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WEIGHTED MAJORITY GAMES HAVE ASYMPTOTIC VALUE*†

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The asymptotic value of a game v with a continuum set of players, I, is defined whenever all the sequences of the Shapley values of finite games that "approximate" v have the same limit. A weighted majority game is a game of the form $f \circ \mu$ where μ is a positive measure and $f(x) = 1$ if $x \geqslant q$ and $f(x) = 0$ otherwise, and q is a real number, $0 < q < \mu(I)$. In this paper we prove that all weighted majority games have asymptotic values. This result is then used further to show that if v is of the form $v = f \circ \mu$, where μ is a probability measure and f is a function of bounded variation on $[0,1]$ that is continuous at 0 and at 1, then v has an asymptotic value. This had previously been known only when f is absolutely continuous, or when μ has at most finitely many atoms or when μ is purely atomic. Thus, the essential novelty is that even when μ has countably many atoms and a nonatomic part, $f \circ \mu$ has an asymptotic value. We also show that $f \circ \mu$ does not necessarily have an asymptotic value when μ is a signed measure.

1. Introduction. The Shapley value is one of the basic solution concepts of cooperative game theory. It measures the payoff that each player can expect to obtain, "on the average," by playing the game. The Shapley value for games with finitely many players was introduced by Shapley (1953), as an operator that associates to each game a corresponding vector of payoffs to the players; this operator is uniquely determined by a number of plausible axioms.

Starting in the late fifties, one of the main lines of study of the behavior of the value has been in the context of "large games" - where there are "many players", some of which are almost insignificant.[1] This models situations that frequently occur in economic and political institutions. A class of such games, called "oceanic" games, has been studied by Milnor and Shapley (1961), Shapiro and Shapley (1960), Shapley (1961) and Hart (1973). These are weighted majority games in which a sizeable fraction of the total vote is controlled by a few large ("major") players, and the rest is distributed among a large number of small ("minor") voters. Shapiro and Shapley (1960), Milnor and Shapley (1961), and Shapley (1961) presented asymptotic results for the values of the major players, when the minor ones become smaller and smaller. A more difficult task turned out to be finding the limit of the values of the minor players. Even in the case where there are no major players at all (thus, each player controls a negligible fraction of the total vote), this was an open problem for many years. This first case was solved by Neyman (1981). It was then extended to the case of finitely many major players in a later paper Neyman (1979).

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[1]Kuhn and Tucker list 14 outstanding research problems in their preface to the first volume of *Contributions to the Theory of Games* (1950). The eleventh urges us "to establish significant asymptotic properties of n-person games, for large n" [12, p. xii].

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A parallel direction of investigation was the extension of the value to weighted majority games with countably many players. Such a game is given as an ordered pair (q, W), where q is a real number and W is a measure on the nonnegative integers $\mathbf{N}$ such that $0 < q < W(\mathbf{N})$. The set of players is $\mathbf{N}$; a coalition $S \subset \mathbf{N}$ is winning if and only if $W(S) \geqslant q$. If the voting measure W has finite support then the weighted majority game (q, W) becomes a finite game, and the value ψ_i of player i is just the probability of i being a pivot in a random order of the players. In the general case (when the voting measure does not have finite support), the value ψ_i of player i is defined similarly, as the probability of i being a pivot in a random order[2] of the countable set of players $\mathbf{N}$. It turns out (Shapiro and Shapley 1971, Theorem 16, or Artstein 1971) that ψ_i is the limit of the values ψ_i^k of i in the truncated games (q, W^k), where $W^k(j) = W(j)$ if $j \leqslant k$ and $W^k(j) = 0$ otherwise. However, whether the resulting ψ is efficient (i.e., $\sum_{i=1}^{\infty}\psi_i = 1$) for all such games was an open problem for may years; it was recently answered in the affirmative by Berbee (1981).

The analysis of the behavior of the value in the n-person games as n becomes large finds a natural and more general setting in the framework of values of games with a continuum of players. Such games are functions v from $\mathscr{C}$ to the reals, with $v(\varnothing) = 0$, where $(I, \mathscr{C})$ is a measurable space isomorphic to $([0,1], \mathscr{B})$ (where $\mathscr{B}$ is the σ-field of Borel sets on $[0,1]$. Here, I is interpreted as the set of players, and $\mathscr{C}$ as the family of possible coalitions. The value for such games is a generalization of the Shapley value for finite games due to Aumann and Shapley (1974). Of special interest are values that are obtained as limits of values of finite approximants. The asymptotic value is the "strongest" possible such value in the sense that, if it exists for a particular game v, then any limiting value[3] will exist for that game and will equal the asymptotic one. Briefly, the asymptotic value is defined on each game v for which all the sequences of Shapley values, corresponding to sequences of finite games that "approximate" v, have the same limit. Any result asserting that a given game has an asymptotic value is essentially a result about the limit behavior of the values of (finite) games with many players.

The asymptotic value has been studied extensively (Aumann and Shapley 1974, Dubey 1980, Fogelman and Quinzii 1980, Hart, 1977, Kannai 1966, Neyman 1979, 1981, 1982). Let μ be a probability measure on the measurable space of players $(I, \mathscr{C})$ and let f be a monotonic function on $[0,1]$ with $f(0) = 0$. It has long been known (Kannai 1966 and Aumann and Shapley 1974) that when μ is nonatomic and f is absolutely continuous, $f \circ \mu$ has an asymptotic value. Fogelman and Quinzii (1980) showed that whenever f is absolutely continuous and μ has at most finitely many atoms, $f \circ \mu$ has an asymptotic value. Neyman (1979, 1981) showed that $f \circ \mu$ has an asymptotic value whenever f is continuous at 0 and 1 and μ has at most finitely many atoms. Berbee's result (1981) implies that $f \circ \mu$ has an asymptotic value whenever f is a jump function and μ is purely atomic. The present paper asserts that $f \circ \mu$ has an asymptotic value for every probability measure μ and all monotonic functions f that are continuous at 0 and 1.

The set of all games having an asymptotic value is a linear space and thus the result implies that f could be any function in bv' where bv' denotes all functions from $[0,1]$ to the reals with $f(0) = 0$ that are continuous at 0 and 1 and are of bounded variation. Moreover, the space, ASYMP, of all games that are of bounded variation (equivalently, that are the difference of two monotonic games) and have an asymptotic value is a closed (in the bounded variation norm) subspace of BV - the space of all games having

[2] An order of a finite set is usually identified with a permutation; this is not so in the countable case.
[3] Like the μ-value [2], [10], [15] and the partition value [19].

bounded variation. Thus, if M stands for all measures on $(I, \mathscr{C})$, and $bv'M$ is the closed subspace of BV generated by games of the form $f \circ \mu$ where $f \in bv'$ and μ is a probability measure in M, then our main result is

THEOREM A. *$bv'M \subset ASYMP$.*

In §2, the formal definitions and statement of results are given. §3 contains formulas for the value for finite games and the formal statement of two previously known results that are essential for our proof. §4 contains the proof of Theorem A. In §5, we show by means of counterexamples that there is no hope to extend our result to signed measures as well, and indicate other possible extensions.

2. Statement of results. We begin by recalling that a *coalitional game*, or *game* for short, is a real valued function v on the σ-field $\mathscr{C}$ of a measurable space $(I, \mathscr{C})$, with $v(\varnothing) = 0$. It is *monotonic* if $S, T \in \mathscr{C}$ and $T \subset S$ imply that $v(S) \leqslant v(T)$; it is of *bounded variation* if it is the difference between two monotonic games. The game v is finite if $\mathscr{C}$ is finite. The Shapley value for a finite game v is the measure on $\mathscr{C}$ given by

$$(\psi v)(a) = \frac{1}{n!} \sum_{\mathscr{R}} \left[v\left(\mathscr{P}_a^{\mathscr{R}} \cup a\right) - v\left(\mathscr{P}_a^{\mathscr{R}}\right) \right]$$

where the sum runs over all orders $\mathscr{R}$ of the players (atoms of $\mathscr{C}$) and $\mathscr{P}_a^{\mathscr{R}}$ is the union of all atoms preceding a (an atom of $\mathscr{C}$) in the order $\mathscr{R}$. Given $T \in \mathscr{C}$, a T-admissible sequence is an increasing sequence $(\Pi_1, \Pi_2, \ldots,)$ of finite fields such that $T \in \Pi_1$ and $\bigcup_i \Pi_i$ generates $\mathscr{C}$. Given a finite subfield Π of $\mathscr{C}$, the restriction of v to Π, v_Π, is a finite game (on (I, Π)). A game ϕv is said to be the *asymptotic value* of v, if for every $T \in \mathscr{C}$ and every T-admissible sequence $(\Pi_i)_{i=1}^{\infty}$, the following limit and equality exists:

$$\lim_{k \to \infty} \psi v_{\Pi_k}(T) = \phi v(T).$$

The set of all games v (on $(I, \mathscr{C})$) of bounded variation and having an asymptotic value is denoted by ASYMP.

