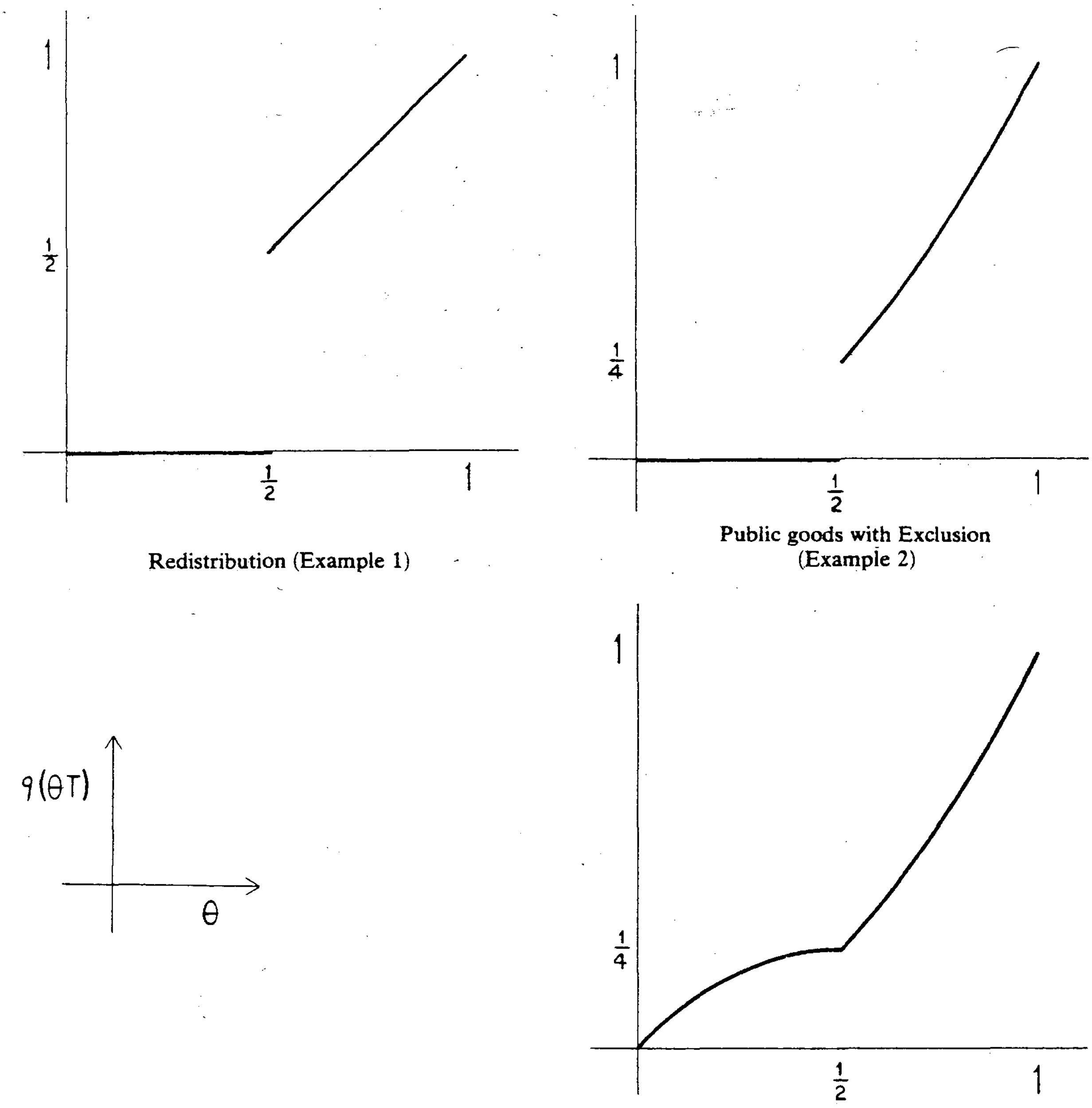
which works out to $\theta(e(dt) + \mu(dt))$ plus an infinitesimal of higher order. Since $\int_{1/2}^{1} \theta d\theta = \frac{3}{8}$, we conclude that

$$\phi q = (1/4)\nu + (3/8)e + (3/8)\mu.$$

Thus the vote measure $\nu$ is again an important component of the value.

*Example* 3: A Non-Voting Game. With this example, we return to the main subject of this paper. Suppose that the specifications are precisely as in the previous example, except that all may produce public goods, and all enjoy any produced. A coalition will produce if and only if it gains more out of such production than its complement; since all the utilities are the same, this means simply that it is larger than its complement. Thus a coalition will produce what it can if it is in the majority, and

FIGURE 1
Three voting games




Redistribution (Example 1)

Public goods with Exclusion (Example 2)

Public goods without Exclusion (Examples 3 and 4)

otherwise will produce nothing; all coalitions will enjoy whatever is produced. Therefore

$$q(S)=\begin{cases}\mu(S)e(S) & \text{if } \mu(S)>\frac{1}{2}\\ \mu(S)(1-e(S)) & \text{if } \mu(S)<\frac{1}{2}.\end{cases}$$

For diagonal coalitions, we get $q(\theta T)=\theta\max(\theta, 1-\theta)$ (see Figure 1). Again using (7), we find that this time the middle integral is an infinitesimal of the second order, and so may be ignored; the other two integrals yield

$$\phi q=(3/4)\mu+(1/4)e.$$

Of course, the vote measure $\nu$ does not figure in the expression for the value, since it does not figure in the description of the game.