The essential part of the main result (of the present paper) is that whenever $f \in bv'$, where bv' is the space of all functions f of bounded variation from $[0, 1]$ to the reals with $f(0) = 0$ and f continuous at 0 and 1, and μ is a probability measure on $(I, \mathscr{C})$, then $f \circ \mu$ is in ASYMP. There are properties of the set ASYMP that enable to deduce a stronger result. The space of all games of bounded variation is denoted BV. The *variation norm* of v is defined by $\|v\| = \inf(u(I) + w(I))$, where the inf ranges over all monotonic functions u and w such that $v = u - w$. The set of all measures on $(I, \mathscr{C})$ is denoted by M. The closed (in the bounded variations norm) linear subspace of BV that is generated by games of the form $f \circ \mu$, $f \in bv'$ and μ is a probability measure in M is denoted $bv'M$. Our main result is

THEOREM A. *$bv'M \subset ASYMP$.*

There are various subspaces of $bv'M$ which were known to be included in $ASYMP$. Let pNA be the closed subspace of BV that is generated by powers of nonatomic measures and pFL stands for the closed space generated by powers of positive measures with at most finitely many atoms. Let NA denote all nonatomic measures, FL all measures with at most finitely many atoms and M_a all purely atomic measures. Each of the spaces, $bv'NA$, $bv'FL$ and $bv'M_a$ is defined as the closed subspace of BV

that is generated by games of the form $f \circ \mu$ where $f \in bv'$ and μ is a probability measure in NA, FL, or M_a respectively.

Kannai (1966) and Aumann and Shapley (1974) show that $pNA \subset ASYMP$. Fogelman and Quinzii (1980) show that $pFL \subset ASYMP$. Neyman (1981) and (1979) show that $bv'NA \subset ASYMP$ and $bv'FL \subset ASYMP$ respectively. Berbee (1981) together with Shapiro and Shapley (1971, Theorem 14) and the proof of [17, Lemma 8 and Theorem A] imply that $bv'M_a \subset ASYMP$. It was further announced in [16] that Berbee's result implies the existence of a partition value on $bv'M$. However, whether $bv'M$ is contained in $ASYMP$ was an open problem.

3. Preliminaries. This section recalls formulas for the Shapley value of finite games, two theorems that are basic for our present proof, and few well-known properties of the value.

Let (A, v) be a finite game, i.e., A is finite set and $v\colon 2^A \to \mathbf{R}$ with $v(\varnothing) = 0$. The Shapley value of the game v to player $a \in A$ is given by

$$\psi v(a) = (1/|A|!)\sum_{\mathscr{R}} v\left(\mathscr{P}_a^{\mathscr{R}} \cup \{a\}\right) - v\left(\mathscr{P}_a^{\mathscr{R}}\right) \tag{3.1}$$

where $|A|$ stands for the number of elements in A, the summation ranges over all orders $\mathscr{R}$ of the player set A and $\mathscr{P}_a^{\mathscr{R}}$ denotes the set of all players that precede a in the order $\mathscr{R}$. An alternative formula for $\psi v(a)$ could be given by means of a family X_a, $a \in A$, of i.i.d random variables that are uniformly distributed on $(0, 1)$. The values of X_a, $a \in A$, induce with probability one an order $\mathscr{R}$ on A; a precedes b if and only if $X_b > X_a$. As X_a, $a \in A$, are i.i.d and nonatomic, all orders are equally likely. Thus,

(3.2)

$$\begin{aligned}\psi v(a) &= E(v(\{b \in A\colon X_b \leqslant X_a\}) - v(\{b \in A\colon X_b < X_a\})) \\ &= \int_0^1 E(v(\{b \in A\colon X_b \leqslant t\} \cup \{a\}) - v(\{b \in A\colon X_b \leqslant t\} \setminus \{a\}))\, dt.\end{aligned}$$

When the game v is a weighted majority game, it is described by a pair (q, W) where q is a real number and W is a measure on A with $0 < q \leqslant \Sigma_{a \in A} W(a)$. For $S \subset A$, $v(S) = 1$ if $\Sigma_{a \in S} W(a) \geqslant q$ and $v(S) = 0$ otherwise. In that case formula (3.2) could be rewritten as

$$\begin{aligned}(\psi v)(a) &= E\left(I\left(q \leqslant \sum_{b \in A} W(b) I(X_b \leqslant X_a) < q + W(a)\right)\right) \\ &= \int_0^1 E\left(I\left(q - W(a) \leqslant \sum_{\substack{b \in A \\ b \neq a}} W(b) I(X_b \leqslant t) < q\right)\right) dt.\end{aligned} \tag{3.3}$$

For a subset B of A the Shapley value $\psi v(B)$ is given by $\psi v(B) = \Sigma_{a \in B} \psi v(a)$ and if v is a weighted majority game then formula (3.3) yields

$$(\psi v)(B) = E\left(\sum_{b \in B} I\left(q \leqslant \sum_{a \in A} W(a) I(X_a \leqslant X_b) < q + W(b)\right)\right). \tag{3.4}$$

The proof of the main result of the paper will make use of two previously known results in value theory, which for completeness will be now stated. The first one is the main theorem of [18].

THEOREM 3.5. *For every* $\epsilon > 0$ *there exists* $K = K(\epsilon)$ *such that if* $v = (q, (W(a))_{a \in A})$ *is a weighted majority game with*

$$K \max_{a \in A} W(a) < q < \sum_{b \in A} W(b) - K \max_{a \in A} W(a),$$

then

$$\sum_{a \in A} \left| \psi v(a) - W(a) \Big\backslash \sum_{b \in A} W(b) \right| < \epsilon.$$

The second is the main theorem of [4].

THEOREM 3.6. *Let* $(\alpha_i)_{i=1}^{\infty}$ *be a countable sequence with* $\alpha_i \geqslant 0$ *and* $\Sigma \alpha_i = 1$. *Assume that* $(X_i)_{i=1}^{\infty}$ *is a sequence of i.i.d random variables that are uniformly distributed on* $[0,1]$. *Then for every* $0 < q < 1$, $\Sigma_{i=1}^{\infty} E(i) = 1$ *where*

$$E(i) = E\left(I\left(q - \alpha_i \leqslant \sum_{j=1}^{\infty} \alpha_j I(X_j < X_i) < q\right)\right.$$

$$= \int_0^1 E\left(I\left(q - \alpha_i \leqslant \sum_{j \neq i} \alpha_j I(X_j < t) < q\right)\right) dt.$$

The dual of a finite game (A, v) is a game (A, v^*) where $v^*(S) =: v(A) - v(A \setminus S)$. It is well known that $\psi v = \psi v^*$. Also if v is an arbitrary game, i.e., $v: \mathscr{C} \to \mathbf{R}$ with $v(\varnothing) = 0$ then the dual of v, v^*, is given by $v^*(S) = v(I) - v(I \setminus S)$ for every $S \in \mathscr{C}$. The game v has an asymptotic value if and only if its dual has, and then the asymptotic values coincide. If f: $[0,1] \to \mathbf{R}$, $f(0) = 0$ and μ is a probability measure then it is easily verified that $(f \circ \mu)^* = f^* \circ \mu$ where $f^*(x) = f(1) - f(1-x)$.

4. The asymptotic value on $bv'M$. In this section we will prove the main result of this paper, namely, that $bv'M \subset$ ASYMP.

For $0 < q < 1$ we denote by f_q the real valued function on $[0,1]$ that is given by $f_q(x) = 1$ if $x \geqslant q$ and $f_q(x) = 0$ if $x < q$.

THEOREM 1. *For any probability measure* μ *on* $(I, \mathscr{C})$ *and for any* $0 < q < 1$, $f_q \circ \mu \in ASYMP$.

PROOF. We have to show the existence of a finitely additive set function $\phi(f_q \circ \mu)$ such that for any increasing sequence $(\Pi_n)_{n=1}^{\infty}$ of finite subfields of $\mathscr{C}$ such that $\bigcup_i \Pi_i$ generates $\mathscr{C}$, and $T \in \Pi_1$ we have

$$\lim_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(T) = \phi(f_q \circ \mu)(T), \tag{2}$$

where ψ denotes the Shapley value for finite games.

We will start by introducing a formula for $\phi(f_q \circ \mu)$. Let $\mu = \mu_A + \mu_{NA}$ be the decomposition of μ into a purely atomic measure μ_A and a nonatomic one μ_{NA}. Let $(y_i)_{i=1}^{\infty}$ be the atoms of the measure μ, with $1 \leqslant i < j < \infty \Rightarrow y_i \neq y_j$. Set $\alpha_i = \mu(\{y_i\}) = \mu_A(\{y_i\})$, $\alpha = 1 - \Sigma_{i=1}^{\infty} \alpha_i$. Let $(X_i)_{i=1}^{\infty}$ be a sequence of i.i.d random variables that are uniformly distributed on $(0,1)$. For $i \in \mathbf{N}$ and $0 < t < 1$, let $E(i, t)$ be defined by

$$E(i,t) = E\left(I\left(q - \alpha_i \leqslant \sum_{j \neq i} \alpha_j I(X_j \leqslant t) + t\alpha < q\right)\right) \tag{3}$$

and let $E(i)$ be defined by

$$E(i) = \int_0^1 E(i,t)\,dt. \tag{4}$$

Then $\phi(f_q \circ \mu)$ is given by: for $T \in \mathscr{C}$

$$\phi(f_q \circ \mu)(T) = \begin{cases} \sum_{\{i:\, y_i \in T\}} E(i) & \text{if } \alpha = 0, \\ \sum_{\{i:\, y_i \in T\}} E(i) + \left(1 - \sum_{i=1}^{\infty} E(i)\right)\mu_{NA}(T)/\alpha & \text{if } \alpha > 0. \end{cases} \tag{5}$$

LEMMA 6. *Let $(\Pi_n)_{n=1}^{\infty}$ be an increasing sequence of finite subfields of $\mathscr{C}$ for which $\bigcup_i \Pi_i$ generates $\mathscr{C}$. For any $i \in \mathbf{N}$ let $a^n(i)$ be the atom of Π_n that contains y_i. Then, for each fixed $i \in \mathbf{N}$,*

$$\lim_{n\to\infty} \psi(f_q \circ \mu)_{\Pi_n}(a^n(i)) = E(i). \tag{7}$$

PROOF. If $\mu_{NA} = 0$ and μ has only finitely many atoms, $\psi(f_q \circ \mu)_{\Pi_n}(a^n(i)) = E(i)$ for sufficiently large n, e.g. by (3.3). Therefore we assume that either $\alpha > 0$ or that μ has infinitely many atoms. As $(\Pi_n)_{n=1}^{\infty}$ is increasing it follows that for any $i \in \mathbf{N}$, $(a^n(i))_{n=1}^{\infty}$ is decreasing, and as $\bigcup_n \Pi_n$ generates $\mathscr{C}$ it follows that $\bigcap_{n=1}^{\infty} a^n(i) = y_i$ and therefore for any $i \in \mathbf{N}$, $\lim_{n\to\infty} \mu(a^n(i)) = \alpha_i$. If there are countably many atoms, then for every t in $(0,1)$

$$E\left(I\left(q - \alpha_i = \sum_{j \neq i} \alpha_j I(X_j \leqslant t) + t\alpha\right)\right) = 0, \tag{8}$$

and if $\alpha > 0$, (8) holds for all but finitely many values of t. Therefore, $E(i,t) = E(I(q - \alpha_i < \sum_{j\neq i} \alpha_j I(X_j \leqslant t) + t\alpha < q))$, for all but finitely many values of t. As $I(q - \alpha_i + \eta < \sum_{j\neq i}\alpha_j I(X_j \leqslant t) + t\alpha < q - \eta)$ increases as $\eta \to 0+$ to $I(q - \alpha_i < \sum_{j \neq i}\alpha_j I(X_j \leqslant t) + t\alpha < q)$, we deduce from Lebesgue monotone convergence theorem that for all but finitely many values of t,

$$E(i,t) = \lim_{\eta \to 0+} E\left(I\left(q - \alpha_i + \eta < \sum_{j\neq i} \alpha_j I(X_j \leqslant t) + t\alpha < q - \eta\right)\right)$$

and therefore for all but finitely many values of t, for every $\epsilon > 0$ there exists $\eta > 0$ such that

$$E\left(I\left(q - \alpha_i + \eta < \sum_{j\neq i} \alpha_j I(X_j \leqslant t) + t\alpha < q - \eta\right)\right) > E(i,t) - \epsilon.$$

There is $k \in \mathbf{N}$ s.t. $k \geqslant i$ and $\sum_{j=k+1}^{\infty} \alpha_j \leqslant \eta/4$, and there is n_0 such that for $n \geqslant n_0$, $1 \leqslant m < j \leqslant k \Rightarrow a^n(m) \neq a^n(j)$ and $1 \leqslant j \leqslant k \Rightarrow \mu(a^n(j)) - \alpha_j < \eta/4k$. Let A_n be all atoms of Π_n and let X_a^n, $a \in A_n$, be i.i.d random variables that are uniformly distributed on $(0,1)$ and we assume without loss of generality that for each fixed $n > n_0$, $X_{a^n(i)}^n = X_i$ for each $1 \leqslant i \leqslant k$. Let $A_{n,k} = A_n \setminus \{a^n(1), \ldots, a^n(k)\}$. Then, recalling that μ_{NA} is the nonatomic part of μ,

$$0 \leqslant \sum_{\substack{a \in A_n \setminus A_{n,k} \\ a \neq a^n(i)}} \mu(a) I(X_a^n \leqslant t) - \sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j I(X_j \leqslant t) \leqslant \eta/4 \quad \text{and}$$

$$0 \leqslant \sum_{a \in A_{n,k}} \mu(a) I(X_a^n \leqslant t) - \sum_{a \in A_{n,k}} \mu_{NA}(a) I(X_a^n \leqslant t) < \eta/4 \quad \text{and}$$

$$0 \leqslant \sum_{a \in A_n} \mu_{NA}(a) I(X_a^n \leqslant t) - \sum_{a \in A_{n,k}} \mu_{NA}(a) I(X_a^n \leqslant t) < \eta/4.$$

Therefore

$$-\eta/4 < \sum_{\substack{a \in A_n \\ a \neq a^n(i)}} \mu(a) I(X_a^n \leqslant t) - \left[\sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j I(X_j \leqslant t) + \sum_{a \in A_n} \alpha_{NA}(a) I(X_a^n \leqslant t) \right]$$

$$\leqslant \eta/2.$$

Thus

$$I\left(q - \alpha_i \leqslant \sum_{\substack{a \in A_n \\ a \neq a^n(i)}} \mu(a) I(X_a^n \leqslant t) < q \right)$$

$$\geqslant I\left(q - \alpha_i + \eta/4 \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j I(X_j \leqslant t) \right.$$

$$\left. + \sum_{a \in A_n} \mu_{NA}(a) I(X_a^n \leqslant t) < q - \eta/2 \right).$$

As $\sum_{a \in A_n} \mu_{NA}(a) I(X_a^n \leqslant t)$ converges in distribution to $t\alpha$ we deduce that for sufficiently large n,

$$E\left(I\left(q - \alpha_i + \eta/4 \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j I(X_j \leqslant t) + \sum_{a \in A_n} \mu_{NA}(a) I(X_a^n \leqslant t) < q - \eta/2 \right) \right)$$

$$\geqslant E\left(I\left(q - \alpha_i + \eta \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j I(X_j \leqslant t) + t\alpha < q - \eta \right) \right) - \epsilon.$$

As $\mu(a^n(i) \geqslant \alpha_i$,

$$I\left(q - \mu(a^n(i)) \leqslant \sum_{\substack{a \in A_n \\ a \neq a^n(i)}} \mu(a) I(X_a^n \leqslant t) < q\right)$$

$$\geqslant I\left(q - \alpha_i \leqslant \sum_{\substack{a \in A_n \\ a \neq a^n(i)}} \mu(a) I(X_a^n \leqslant t) < q\right).$$