*Example* 4: A Voting Game. Like the previous example, except that coalitions with a minority of the vote measure $\nu$ may not produce public goods. This time a coalition will produce if and only if it wants to *and* may; i.e. iff $\nu(S) > \frac{1}{2}$ and $\mu(S) > \frac{1}{2}$. Denoting the $q$ of the previous example by $q^A$, we find that

$$q(S) = \begin{cases} q^A(S) & \text{if } (\mu(S) - \frac{1}{2})(\nu(S) - \frac{1}{2}) > 0 \\ 0 & \text{otherwise} \end{cases}$$

(see Figure 2). Since the diagonal is wholly within the area in which $q = q^A$, we get the same worth for diagonal coalitions as before (see Figure 1).

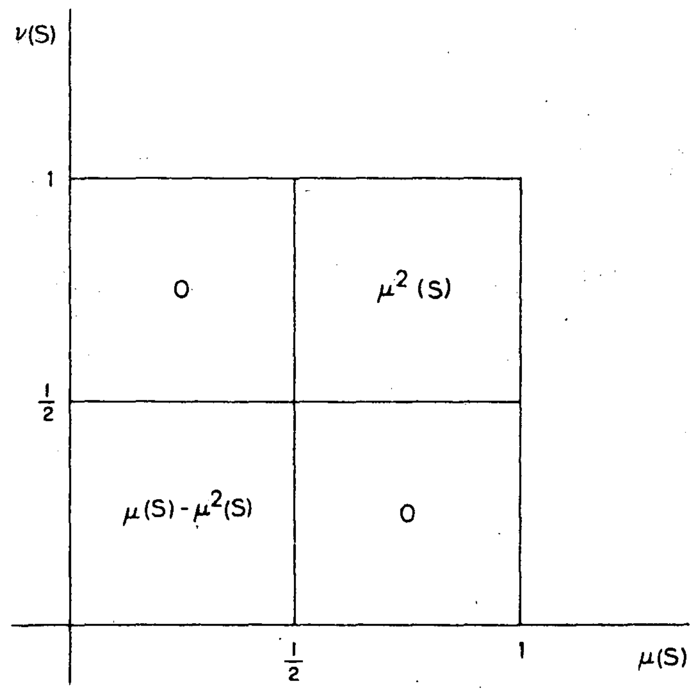



FIGURE 2

$q(S)$ in Example 4 when $e \equiv 1$

By the theorem, the value $\phi q$ is the same as in the previous example,[12] i.e. $(3/4)\mu + (1/4)e$.

The characteristic feature of the first two examples, which is absent from the last two, is the jump in $q(\theta T)$ at $\theta = \frac{1}{2}$. It is because of this jump that the middle integral in (7) contributes non-negligibly to $(\phi q)(dt)$. The contribution is precisely $\nu(dt)$ times the size of the jump; only here does the vote measure put in an appearance. Thus the discontinuity in $q(\theta T)$ at $\frac{1}{2}$ is intimately associated with the relevance of the vote measure. A little thought will convince the reader that this makes economic sense as well. An individual's vote is only significant because it may pivot, i.e. turn a minority into a majority; if nothing much happens to anybody even when pivoting occurs, the vote can't be very important.

Example 4 bears further examination because though $q(\theta T)$ is continuous, $q$ itself has an essential discontinuity at $(1/2)T$. To enable the discussion to take place in two dimensions, let us take $e(t) \equiv 1$, i.e. $e(S) = \mu(S)$. If in Figure 2 one approaches the centre

$(1/2)T$ of the diagonal from the southwest or notheast, then $q \to \frac{1}{4}$, whereas $q \to 0$ if it is approached from the northwest or southeast. If one considers $v$ instead of $q$, one finds

$$v(S) = \begin{cases} (2\mu^2(S) - \mu(S) + 1)/2 & \text{if } \mu(S) > \frac{1}{2} \text{ and } \nu(S) > \frac{1}{2} \\ (\mu(S) - 2\mu^2(S) + 1)/2 & \text{if } \mu(S) < \frac{1}{2} \text{ and } \nu(S) < \frac{1}{2} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Note that whereas $v(S)$, too, is discontinuous along almost all of the "voting line" $\nu(S) = \frac{1}{2}$, it is continuous along the line $\mu(S) = \frac{1}{2}$; in particular, the discontinuity at $(1/2)T$ itself has disappeared.

In a sense, $q(S)$ is an unnatural object, because it is the payoff to $S$ when $S$ is acting with a different objective in mind: not its own payoff, but the difference between the payoffs to it and to its complement. This accounts for $q$'s bad behaviour. However, it is not this bad behaviour that causes the discontinuities in $q(\theta T)$; in each of our three examples, $v(\theta T)$ has the same jump (or lack of it) at $\theta = \frac{1}{2}$ as $q(\theta T)$.