Altogether, we conclude that

$$\varliminf_{n \to \infty} E\left(I\left(q - \mu(a^n(i)) \leqslant \sum_{\substack{a \in A_n \\ a \neq a^n(i)}} \mu(a) I(X_a^n \leqslant t) < q\right)\right)$$

$$\geqslant \varliminf_{n \to \infty} E\left(I\left(q - \alpha_i \leqslant \sum_{\substack{a \in A_n \\ a \neq a^n(i)}} \mu(a) I(X_a^n \leqslant t) < q\right)\right)$$

$$\geqslant \varliminf_{n \to \infty} E\left(I\left(q - \alpha_i + \eta/4 \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j I(X_j \leqslant t)\right)\right.$$

$$\left. + \sum_{a \in A_n} \mu_{NA}(a) I((X_a^n \leqslant t) < q - \eta/2\right) > E(i, t) - 2\epsilon.$$

As

$$\psi(f_q \circ \mu)_{\Pi_n}(a^n(i)) = \int_0^1 E\left(I\left(q - \mu(a^n(i)) \leqslant \sum_{\substack{a \in A_n \\ a \neq a^n(i)}} \mu(a) I(X_a^n \leqslant t) < q\right)\right) dt$$

we obtain by applying Fatou's Lemma that $\varliminf_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(a^n(i)) \geqslant E(i) - 2\epsilon$. A similar argument shows that

$$\varlimsup_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(a^n(i)) \leqslant E(i) + 2\epsilon.$$

As this holds for any $\epsilon > 0$ the lemma is proved. ∎

The next lemma uses the previous one together with a result of Berbee to prove Theorem 1 in the case that μ is purely atomic.

LEMMA 9. *If* $\mu_{NA} = 0$, *then* (2) *holds with* $\phi(f_q \circ \mu)(T) = \sum_{\{i:\, y_i \in T\}} E(i)$.

The Lemma is an almost direct corollary of [4] and [21, Theorem 14]; for completeness a proof is included here.

PROOF. Let $(\Pi_n)_{n=1}^{\infty}$ be an increasing sequence of subfields of $\mathscr{C}$ with $T \in \Pi_1$ and $\bigcup_n \Pi_n$ generating $\mathscr{C}$. Given $\epsilon > 0$ there is k s.t. $\sum_{\{1 \leqslant i \leqslant k:\, y_i \in T\}} E(i) \geqslant \sum_{\{i:\, y_i \in T\}} E(i) - \epsilon$. There is n_0 s.t. for $n \geqslant n_0$, $1 \leqslant i < j \leqslant k \Rightarrow a^n(i) \neq a^n(j)$. As $\psi(f_q \circ \mu)_{\Pi_n}(\cdot)$ is finitely additive and monotonic, the inclusion $\bigcup_{i=1\ y_i \in T}^{k} a^n(i) \subset T$ implies that $\psi(f_q \circ \mu)_{\Pi_n}(T) \geqslant \psi(f_q \circ \mu)_{\Pi_n}(\bigcup_{i=1\ y_i \in T}^{k} a^n(i))$ and as for $n \geqslant n_0$, $1 \leqslant i < j \leqslant k$, $a^n(i) \neq a^n(j)$ we deduce from the finite additivity of $\psi(f_q \circ \mu)(\cdot)$ that for $n \geqslant n_0$

$$\psi(f_q \circ \mu)_{\Pi_n}(T) \geqslant \sum_{\{i \leqslant k:\, y_i \in T\}} \psi(f_q \circ \mu)_{\Pi_n}(a^n(i))$$

and therefore by Lemma 6, that

$$\varliminf_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(T) \geqslant \sum_{\{i \leqslant k:\, y_i \in T\}} E(i)$$

which by the selection of k is $\geqslant (\sum_{\{i:\, y_i \in T\}} E(i)) - \epsilon$. As this holds for all $\epsilon > 0$, we conclude that for any $T \in \Pi_1$

$$\varliminf_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(T) \geqslant \sum_{\{i:\, y_i \in T\}} E(i) \tag{10}$$

and similarly that

$$\varliminf_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(T^c) \geqslant \sum_{\{i:\, y_i \in T\}} E(i).$$

By the efficiency of ψ, $\psi(f_q \circ \mu)_{\Pi_n}(T) = 1 - \psi(f_q \circ \mu)_{\Pi_n}(T^c)$ and therefore

$$\varlimsup_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(T) = 1 - \varliminf_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(T^c) \leqslant 1 - \sum_{\{i:\, y_i \in T^c\}} E(i).$$

By Berbee's result $\sum_{i=1}^{\infty} E(i) = 1$ and therefore $1 - \sum_{\{i:\, y_i \in T^c\}} E(i) = \sum_{\{i:\, y_i \in T\}} E(i)$, which implies that $\varlimsup_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(T) \leqslant \sum_{\{i:\, y_i \in T\}} E(i)$ which together with (10) implies that

$$\lim_{n \to \infty} \psi(f_q \circ \mu)_{\Pi_n}(T) = \sum_{\{i:\, y \in T\}} E(i)$$

which completes the proof of Lemma 9. ∎

Recall that $(\Pi_n)_{n=1}^{\infty}$ is an increasing sequence of subfields such that $\bigcup_i \Pi_i$ generates $\mathscr{C}$, and that A_n denotes the set of atoms of the subfield Π_n. For each $n \in \mathbf{N}$, let X_a^n, $a \in A_n$ be i.i.d random variables that are uniformly distributed on $(0,1)$, and let $S_n(t) \doteq \{a \in A_n: X_a^n \leqslant t\}$, $0 \leqslant t \leqslant 1$. Note that $S_n(t)$ is a stochastic process on $[0,1]$ with values in the power set of A_n, such that $S_n(0) = \varnothing$ and $S_n(1) = A_n$. Note that the restriction of any measure v on $\mathscr{C}$ to Π_n is a measure on (I, Π_n) and thus induces a measure v_{Π_n} on $(A_n, 2^{A_n})$. No confusion should result if we denote this induced measure also by v, i.e., for any $B \subset A_n$, $v(B) = \sum_{b \in B} v(b)$. Using these notations we observe that $v(S_n(t))$ is a real valued (nondecreasing if $v \geqslant 0$) stochastic process on $[0,1]$ with $v(S_n(0)) = 0$ and $v(S_n(1)) = v(I)$.

We will derive now some general inequalities for the process $v(S_n(t))$. The present paper will apply these inequalities only when v is nonnegative. As there is almost no extra cost to drive the inequalities for a signed measure v, we have chosen to state and prove the inequalities for signed measures. If v is a signed measure, we denote by $|v|$ the measure which is the sum of the positive v^+ and negative v^- parts of the measure v.

LEMMA 11. *For any measure v on $(I, \mathscr{C})$ with $v \neq 0$, and any $0 \leqslant t < \bar{t} \leqslant 1$ and any $c > 0$,*

$$\operatorname{Prob}\left(|v(S_n(\bar{t})) - v(S_n(t)) - (\bar{t} - t)v(I)| > c|v|(I)\right) < \frac{(\bar{t} - t)\max\{|v(a)|: a \in A_n\}}{c^2(|v|(I))}.$$

PROOF. It is a direct application of Chebyschev's inequality to the random variable $v(S_n(\bar{t})) - v(S_n(t)) = \sum_{a \in A_n} v(a) I(t < X_a^n \leqslant \bar{t})$ which is a sum of the independent random variables $v(a)I(t < X_a^n \leqslant \bar{t})$, $a \in A_n$. $E(v(a)I(t < X_a^n \leqslant \bar{t})) = (\bar{t} - t)v(a)$ and thus

$$E\left(v(S_n(\bar{t})) - v(S_n(t))\right) = \sum_{a \in A_n} (\bar{t} - t)v(a) = (\bar{t} - t)v(I),$$

while $\operatorname{Var}(v(a)I(t < X_a^n \leqslant \bar{t})) = (\bar{t} - t)[1 - (\bar{t} - t)](v(a))^2$ and therefore using the independence of the summands,

$$\begin{aligned}\operatorname{Var}\left(v(S_n(\bar{t})) - v(S_n(t))\right) &= (\bar{t} - t)(1 - (\bar{t} - t)) \sum_{a \in A_n} (v(a))^2 \\ &\leqslant (\bar{t} - t) \sum_{a \in A_n} \max\{|v(b)|: b \in A_n\}|v(a)| \\ &\leqslant (\bar{t} - t)\Big(\sum_{a \in A_n} |v(a)|\Big)\max\{|v(b)|: b \in A_n\} \\ &\leqslant (\bar{t} - t)\max\{|v(a)|: a \in A_n\} \cdot |v|(I).\end{aligned}$$