*Example* 5: The Optimal Quantity of a Public Good. Examples 3 and 4 are instructive from the point of view of understanding the TU analysis in this paper, which underlies the NTU analysis. But the NTU analysis itself is in these particular games trivial, since there is only one public good, so that in the end all agents will agree to produce a maximum amount of it. To obtain a non-trivial NTU example, we require at least two public goods, so that there can be some difference of opinion as to how much of each one should be produced. Since the comparison measure will now be endogenous, we abandon the convention that $\lambda = \mu$. We do adopt the normalization

$$\lambda(T) = 1. \tag{8}$$

Our public goods economy has a single resource ($l = 1$). There are two kinds of public goods, and the utility functions are of the form

$$u_t(y) = f_t(y^1) + y^2,$$

where $f_t$ is concave, differentiable, increasing on the non-negative reals, and satisfies $f_t'(0) > 1 > f_t'(1/2)$ ($f_t'(0) = \infty$ is not excluded); it will be convenient to call the first good concave, and the second one linear. Given an amount $z$ of resource, any combination of public goods totalling up to $z$ may be produced; i.e.

$$G(z) = \{y \in E_+^2 : y^1 + y^2 \leqq z\}.$$

The total endowment $e(T)$ of the resource is 1.

Intuitively, one may think of the concave good as representing some particular government activity on which one wishes to focus attention, say defence; the problem is to determine its budget. The linear good represents an amalgam of all other government activity, and the resource $z$ an amalgam of all resources; one may think of $z$ in units of money. We consider the voting and the non-voting games, which as we know have the same value outcomes.

The point of the example is that the agents differ in their assessment of the concave good. Since $f_t'(0) > 1$, all agents would like some of this good to be produced, but they differ as to how much. If $dt$ could decide by himself what is to be done with the total amount of resource available to Society (namely 1), he would produce that amount $y_t^1$

of the concave good for which $f_t'(y_t^1)=1$; the remaining resources would be allocated to production of the linear good. Since $f_t'(1/2)<1$, it follows that $y_t^1<\frac{1}{2}$.

Let $\lambda$ be given. We have

$$(\phi v_\lambda)(dt)=(\phi q_\lambda)(dt)=\int_0^1 (q_\lambda(\theta T\cup dt)-q_\lambda(\theta T))d\theta.$$

From Section 5 we know that if $\theta>\frac{1}{2}$, then the optimal strategy for $\theta T$ is to produce a $y$ that will maximize

$$\int_{\theta T} u_t(y)\lambda(dt)=\theta\int_T (f_t(y^1)+y^2)\lambda(dt)=\theta y^2+\theta F_\lambda(y^1),$$

where

$$F_\lambda(y^1)=\int_T f_t(y^1)\lambda(dt).$$

From $f_t'(0)>1>f_t'(1/2)$ and $\lambda(T)=1$ it follows that $F_\lambda'(0)>1>F_\lambda'(1/2)$; hence there is a $y_\lambda^1$ between 0 and $\frac{1}{2}$ with $F_\lambda'(y_\lambda^1)=1$. If $\theta>\frac{1}{2}$, then the optimal strategy for $\theta T$ is to produce precisely $y_\lambda^1$ of the concave good; the remainder of the resource will be used for producing the linear good, yielding $\theta-y_\lambda^1$ of the linear good. The optimal strategy for the complementary coalition $(1-\theta)T$ is to produce nothing.

Consider next the coalition $\theta T\cup ds$, where still $\theta>\frac{1}{2}$. The arrival of $ds$ places additional resources in the amount of $e(ds)$ at the disposal of the coalition. Since $y_\lambda^1<\frac{1}{2}$, the coalition is already producing all it wants of the concave good, so that this additional amount will be used to produce the linear good. Each agent $dt$ in $\theta T$ will therefore obtain an additional utility of $e(ds)\lambda(dt)$; all together, they obtain an additional utility of

$$e(ds)\lambda(\theta T)=\theta e(ds)\lambda(T)=\theta e(ds),$$

by the normalization condition (8). By joining the coalition $\theta T$, though, $ds$ causes an increase in utility in another way, namely by adding his own utility for the entire bundle now present. This is

$$(f_s(y_\lambda^1)+\theta-y_\lambda^1)\lambda(ds).$$

Hence we conclude that for $\theta>\frac{1}{2}$,

$$q(\theta T\cup ds)-q(\theta T)=\theta e(ds)+(f_s(y_\lambda^1)+\theta-y_\lambda^1)\lambda(ds). \tag{9}$$