Therefore, by Chebyshev's inequality,

$$\begin{aligned}&\operatorname{Prob}\left(|v(S_n(\bar{t})) - v(S_n(t)) - (\bar{t} - t)v(I)| > c(|v|(I))\right) \\ &\quad < \frac{(\bar{t} - t)\max\{|v(a)|: a \in A_n\}(|v|(I))}{c^2(|v|(I))^2} = \frac{(\bar{t} - t)\max\{|v(a)|: a \in A_n\}}{c^2(|v|(I))}.\end{aligned}$$

COROLLARY 12. *For any $\epsilon > 0$ and $0 < q < 1$ there exists $\beta > 0$ such that for any probability measure μ on $(I, \mathscr{C})$ and any finite subfield Π_n of $\mathscr{C}$,*

$$\operatorname{Prob}\left(\mu(S_n(1 - \beta)) < \frac{q + 1}{2}\right) + \operatorname{Prob}\left(\mu(S_n(\beta)) > q/2\right) < \epsilon/2.$$

PROOF. Apply the previous lemma to $v = \mu$, $\bar{t} - t = \beta$ and $c = [q/2 \wedge (1 - q)/2] - \beta$ and deduce that each of summands is $\leqslant \beta/c^2$ which converges to zero as $\beta \to 0$.

COROLLARY 13. *For any probability measure μ on $(I, \mathscr{C})$ and $\epsilon > 0$ and any $\beta_1 > 0$, there exist $0 < \beta < \beta_1$ and n_1 such that for all $n \geqslant n_1$,*

$$\begin{aligned}&\operatorname{Prob}\big(\mu_{NA}(I) - \mu_{NA}(S_n(1-\beta)) < \beta\mu_{NA}(I)/2\big)\\&\quad < 4\max\{\mu_{NA}(a): a \in A_n\}/\beta(\mu_{NA}(I)).\end{aligned}$$

PROOF. Apply Lemma 11 to the measure $v = \mu_{NA}$, $1 - \beta = t < \bar{t} = 1$, $c = \beta/2$ to bound the left-hand side by $\max\{\mu_{NA}(a): a \in A_n\}\beta/(\mu_{NA}(I)\beta^2/4) = 4\max\{\mu_{NA}(a): a \in A_n\}/[\beta(\mu_{NA}(I))]$.

LEMMA 14. *Let v be a positive measure on $(I, \mathscr{C})$. Then*

$$\operatorname{Prob}\Big(\max_{0 \leqslant t \leqslant 1}\{|v(S_n(t)) - tv(I)| > \epsilon(v(I))\Big) \leqslant 8\max\{v(a): a \in A_n\}/(\epsilon^3 v(I)).$$

PROOF. There exists $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k = 1$ with $k - 1 < 2/\epsilon$ such that $t_i - t_{i-1} \leqslant \epsilon/2$. By Lemma 11, for every $1 \leqslant i \leqslant k$

$$\operatorname{Prob}\Big(|v(S_n(t_i)) - t_i v(I)| > \frac{\epsilon}{2}v(I)\Big) < \frac{4\max\{v(a): a \in A_n\}}{\epsilon^2 v(I)}.$$

Therefore

$$\begin{aligned}&\operatorname{Prob}(\exists 1 \leqslant i < k \text{ s.t.} |v(S_n(t_i)) - t_i v(I)| > \epsilon v(I)/2)\\&\quad < (k-1)\frac{4\max\{v(a): a \in A_n\}}{\epsilon^2 v(I)} < \frac{8\max\{v(a): a \in A_n\}}{\epsilon^3 v(I)}.\end{aligned}$$

As $v(S_n(t))$ is nondecreasing, if $t_{i-1} \leqslant t \leqslant t_i$ and $|v(S_n(t)) - tv(I)| > \epsilon v(I)$ then either $v(S_n(t_i)) > t_i v(I) + (\epsilon/2)v(I)$ or $v(S_n(t_{i-1})) < t_{i-1}v(I) - (\epsilon/2)v(I)$. Therefore,

$$\begin{aligned}&\operatorname{Prob}\Big(\sup_{0 \leqslant t \leqslant 1}|v(S_n(t)) - tv(I)| > \epsilon v(I)\Big)\\&\quad \leqslant \operatorname{Prob}(\exists 1 \leqslant i < k, |v(S_n(t_i)) - t_i v(I)| > (\epsilon/2)v(I))\\&\quad \leqslant \frac{8\max\{v(a): a \in A_n\}}{\epsilon^3 v(I)}.\end{aligned}$$

We turn now to the essential preparation for the proof that $f_q \circ \mu \in$ ASYMP wherever $0 < q < 1$ and μ is a probability measure on $(I, \mathscr{C})$ with $\mu_{NA}(I) > 0$. Recall that we denoted the sequence of atoms of the probability measure μ by $(y_i)_{i=1}^{\infty}$ (with $i \neq j \Rightarrow y_i \neq y_j$). We assume without loss of generality that $\alpha_k = \mu(\{y_k\})$ is monotonic nonincreasing, i.e., $\alpha_{k-1} \geqslant \alpha_k \geqslant 0$. In what follows $0 < q < 1$, the probability measure μ and the increasing sequence of partitions $(\Pi_i)_{i=1}^{\infty}$ that generates $\mathscr{C}$ are held fixed.

Let $(q_k)_{k=1}^{\infty}$ be a sequence such that $q_k < q$, $q_k \to q$ as $k \to \infty$, $\alpha_k/(q - q_k) \to_{k \to \infty} 0$ and $k(q - q_k) \to_{k \to \infty} 0$. To show that such a sequence exists one has to verify that $k\alpha_k \to_{k \to \infty} 0$ which follows from $\Sigma\alpha_k < \infty$ and the monotonicity of the α_ks. For instance, take $q_k = q - \sqrt{\alpha_k/k}$ if there are infinitely many values of k with $\alpha_k > 0$, and $q_k = q - 1/k^2$ otherwise. For $0 \leqslant x \leqslant 1$ we define the random variable $t_n(x) = \inf\{t: \mu(S_n(t)) \geqslant x\}$, and the σ-field $\mathscr{F}_n(x)$ is the σ-field generated by the random variables $X_a^n I(X_a^n \leqslant t_n(x))$. For any subset B of A_n and a measure v on $(I, \mathscr{C})$ we denote by $\rho(v, B) = \max\{|v(a)|: a \in B\}$. Recall that $A_{n,k} = \{a \in A_n | \forall 1 \leqslant i \leqslant k, y_i \notin a\}$.

Note that for each $\beta > 0$, each one of the following random variables is measurable with respect to $\mathscr{F}_n(q_k)$.

$$I_1 = I\big(\mu(S_n(t_n(q_k))) - q_k < \rho(\mu, A_{n,k}) \wedge (q - q_k)\big),$$

$$I_2 = I\big(\mu(A_{n,k} \setminus S_n(t_n(q_k))) > (\beta/2)\mu(A_{n,k})\big)$$

$$I_3 = I(t_n(q_k) \leqslant 1 - \beta) = I\big(\mu(S_n(1-\beta)) \geqslant q_k\big).$$

For every $U \subset A_n$ we denote by $F_n(U)$ the random variable

$$F_n(U) = \sum_{a \in U} I\big(q \leqslant \mu(S_n(X_a^n)) < q + \mu(a)\big).$$

As $\operatorname{Prob}(X_a^n = X_b^n) = 0$ for $a \neq b$, we may as well assume that for $a \neq b$, $X_a^n \neq X_b^n$. Therefore, at most one of the summands defining F_n is nonzero and thus F_n is a $\{0, 1\}$-valued random variable.