Suppose next that $\theta<\frac{1}{2}$. The optimal strategy for $\theta T$ is to produce nothing; and for its complement $(1-\theta)T$, it is to produce $(y_\lambda^1, (1-\theta)-y_\lambda^1)$, which all agents will enjoy, including those in $\theta T$. If $ds$ joins $\theta T$, then the resources of the complement *decrease* by $e(ds)$. Since $1-\theta>\frac{1}{2}$, we are in the interior of the range where the linear good is being produced, so that the loss in utility to $\theta T$ by having $ds$ join it is

$$e(ds)\lambda(\theta T)=\theta e(ds),$$

i.e. a gain of $-\theta e(ds)$. On the other hand, there is a *gain* to $\theta T$ of $ds$'s own utility, given by

$$(f_s(y_\lambda^1)+(1-\theta)-y_\lambda^1)\lambda(ds).$$

Hence we conclude that for $\theta<\frac{1}{2}$,

$$q(\theta T\cup ds)-q(\theta T)=-\theta e(ds)+(f_s(y_\lambda^1)+(1-\theta)-y_\lambda^1)\lambda(ds). \tag{10}$$

The analysis of Section 5 shows that the immediate neighbourhood of $(1/2)T$ may be ignored. Hence using (9) and (10), we find

$$\begin{aligned}(\phi v_\lambda)(ds) &= \int_0^1 (q(\theta T \cup ds) - q(\theta T))d\theta \\ &= e(ds)\Big(\int_0^{1/2} (-\theta)d\theta\Big) + \Big(\int_{1/2}^1 \theta d\theta\Big) + [f_s(y_\lambda^1) + (1-y_\lambda^1)]\lambda(ds) \\ &\quad -\lambda(ds)\Big(\int_0^{1/2} \theta d\theta + \int_{1/2}^1 (1-\theta)d\theta\Big) \\ &= [f_s(y_\lambda^1) + (1-y_\lambda^1)]\lambda(ds) + \tfrac{1}{4}e(ds) - \tfrac{1}{4}\lambda(ds).\end{aligned} \tag{11}$$

Suppose now that we are at a value outcome. This means that we have found a $\lambda$ such that the utility to each $ds$ of the optimal outcome produced by the all-player set $T$ is precisely equal to the value of $ds$. The optimal outcome produced by $T$ is $(y_\lambda^1, 1-y_\lambda^1)$, and its utility to $ds$ is

$$[f_s(y_\lambda^1) + (1-y_\lambda^1)]\lambda(ds).$$

Equating this with $(\phi v_\lambda)(ds)$ and using (11), we find that for all $s$,

$$\tfrac{1}{4}e(ds) - \tfrac{1}{4}\lambda(ds) = 0,$$

which means $e = \lambda$. Thus in this example, the weights turn out to be the initial resources; there is a unique value outcome, found by maximizing $\int_T u_t(y)e(dt)$. More precisely, the value outcome is $(y_e^1, 1-y_e^1)$, where $y_e^1$ is that amount of the concave good for which $\int_T f_t'(y_e^1)e(dt) = 1$; i.e. a sort of average of the $y_t^1$, weighted by the $e(dt)$ and taking the utilities into account.

For an instance of an explicit calculation, take

$$f_t(z) = w(t)\log(1+z),$$

where $1 < w(t) < \frac{3}{2}$ (this comes from $f_t'(0) > 1 > f_t'(1/2)$). Then $y_t^1 = w(t) - 1$, and the unique value outcome is given by $y_e^1 = \int_T y_t^1 e(dt)$.

In this example the utility functions are separable, with a linear utility for one of the goods. This is reminiscent of the utility function traditionally used to get a TU effect in an NTU pure exchange private goods economy (see Aumann and Shapley (1974), Sections 30 and 34, and the sources quoted there). The resemblance is, however, superficial. The presence of a private good with a separable linear utility is tantamount to having side payments, and in particular $\lambda$ must coincide with $\mu$. Here the linear good is public, not private, and so cannot be used for side payments; and we have already seen that $\lambda$ need not coincide with $\mu$.

*Example* 6: Public Bads. Without the monotonicity assumption, Assumption 3, our theorem fails. To show this, we modify Example 5 by adding a public good $y^0$ with a negative utility (called a "bad"), and a resource $z^0$ that can be used only to produce the bad. Formally,

$$u_t(y^0, y) = f_t(y^1) + y^2 - y^0$$
$$G(z^0, z) = \{(y^0, y) \in E_+^3 : y^1 + y^2 \leqq z,\ y^0 \leqq z^0\},$$

where $y$, $z$, $f_t$, and the endowment $\boldsymbol{e}$ of the "old" resource $z$ are exactly as in Example 5. The total endowment $e^0(T)$ of the new (or "nuisance") resource is 1, but no particular relationship between $\boldsymbol{e}$ and $\boldsymbol{e}^0$ is assumed.

In both the voting and the non-voting games, majority coalitions near the diagonal will produce no bads, and will produce public goods in exactly the same amounts as in Example 5. But a minority coalition near the diagonal will behave differently in the voting and non-voting games. In the voting game, it has no licence to produce, and must simply consume what is produced by the majority. But in the non-voting game, it will produce all it can of the public bad, since it suffers less than the majority from it (because it is smaller than the majority), and its aim is to minimize the difference between the majority's payoff and its own; and as in Example 5, it will produce nothing of the original public "goods" $y^1$ and $y^2$.