LEMMA 15. *For any $\beta > 0$, if $\theta = 8(q - q_k)/[\beta\mu(A_{n,k})]$ and if I_4, I_5 are the random variables,*

$$I_4 = I\big(\mu(A_{n,k} \cap S_n(t_n(q_k) + \theta) \setminus S_n(t_n(q_k))) > q - q_k\big),$$

$$I_5 = I\big(\forall 1 \leqslant i \leqslant k, X_{a^n(i)}^n \notin (t_n(q_k), t_n(q_k) + \theta)\big),$$

then, on $I_2 = 1$,

$$E(I_4 | \mathscr{F}_n(q_k)) \geqslant 1 - \rho(\mu, A_{n,k})/(q - q_k), \tag{16}$$

and on $I_3 = 1$,

$$E(I_5 | \mathscr{F}_n(q_k)) \geqslant 1 - 8k(q - q_k)/\big(\beta^2\mu(A_{n,k})\big), \tag{17}$$

and on $I_2 = I_3 = 1$,

$$\begin{aligned} &E(I_4 \cdot I_5 | \mathscr{F}_n(q_k)) \\ &\quad\geqslant \big(1 - \rho(\mu, A_{n,k})/(q - q_k)\big)\big(1 - 8k(q - q_k)/\beta^2\mu(A_{n,k})\big). \end{aligned} \tag{18}$$

PROOF. We apply Lemma 11, conditionally to $\mathscr{F}_n(q_k)$, to the measure μ on $\bar{A}_{n,k} = \{a \in A_{n,k}: X_a^n > t_n(q_k)\}$. Noting that $\bar{A}_{n,k}$ is measurable w.r.t. $\mathscr{F}_n(q_k)$ and that the random variables $Y_a^n = (X_a^n - t_n(q_k))/(1 - t_n(q_k))$, $a \in \bar{A}_{n,k}$, are conditionally to $\mathscr{F}_n(q_k)$ i.i.d, uniformly distributed on $(0,1)$ and that

$$I_4 = I\left(\sum_{a \in \bar{A}_{n,k}} I(Y_a^n \leqslant \theta/(1 - t_n(q_k)))\mu(a) > q - q_k\right)$$

which is

$$\geqslant I\left(\sum_{a \in \bar{A}_{n,k}} I(Y_a^n \leqslant \theta)\mu(a) > q - q_k\right)$$

$$= I\left(\sum_{a \in \bar{A}_{n,k}} I(Y_a^n \leqslant \theta)\mu(a) > \theta\beta\mu(A_{n,k})/8\right).$$

On $I_2 = 1$, $\beta\mu(A_{n,k}) < 2\mu(\bar{A}_{n,k})$ and therefore on $I_2 = 1$,

$$I_4 \geqslant I\left(\sum_{a \in \bar{A}_{n,k}} I(Y_a^n \leqslant \theta)\mu(a) > \theta\mu(\bar{A}_{n,k})/4\right).$$

By Lemma 11,

$$E\left(I\left(\sum_{a \in \bar{A}_{n,k}} I(Y_a^n \leqslant \theta)\mu(a) > \frac{\theta}{4}\mu(\bar{A}_{n,k})\right)|\mathscr{F}_n(q_k)\right)$$

$$\geqslant 1 - \frac{\theta\rho(\mu, \bar{A}_{n,k})}{(3\theta/4)^2\mu(\bar{A}_{n,k})}$$

which on $I_2 = 1$ is

$$\geqslant 1 - \frac{16}{9\theta}\frac{\rho(\mu, A_{n,k})}{(\beta/2)\mu(A_{n,k})}$$

$$= 1 - \frac{32}{9\theta\beta}\frac{\rho(\mu, A_{n,k})}{\mu(A_{n,k})} = 1 - \frac{32 \cdot \beta \cdot \mu(A_{n,k}) \cdot \rho(\mu, A_{n,k})}{9\beta \cdot 8(q - q_k)\mu(A_{n,k})}$$

$$\geqslant 1 - \frac{\rho(\mu, A_{n,k})}{(q - q_k)}.$$

Altogether, we deduce that on $I_2 = 1$, $E(I_4|\mathscr{F}_n(q_k)) \geqslant 1 - \rho(\mu, A_{n,k})/(q - q_k)$ which proves (16). Let $\bar{A}_n = \{a \in A_n\colon X_a^n > t_n(q_k)\}$ and note that $\bar{A}_n$ is measurable w.r.t. $\mathscr{F}_n(q_k)$. To prove (17), note that the random variables $Y_a^n = (X_a^n - t_n(q_k))/(1 - t_n(q_k))$, $a \in \bar{A}_n \setminus A_{n,k}$, are conditionally to $\mathscr{F}_n(q_k)$, i.i.d. uniformly distributed on $(0,1)$ and that on $I_3 = I$, the random variable $I_5 = I(\forall a \in \bar{A}_n \setminus A_{n,k}, Y_a^n \geqslant \theta/(1 - t_n(q_k)))$ is $\geqslant I(\forall a \in \bar{A}_n \setminus A_{n,k}, Y_a^n \geqslant \theta/\beta)$. Therefore, on $I_3 = 1$,

$$E(I_5|\mathscr{F}_n(q_k)) \geqslant I(\theta \leqslant \beta)(1 - \theta/\beta)^{|\bar{A}_n \setminus A_{n,k}|}$$

$$\geqslant 1 - k\theta/\beta = 1 - 8k(q - q_k)/(\beta^2\mu(A_{n,k})),$$

which completes the proof of (17). As I_2 and I_3 are $\{0,1\}$-valued random variables that are measurable w.r.t. $\mathscr{F}_n(q_k)$, (18) is an immediate corollary of (16) and (17). Note that I_4 and I_5 are independent conditionally to $\mathscr{F}_n(q_k)$.

LEMMA 20. *For any $\beta > 0$ and any probability measure μ,*

$$E\big(F_n(A_{n,k})|\mathscr{F}_n(q_k)\big) \geqslant I_1 \cdot I_2 \cdot I_3\left(1 - \frac{\rho(\mu, A_{n,k})}{q - q_k} - 8k(q - q_k)/\big(\beta^2\mu(A_{n,k})\big)\right)$$

PROOF. As I_i, $i = 1,2,3$ are $\{0,1\}$-valued random variables that are measurable w.r.t. $\mathscr{F}_n(q_k)$, it is enough to prove that on $I_1 = I_2 = I_3 = 1$,

$$E\big(F_n(A_{n,k})|\mathscr{F}_n(q_k)\big) \geqslant 1 - \rho(\mu, A_{n,k})/(q - q_k) - 8k(q - q_k)/\big(\beta^2\mu(A_{n,k})\big).$$

On $I_1 = 1$, $F_n(A_{n,k}) \geqslant I_4 \cdot I_5$ and therefore on $I_1 = I_2 = I_3 = 1$, $E(F_n(A_{n,k})|\mathscr{F}_n(q_k)) \geqslant E(I_4 \cdot I_5|\mathscr{F}_n(q_k))$ which by Lemma 15 is $\geqslant 1 - \rho(\mu, A_{n,k})/(q - q_k) - 8k(q - q_k)/(\beta^2\mu(A_{n,k}))$.

LEMMA 21. *Let $F_n^k(A_{n,k}) = \sum_{a \in A_{n,k}} I(q_k < \mu(S_n(X_a^n)) \leqslant q_k + \mu(a))$. Then for every $\epsilon > 0$, there exists k_1 such that for all $k \geqslant k_1$*

$$\limsup_{n\to\infty} E\big(\big|F_n^k(A_{n,k}) - F_n(A_{n,k})\big|\big) \leqslant 2\epsilon \quad \text{and}$$

$$\limsup_{n\to\infty} \big|E\big(F_n^k(A_{n,k})\big) - E\big(F_n(A_{n,k})\big)\big| \leqslant \varepsilon.$$

PROOF. Let $\epsilon > 0$. Choose $\beta > 0$ such that $\text{Prob}(I_3 = 1) \geqslant 1 - \epsilon/3$ for all n and k. The existence of such $\beta > 0$ follows from Corollary 12. Note that $I_3 \cdot I_2 \geqslant I_3 \cdot I(\mu(A_{n,k} \setminus S_n(1 - \beta)) > (\beta/2)\mu(A_{n,k}))$, and that the right-hand side of the inequality is a product of two $\{0,1\}$-valued random variables. By Lemma 11, $E(I(\mu(A_{n,k} \setminus S_n(1 - \beta)) > (\beta/2)\mu(A_{n,k}))) \geqslant 1 - 4\rho(\mu, A_{n,k})/(\beta\mu(A_{n,k}))$ and therefore $E(I_3 \cdot I_2) \geqslant 1 - \epsilon/3 - 4\rho(\mu, A_{n,k})/(\beta\mu(A_{n,k}))$. As

$$\mathit{lim\ sup}_{n\to\infty}\rho(\mu, A_{n,k})/\mu(A_{n,k}) \leqslant \alpha_k/\alpha$$

we deduce that if k is such that $\alpha_k < \epsilon\beta\alpha/12$ then for sufficiently large n,

$$E(I_3 \cdot I_2) \geqslant 1 - 2\epsilon/3. \tag{22}$$

As $\limsup_{n\to\infty}\rho(\mu, A_{n,k}) = \alpha_{k+1} \leqslant \alpha_k = o(q - q_k)$ as $k \to \infty$, we deduce that for sufficiently large k, $\limsup_{n\to\infty}\rho(\mu, A_{n,k})/(q - q_k) < \epsilon/6$. Also $k(q - q_k) \to 0$ as $k \to \infty$ and $\lim_{n\to\infty}\mu(A_{n,k}) \geqslant \alpha$ and therefore, for sufficiently large k,

$$\limsup_{n\to\infty} 8k(q - q_k)/\big[\beta^2\mu(A_{n,k})\big] < \epsilon/6.$$

The two last strict inequalities imply that for sufficiently large k,

(23)

$$\liminf_{n\to\infty}\Big(1 - \big[\rho(\mu, A_{n,k})/(q - q_k)\big] - 8k(q - q_k)/\big[\beta^2\mu(A_{n,k})\big]\Big) > 1 - \epsilon/3.$$