In the voting game, therefore, the value outcome is exactly the same as in Example 5. In the non-voting game, we obtain for $\theta > \frac{1}{2}$

$$q(\theta T \cup ds) - q(\theta T) = \theta e(ds) + \theta e^0(ds) + (f_s(y^1_\lambda) + \theta - y^1_\lambda - (1-\theta))\lambda(ds).$$

The right side here differs from that in (9) in two places: first, in the term $\theta e^0(ds)$, which is the total increment caused to the entire coalition by having $ds$ deny its "bad" resource $e^0(ds)$ to the minority opposition coalition; and second, in the term $-(1-\theta)$ which now appears in the terms describing the utility of $ds$ for the entire bundle now present, and which is due to the production of public bad by the minority coalition. In a similar manner, we obtain

$$q(\theta T \cup ds) - q(\theta T) = -\theta e(ds) - \theta e^0(ds) + \lambda(s)(f_s(y^1_\lambda) + (1-\theta) - y^1_\lambda - \theta)\lambda(ds)$$

when $\theta < \frac{1}{2}$. Proceeding as in (11), we obtain

$$(\phi v_\lambda)(ds) = [f_s(y^1_\lambda) + (1 - y^1_\lambda)]\lambda(ds) + \tfrac{1}{4}e(ds) + \tfrac{1}{4}e^0(ds) - \tfrac{1}{2}\lambda(ds). \tag{12}$$

Since a value outcome is Pareto optimal, no public bad is produced in the end; therefore the expression for the value outcome, is, as in Example 5, of the form $(0, y^1_\lambda, 1 - y^1_\lambda)$, and hence its utility to $ds$ is given by

$$[f_s(y^1_\lambda) + (1 - y^1_\lambda)]\lambda(ds). \tag{13}$$

(The value outcome itself is of course not as in Example 5, since $\lambda$ is different, as we shall soon see.) Equating (13) with $(\phi v_\lambda)(ds)$ and using (12), we deduce $\lambda = (e + e^0)/2$; this is quite different from the comparison measure obtained for the voting game, namely $\lambda = e$. Specifically, in the logarithmic example calculated at the end of Example 5, we obtain

$$y^1_\lambda = \begin{cases} \int_T y^1_t e(dt) & \text{in the voting game} \\ \int_T y^1_t (e(dt) + e^0(dt))/2 & \text{in the non-voting game.} \end{cases}$$

*Example* 7. Our theorem also fails if we modify the voting game so that the majority is allowed to expropriate some of the resources of the minority against its will. To show this, we change Example 6 slightly, by making the "bad" $y^0$ into a "good", i.e., setting

$$u_t(y^0, y) = f_t(y^1) + y^2 + y^0,$$

and specifying that the resource $z^0$ is expropriable against the minority's will. The production function $G$ and all other features of the example are as before.

If he wishes, the reader may think of $z$ as labour (or "time"), and $z^0$ as land. A person can "destroy" his productive time simply by refusing to work, and thus avoid taxation; but land cannot be destroyed (compare Section 8a).

In the non-voting game, land and labour play similar roles; the calculations are like those of Examples 5 and 6, and yield $\lambda = (e + e^0)/2$. In the modified voting game, though, private ownership of the land is essentially meaningless, since the majority can—and therefore will—always expropriate the land. The calculations then yield $\lambda = e$. Thus the result is as in Example 6; the formula for the value outcome $y^1_\lambda$ in the logarithmic case is of course also the same.

Both this example and the previous one are examples of "public goods games" in the sense of AKN. This implies that the outcomes are independent of the voting weights, as indeed is apparent from the results ($\nu$ does not enter the formulas). But as these examples show, the stronger theorem of this paper fails for these games. In a sense, resources that can be expropriated by the majority against the will of the minority are to all intents and purposes public property, and cannot enter the calculations like privately held resources.

## 7. FORMAL PROOF

The proof of our theorem uses much of the machinery developed in connection with the proof of the main theorem of AKN, in Section 7 of that paper. Rather than reviewing all this material, we simply assume that it is before us, and continue the development from there. For convenience, we use a separate numeration of formulas in this section, starting from (7.10); Formulas (7.1) through (7.9) are in AKN.

Denote the non voting and voting games by $A$ and $B$ respectively. Set $r = v^A - v^B$. As in Section 7 of AKN, we may assume that $u_t(y)$ is strictly positive. Let $\varepsilon > 0$ be given.

First assume (i) in Assumption 5. For each $j$ with $1 \leqq j \leqq m$ and each $y$ in $C$, define

$$U^y_j\,(S) = \int_S u^j_t(y)(dt),$$

where $u^j_t(y) = \partial u_t(y)/\partial y^j$. Note that $U^y_j\,(T) > 0$, since $u^j_t(y) > 0$ for all $j$; define a probability measure $\hat{U}^y_j$ by $\hat{U}^y_j = U^y_j/U^y_j\,(T)$. Let $\Psi$ consist of all the measures $\hat{U}^y_j$, together with the $\hat{U}^y$ defined in Section 7 of AKN, and the voting measure $\nu$; let $\mathcal{D} = U(\Psi, \varepsilon)$. By Lemma 7.7, $\Psi$ is compact, and hence by Lemma 7.8, $\mathcal{D}$ is a diagonal neighbourhood.