By Lemma 20, (22) and (23), using the fact that I_1 and $I_2 \cdot I_3$ are $\{0,1\}$-valued random variables, we deduce that for sufficiently large k,

$$\begin{aligned}
&\limsup_{n\to\infty} E\left(\left(I_1 - F_n(A_{n,k})\right)^+\right)\\
&\quad = \limsup_{n\to\infty} E\left(I_1\left(I_1 - F_n(A_{n,k})\right)\right)\\
(24)\quad &\quad = \limsup_{n\to\infty} E\left(E\left(\left(I_1 - F_n(A_{n,k})\right)I_1 | \mathscr{F}_n(q_k)\right)\right)\\
&\quad = \limsup_{n\to\infty} E\left(I_1 \cdot \left[E(I_1|\mathscr{F}_n(q_k)) - E\left(F_n(A_{n,k})|\mathscr{F}_n(q_n)\right)\right]\right) \leqslant \epsilon.
\end{aligned}$$

Recall that $\limsup_{n\to\infty}\rho(\mu, A_{n,k}) \leqslant \alpha_k = o(q - q_k)$ as $k \to \infty$, and observe that when $\rho(\mu, A_{n,k}) < q - q_k$, $I_1 \geqslant F_n^k(A_{n,k})$. Therefore we conclude that for sufficiently large k,

$$\limsup_{n\to\infty} E\left(\left(F_n^k(A_{n,k}) - F_n(A_{n,k})\right)^+\right) \leqslant \epsilon. \tag{25}$$

In all the above computations that were leading to (25), there was no essential use (aside for notational convenience) to the fact that q was held fixed and q_k was varying. Therefore in a similar way we obtain that for sufficiently large k,

$$\limsup_{n\to\infty} E\left(\left(F_n(A_{n,k}) - F_n^k(A_{n,k})\right)^+\right) \leqslant \epsilon. \tag{26}$$

The two inequalities, (25) and (26), complete the proof of Lemma 21. For any T in Π_1, we denote by $T_{n,k} = \{a \in A_{n,k}: a \subset T\}$.

LEMMA 27. *For any $\epsilon > 0$, there is k_1, s.t. for every $k \geqslant k_1$,*

$$\liminf_{n\to\infty}\left[E\left(F_n(T_{n,k})\right) - E(I_1)\mu(T_{n,k})/\mu(A_{n,k})\right] > -\epsilon.$$

PROOF. By the renewal theory for sampling without replacement ([18] or Theorem 3.5), there is $K = K(\epsilon) \geqslant 1$ such that for every finite weighted majority game $v = [r, (w_i)_{i\in\{1,\ldots,m\}}]$, if $K(\epsilon)\max_{i=1}^m w_i < r < \sum_{j=1}^m w_i - K(\epsilon)\max_{i=1}^m w_i$, then $\sum_i|\psi v(i) - w_i/\sum_{j=1}^m w_j| < \epsilon/2$, which in particular implies that for any subset Q of the set of players $\{1,\ldots,m\}$,

$$\psi v(Q) \geqslant w(Q)/\sum_{j=1}^m w_i - \epsilon/4. \tag{28}$$

Let k_2 be such that for $k \geqslant k_2$,

$$(K(\epsilon) + 1)\alpha_k < q - q_k \quad\text{and}\quad K(\epsilon)\alpha_k < 1 - q. \tag{29}$$

Choose $\beta > 0$ such that $\text{Prob}(I_3 = 1) > 1 - \epsilon/16$ for all n and k. As in the proof of Lemma 21, there is k_3 such that for $k \geqslant k_3$, and for sufficiently large n,

$$E(I_2 \cdot I_3) \geqslant 1 - \epsilon/16 - 4\rho(\mu, A_{n,k})/\left[\beta\mu(A_{n,k})\right] \geqslant 1 - \epsilon/8. \tag{30}$$

Choose $\theta = 8(q - q_k)/[\beta\mu(A_{n,k})]$ and let I_4 and I_5 be the $\{0,1\}$-valued random variables defined in Lemma 15. Let $\bar{A}_{n,k} = \{a \in A_{n,k} | X_a^n > t_n(q_k)\}$ and $\bar{T}_{n,k} = T_{n,k} \cap \bar{A}_{n,k}$. Set

$$I_6 = \sum_{a \in \bar{T}_{n,k}} I\big(q - \mu(S_n(t_n(q_k))) \leqslant \mu\big(S_n(X_a^n) \cap \bar{A}_{n,k}\big) < q - \mu(S_n(t_n(q_k))) + \mu(a)\big).$$

Setting $Y_a^n = [X_a^n - t_n(q_k)]/[1 - t_n(q_k)]$ we note that conditionally to $\mathscr{F}_n(q_k)$, the random variables Y_a^n, $a \in \bar{A}_{n,k}$ are i.i.d. uniformly distributed on $(0,1)$, and that

$$I_6 = \sum_{a \in \bar{T}_{n,k}} I\big(q - \mu(S_n(t_n(q_k))) \leqslant \mu\big(S_n(X_a^n) \cap \bar{A}_{n,k}\big)$$
$$< q - \mu(S_n(t_n(q_k))) + \mu(a)\big).$$

Therefore, on $I_1 = 1$, $E(I_6 | \mathscr{F}_n(q_k)) = (\psi v)(\bar{T}_{n,k})$ where v is the weighted majority game $[q - \mu(S_n(t_n(q_k))), (\mu(a))_{a \in \bar{A}_{n,k}}]$. Applying (28) to the weighted majority game v we deduce that on $I_1 = 1$, $k \geqslant k_2$ and n sufficiently large

$$E(I_6 | \mathscr{F}_n(q_k)) \geqslant \mu(\bar{T}_{n,k})/\mu(\bar{A}_{n,k}) - \epsilon/4. \tag{31}$$

Note that $F_n(T_{n,k}) \geqslant I_1 \cdot I_4 \cdot I_5 \cdot I_6 \geqslant I_1 \cdot I_2 \cdot I_3 \cdot I_4 \cdot I_5 \cdot I_6$. As I_1, I_2, I_3 are $\{0,1\}$-valued random variables that are measurable with respect to $\mathscr{F}_n(q_k)$ we deduce that

$$E\big(F_n(T_{n,k}) | \mathscr{F}_n(q_k)\big) \geqslant I_1 \cdot I_2 \cdot I_3 E(I_4 \cdot I_5 \cdot I_6 | \mathscr{F}_n(q_k)). \tag{32}$$

As $I_4 \cdot I_5$ and I_6 are $\{0,1\}$-valued we deduce that

$$E(I_4 \cdot I_5 \cdot I_6 | \mathscr{F}_n(q_k)) \geqslant E(I_6 | \mathscr{F}_n(q_k)) + E(I_4 \cdot I_5 | \mathscr{F}_n(q_k)) - 1$$

which by (31) and (18) is on $I_1 = I_2 = I_3 = 1$, $k \geqslant k_2$ and n sufficiently large

$$\geqslant \mu(\bar{T}_{n,k})/\mu(\bar{A}_{n,k}) - \epsilon/4 - \frac{\rho(\mu, A_{n,k})}{q - q_k} - \frac{8k(q - q_k)}{\beta^2\mu(A_{n,k})}$$

and therefore using (32) we have

$$E\big(F_n(T_{n,k}) | \mathscr{F}_n(q_k)\big)$$
$$\geqslant I_1 \cdot I_2 \cdot I_3 \mu(\bar{T}_{n,k})/\mu(\bar{A}_{n,k}) - \epsilon/4 - \frac{\rho(\mu, A_{n,k})}{q - q_k} - \frac{8k(q - q_k)}{\beta^2\mu(A_{n,k})}.$$