Before proceeding, we note that

$$\frac{\partial}{\partial y^j} U^y(S) = U^y_j\,(S); \tag{7.10}$$

this follows from Lebesgue's dominated convergence theorem and the mean value theorem (which implies that the difference ratios tending to $\partial u_t(y)/\partial y^j$ are uniformly bounded). Again using the mean value theorem, we obtain from (7.2) and (7.10) that if $x$, $y$, and $x+y$ are in $C$, then for any $S$ there is a point $z$ on the line segment connecting $x$ to $x+y$ such that

$$H^{x+y}(S) - H^x(S) = \textstyle\sum_{j=1}^m y^j (U^z_j\,(S) - U^z_j\,(T\backslash S)). \tag{7.11}$$

Now let $\varnothing = S_0 \subset \cdots \subset S_k = T$ be a chain in $\mathcal{D}$, which we call $\Omega$. Let $i_1$ be the greatest index for which $\nu(S_1) < \frac{1}{2} - \varepsilon$, and $i_2$ the smallest index for which $\nu(S_i) > \frac{1}{2} + \varepsilon$. Define five chains $\Omega_1, \ldots, \Omega_5$ exactly as in Section 7 of AKN. First, let $i \geqq i_2$. From the definition of $\mathcal{D}$ it follows that $\hat{U}^z_j\,(S_i) > 1/2$ for all $z$ in $C$ and all $j$; hence $\hat{U}^z_j\,(T\backslash S_i) < \frac{1}{2}$, and therefore $U^z_j\,(S_i) - U^z_j\,(T\backslash S_i) > 0$. Thus if $x$, $y$, and $x+y$ are in $C$, then (7.11) yields

$$H^{x+y}(S_i) \geqq H^x(S_i). \tag{7.12}$$

Since $\nu(S_i) > \frac{1}{2}$, the coalition $S_i$—but not its complement $T\backslash S_i$—may produce public goods in the voting game $B$, and hence

$$w^B(S_i) = \max_{x \in G(e(S_i))} H^x(S_i). \tag{7.13}$$

In the non-voting game $A$, both $S_1$ and $T\backslash S_i$ may produce; hence by (7.12), by

$0 \in G(e(T\backslash S_i))$ (Assumptions 3 and 4), and by (7.13),

$$w^A(S_i) = \max_{x \in G(e(S_i))} \min_{y \in G(e(T\backslash S_i))} H^{x+y}(S_i)$$
$$= \max_{x \in G(e(S_i))} H^x(S_i) = w^B(S_i). \tag{7.14}$$

Hence $v^A(S_i) = v^B(S_i)$, hence $r(S_i) = 0$, and so $\|r\|_{\Omega_5} = 0$. If $i \leqq i_1$, the same proof applied to $T\backslash S_i$ instead of $S_i$ shows that $r(S_i) = -r(T\backslash S_i) = 0$, whence $\|r\|_{\Omega_1} = 0$.

Next, proceeding exactly as in Section 7 of AKN (using $v^A$ and $v^B$ instead of $v^\nu$ and $v^\zeta$), one shows that $\|r\|_{\Omega_3} < 16\varepsilon K$, $\|r\|_{\Omega_2} < 8\varepsilon K$, and $\|r\|_{\Omega_4} < 8\varepsilon K$. Thus

$$\|v^A - v^B\|_\Omega = \|r\|_\Omega < 0 + 8\varepsilon K + 16\varepsilon K + 8\varepsilon K + 0 = 32\varepsilon K.$$

Hence by Lemmas 7.5 and 7.6, the proof of our theorem under the differentiability assumption, Assumption 5(i) is complete.

Finally, assume (ii) in Assumption 5, i.e. that there are only finitely many utility types $T_1 \ldots, T_h$; thus all agents in a fixed $T_j$ have the same utility function $u_j$, and $\bigcup_{j=1}^h T_j = T$. Define

$$\lambda_j(S) = \begin{cases} \lambda(S \cap T_j)/\lambda(T_j) & \text{if } \lambda(T_j) > 0, \\ 0 & \text{if } \lambda(T_j) = 0. \end{cases}$$

Let $\Psi$ consist of the voting measure $\nu$ and all the $\lambda_j$, and let $\mathcal{D} = U(\Psi, \varepsilon)$. Since $\Psi$ is finite, $\mathcal{D}$ is by definition a diagonal neighbourhood. Let $\varnothing = S_0 \subset \cdots \subset S_k = T$ be a chain in $\mathcal{D}$, which we call $\Omega$. Let $i_1$ be the greatest index for which $\nu(S_i) < \frac{1}{2} - \varepsilon$, and $i_2$ the smallest index for which $\nu(S_i) > (1/2) + \varepsilon$. Define five chains $\Omega_1, \ldots, \Omega_5$ exactly as in Section 7 of AKN.

First let $i > i_2$. From the definition of $\mathcal{D}$ it follows that when $\lambda(T_j) > 0$, then $\lambda_j(S_i) > \frac{1}{2}$, hence $\lambda_j(T\backslash S_i) < \frac{1}{2}$, and hence $\lambda_j(S_i) - \lambda_j(T\backslash S_i) > 0$. Hence for all $x$ and $y$ in $E_+^m$, the monotonicity of the utilities (Assumption 3) yields

$$\begin{aligned} H^{x+y}(S_i) &= U^{x+y}(S_i) - U^{x+y}(T\backslash S_i) \\ &= \textstyle\sum_{j=1}^h \lambda(T_j) u_j(x+y)(\lambda_j(S_i) - \lambda_j(T\backslash S_i)) \\ &\geqq \textstyle\sum_{j=1}^h \lambda(T_j) u_j(x)(\lambda_j(S_i) - \lambda_j(T\backslash S_i)) = H^x(S_i). \end{aligned}$$

Now this is precisely (7.12), and the remainder of the proof is as in the differentiable case.

## 8. DISCUSSION

a. *Description of the Voting Game.* In describing the voting game, we specified that only majority coalitions were permitted to produce, using their own resources only. This appears different from the corresponding description in Aumann and Kurz (1977*a*), in which it was specified that the majority may expropriate resources from the minority, but that faced with expropriation, the minority may destroy part or all of its resources.

In fact, the descriptions are equivalent. Since the minority may destroy its own resources, and the majority may also effectively destroy the minority's resources (simply by refusing to use them), destruction of the minority's resources is an option available to either side. But the zero-sum nature of the threat game implies that for some pair of optimal strategies, any option available to both sides will be taken up by at least one of them, since at least one of the sides will gain (or at least not lose) by doing so. Thus there is no loss of generality in specifying that the minority's resources will in fact be destroyed.

b. *The Coase Theorem and Related Issues.* The "Theorem" of Coase (1960) asserts, among other things, that property rights do not affect the level at which economic activities that generate externalities are performed.[13] For example, suppose steam locomotives emit sparks that damage crops on nearby land. The Coase Theorem states that whether the train runs is independent of whether the railroad must recompense the farmer for lost crops. If profits from running the train exceed the value of the lost crops, it will run; otherwise it won't. Property rights determine the level (and direction) of side payments, but not that of the actual activities.

This is subject to a lot of "ifs": no transaction costs, no income effects,[14] and so on. In effect, one must assume a transferable utility (TU) context. But in that case ordinary Pareto optimality dictates that the protagonists always engage in whatever activities are necessary to maximize the sum of the payoffs, regardless of the capabilities (or "rights") of individuals or subcoalitions (who can always be compensated by transfers). Thus the Coase Theorem is simply an expression of Pareto optimality in the context of externalities.[15]

Public goods are a classic instance of externalities, and the vote may be considered analogous to property rights. Thus our result, which implies that the choice of public goods is independent of who has the vote, sounds like a version of the Coase Theorem. But the resemblance is superficial; our result goes much further.

To clarify this issue, note first that as stated, our model permits no side payments. When side payments are impossible, the Coase Theorem predicts only Pareto optimality, and, of course, there are in general many Pareto optimal outcomes. The choice among these outcomes may very well be affected by property rights; if the railroad is prevented from compensating the farmer, his property rights may be decisive in determining whether the train will run. Similarly, one would expect that when side payments are impossible, the vote *does* affect the choice of public goods. But our theorem says that it does not.

Consider next the TU version of our model, in which it is possible to make side payments so that the utility lost by the payor always precisely equals that gained by the payee. This yields a model like the one in this paper, but with an exogenous comparison measure $\lambda$ that expresses actual rates of exchange. In that case the Coase principle (i.e. the principle of Pareto optimality) leads us to expect a choice of public goods that is independent of the voting weights, but with compensation between the agents that *does* depend on the voting weights. Here again our theorem goes much further: it says that the choice of public goods *and* the schedule of compensations between the agents is independent of the vote.

Finally, consider the possibility of making side payments that cause the payor to lose and the payee to gain utility, but not necessarily in equal amounts.[16] In spite of appearances this is essentially an NTU (non-transferable utility) situation, much like that in which no side payments at all are permitted. While we have not considered this model explicitly, our methods can be applied to it in a straightforward fashion; the conclusion is that as in the cardinal TU case, both the choice of public goods *and* the schedule of side payments are independent of the vote. This is to be contrasted with the Coase Theorem, which with similar side payments allows both the level of externality-producing activity and the schedule of side payments to depend on property rights, stipulating only that the overall outcome be Pareto optimal.

The question arises, what economic factors that are absent in the Coase case account for our strong results. Part of the answer is that we deal with a large number of individually insignificant agents—"a perfect competition" context, so to speak; whereas Coase considers any number of agents, and in fact most of his examples have just two. There is more to it than that, though; the reader is invited to consult Section 6 of AKN.

c. *Existence of Value Outcomes*. This paper has concentrated on equivalence results—on relationships between value outcomes of different games—and has avoided questions of existence. We have proved a statement of the form "two games have the same set of value outcomes," it being understood that the set may be empty. Here we briefly address the existence problem.

This problem divides naturally into two parts: (i) Existence of an asymptotic value for the game $v_\lambda$ with given $\lambda$, and (ii) given a positive solution to (i), finding equilibrium $\lambda$ (i.e. solving (1)).

As far as (i) is concerned, it is likely that a differentiability assumption in the spirit of Assumption 5(i) would be sufficient to ensure existence of an asymptotic value for the non-voting and voting games $v_\lambda^A$ and $v_\lambda^B$. The proof would perhaps use methods similar to those used for exchange economies (Aumann and Shapley (1974), Chapters VI and VII), and the finite dimensionality of the outcome space would further simplify matters.[17]

Without differentiability matters become more problematic. Even with a finite type assumption like Assumption 5(ii), the asymptotic value in general fails to exist in the case of exchange economies (op. cit., Section 19), and presumably for our public goods economies as well. Hart (1977) has demonstrated the existence of an asymptotic value for non-differentiable exchange economies obeying a certain symmetry condition; perhaps a similar result could be proved here.

For (ii), the main problem would be continuity of the value $\phi v_\lambda$ as a function of $\lambda$. With differentiability, this would probably be OK. But without differentiability, it would cause difficulties even when (i) is satisfactorily resolved; the value, which is a kind of average derivative, would not in general be continuous in $\lambda$ when one passes over a kink.

Summing up, it appears that an appropriate differentiability condition is probably sufficient to ensure existence of a value outcome, and that for a general existence theorem, one cannot get away with much less. On the other hand, a *generic* existence theorem may well be provable without any kind of differentiability condition.

We stress that we do not have any existence proof; the remarks in this subsection should be considered conjectures.

d. *The Role of Differentiability*. The proof of the theorem depends on Formula (7.14), which asserts that $w^A(S)=w^B(S)$ for any coalition $S$ that is "close to the diagonal" and has a "considerable" majority; more precisely, that

$$\text{for any } \delta>0 \text{ there is a diagonal neighbourhood } \mathcal{D} \text{ such that } w^A(S)=w^B(S) \text{ whenever } S\in\mathcal{D} \text{ and } \nu(S)>\tfrac{1}{2}+\delta. \tag{13}$$

Assumption 5—that the utilities are either differentiable or are of finitely many different types—is essential to prove (13): It is possible[18] to construct a public goods economy satisfying Assumptions 1 through 4—but not 5—and violating (13). Thus Assumption 5 is essential for our line of proof. It is not known whether the theorem is actually false without it.

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NOTES

1. See Section 9a for an alternative description of the voting game, parallel to that used for the redistribution games of Aumann and Kurz (1977*a*, *b*) (i.e. allowing for expropriation of the minority's resources by the majority, and their destruction by the minority).
2. Shapley (1969).
3. An isomorphism is a one-to-one correspondence that is measurable in both directions.
4. $A$ function is $C^1$ on a closed set $A$ if it can be extended to a $C^1$ function on an open set containing $A$.
5. See Section 4 of Aumann and Kurz (1977*b*).
6. Kannai (1966).
7. $v_\lambda^\Gamma$ is formally defined in Section 7 of AKN; for an informal definition, see (3).

8. In general, the outcomes arising in the analysis of the non-voting game are "mixed", i.e. random variables whose values are pure outcomes; but for simplicity, the informal discussion of this section is restricted to pure outcomes. For the general case, one need only replace pure outcomes $y$ by mixed outcomes $y$, and the utilities $u_t(y)$ by expected utilities $Eu_t(y)$. The voting game always leads to pure outcomes.

9. i.e., has similar characteristics, is statistically similar.

10. Example 7.1 there; it differs from this example only in that there, the vote and population measures are the same.

11. This formulation of the strategic game is different from, but equivalent to, that of Aumann and Kurz (1977*a*). See the discussion in Section 8a.

12. The perspicacious reader will have observed that (5) (or (7) applied directly to this $q$ yields a result different from $(3/4)\mu+(1/4)e$, in fact one that is obviously "wrong" in that it does not satisfy the efficiency axiom for values $((\phi q)(T)=q(T))$. This demonstrates once more that rough, intuitive methods have their limitations, and are no substitute for careful proofs.

13. The assertion concerns rational economic agents who are permitted to trade in their property rights.

14. Dolbear (1967).

15. We are referring to the principle expounded by Coase (1960), not to later developments that discuss the formation of markets in externalities.

16. We called this "ordinal TU" in Section 9 of AKN, to distinguish it from the previous case, which we called "cardinal TU" there, and which is the plain "TU" of most of the literature.

17. Exchange economies have infinite dimensional outcome spaces.

18. See Example 6.18, p. 36 of "Public Goods and Power", by R. Aumann, M. Kurz, and A. Neyman, TR 273 (revised) of the Institute for Mathematical Studies in the Social Sciences (Economics), Stanford University, September 1980.

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