Let k_4 be such that $\alpha_k/(q - q_k) + 8k(q - q_k)/[\beta^2\mu_{NA}(I))] < \epsilon/4$. Then for $k \geqslant k_4$,

$$\lim_{n \to \infty} \frac{\rho(\mu, A_{n,k})}{q - q_k} + \frac{8k(q - q_k)}{\beta^2\mu(A_{n,k})}$$
$$\leqslant \alpha_k/(q - q_k) + 8k(q - q_k)/[\beta^2\mu_{NA}(I)] < \epsilon/4,$$

and therefore for $k \geqslant k_4 \vee k_2$, for sufficiently large n we have

$$E\big(F_n(T_{n,k})|\mathscr{F}_n(q_k)\big) \geqslant I_1 \cdot I_2 \cdot I_3 \mu(\bar{T}_{n,k})/\mu(\bar{A}_{n,k}) - \epsilon/2. \tag{33}$$

By Lemma 14,

$$\operatorname{Prob}\big(\mu(\bar{T}_{n,k}) < (1 - t_n(q_k))\mu(T_{n,k}) - \delta\mu(A_{n,k})\big)$$
$$\leqslant 8\rho(\mu, T_{n,k})/\big(\delta^3\mu(A_{n,k})\big) \quad \text{and}$$
$$\operatorname{Prob}\big(\mu(\bar{A}_{n,k}) > (1 - t_n(q_k) + \delta)\mu(A_{n,k})\big) \leqslant 8\rho(\mu, A_{n,k})/\big(\delta^3\mu(A_{n,k})\big).$$

As $\operatorname{Prob}(t_n(q_k) > 1 - \beta) < \epsilon/8$ we deduce that

$$\operatorname{Prob}\left(\frac{\mu(\bar{T}_{n,k})}{\mu(\bar{A}_{n,k})} < \frac{\mu(T_{n,k})}{\mu(A_{n,k})} - \frac{2\delta}{\beta}\right) \leqslant 16\rho(\mu, A_{n,k})/\delta^3\mu(A_{n,k}) + \epsilon/8$$

and as $I_1 \cdot I_2 \cdot I_3$ is a $\{0,1\}$-valued random variable we deduce that

$$\begin{aligned} &E\big(I_1 \cdot I_2 \cdot I_3 \mu(\bar{T}_{n,k})/\mu(\bar{A}_{n,k})\big) \\ &\quad \geqslant E(I_1 \cdot I_2 \cdot I_3)\mu(T_{n,k})/\mu(A_{n,k}) \\ &\quad\quad -2\delta/\beta - 16\rho(\mu, A_{n,k})/\delta^3\mu(A_{n,k}) - \epsilon/8. \end{aligned} \tag{34}$$

Setting $\delta = \epsilon\beta/16$, observing that $\limsup_{n\to\infty}\rho(\mu, A_{n,k})/\mu(A_{n,k}) \leqslant \alpha_k/\alpha$ and therefore if $16\alpha_k/\delta^3\alpha < \epsilon/8$ (i.e., $k \geqslant k_5$ where $16\alpha_{k_5} < \epsilon\delta^3\alpha/8$), then, for sufficiently large n, $16\rho(\mu, A_{n,k})/\delta^3\mu(A_{n,k}) \leqslant \epsilon/8$ and

$$\begin{aligned} E\big(I_1 \cdot I_2 \cdot I_3\mu(\bar{T}_{n,k})/\mu(\bar{A}_{n,k})\big) &\geqslant E\big(I_1 \cdot I_2 \cdot I_3 \cdot \mu(T_{n,k})/\mu(A_{n,k})\big) \\ &\quad -\epsilon/8 - \epsilon/8 - \epsilon/8. \end{aligned} \tag{35}$$

If also $k \geqslant k_3$, then by (30), $E(I_1 \cdot I_2 \cdot I_3) \geqslant E(I_1) - \epsilon/8$ and therefore using (35), (33) if $k \geqslant k_3 \vee k_5 \vee k_2 \vee k_4$, we have for sufficiently large n,

$$\begin{aligned} E\big(F_n(T_{n,k})\big) &\geqslant E(I_1 \cdot I_2 \cdot I_3)\big(\mu(T_{n,k})/\mu(A_{n,k})\big) - \epsilon/2 - \epsilon/8 - \epsilon/8 - \epsilon/8 \\ &\geqslant E(I_1)\big(\mu(T_{n,k})/\mu(A_{n,k})\big) - \epsilon, \end{aligned}$$

which completes the proof of the Lemma 27. ■

Let $\epsilon > 0$. Let k_1 be sufficiently large so that for all $k \geqslant k_1$,

$$\text{for } n \text{ sufficiently large} \quad \mu(T_{n,k})/\mu(A_{n,k}) \geqslant \mu_{NA}(T)/\alpha - \epsilon, \tag{36}$$

$$\begin{aligned} \text{for } n \text{ sufficiently large} \quad E(I_1) &\geqslant E\big(F_n^k(A_{n,k})\big) > E\big(F_n(A_{n,k})\big) - \epsilon \\ &> 1 - \sum_{i=1}^{\infty} E(i) - 2\epsilon, \quad \text{and} \end{aligned} \tag{37}$$

(38) for n sufficiently large $E\big(F_n(T_{n,k})\big) \geqslant E(I_1)\mu(T_{n,k})/\mu(A_{n,k}) - \epsilon.$

To show that such k_1 exists, note that $\lim_{n\to\infty}\mu(A_{n,k}) \doteq \alpha + \sum_{i>k}\alpha_i$ and that $\lim_{n\to\infty}\mu(T_{n,k}) \geqslant \mu_{NA}(T)$ and therefore for k sufficiently large (36) holds. That (37) holds for k sufficiently large, recall that $I_1 \geqslant F_n^k(A_{n,k})$ whenever $\rho(\mu, A_{n,k}) < q - q_k$, use Lemma 21 and for the right inequality use Lemma 6 which implies that $\lim_{n\to\infty}E(F_n(A_{n,k})) = 1 - \sum_{i=1}^k E(i) > 1 - \sum_{i=1}^\infty E(i) - \epsilon$. Using (36), (37) and (38) we obtain that for sufficiently large n,

$$E\big(F_n(T_{n,k})\big) \geqslant \left(1 - \sum_{i=1}^{\infty} E(i)\right)\mu_{NA}(T)/\alpha - 4\epsilon. \tag{39}$$

If n is sufficiently large, so that $1 \leqslant i \leqslant j \leqslant k \Rightarrow a^n(i) \neq a^n(j)$, then

$$\psi(f_q\circ\mu)_{\Pi_n}(T) = \sum_{\substack{i=1\\ y_j\in T}}^{k} \psi(f_q\circ\mu)_{\Pi_n}(a^n(i)) + E\big(F_n(T_{n,k})\big)$$

and therefore by applying Lemma 6 and (39) we conclude that for k sufficiently large,

$$\begin{aligned}\liminf_{n\to\infty}\psi(f_q\circ\mu)_{\Pi_n}(T) &\geqslant \sum_{\substack{i=1\\ y_j\in T}}^{k} E(i) + \left(1 - \sum_{i=1}^{\infty} E(i)\right)\mu_{NA}(T)/\alpha - 4\epsilon\\ &\geqslant \phi(f_q\circ\mu)(T) - 4\epsilon - \sum_{i>k} E(i).\end{aligned}$$

As this holds for all $\epsilon > 0$,

$$\liminf_{n\to\infty}\psi(f_q\circ\mu)_{\Pi_n}(T) \geqslant \phi(f_q\circ\mu)(T) - \sum_{i>k} E(i),$$

and as this holds for all k sufficiently large we conclude that

$$\liminf_{n\to\infty}\psi(f_q\circ\mu)_{\Pi_n}(T) \geqslant \phi(f_q\circ\mu)(T).$$

As this holds for all $T \in \Pi_1$ we have in particular that

$$\liminf_{n\to\infty}\psi(f_q\circ\mu)_{\Pi_n}(T^c) \geqslant \phi(f_q\circ\mu)(T^c)$$

and as $\psi(f_q\circ\mu)_{\Pi_n}(T) + \psi(f_q\circ\mu)(T^c) = 1 = \phi(f_q\circ\mu)(T) + \phi(f_q\circ\mu)(T^c)$ we deduce that $\lim_{n\to\infty}\psi(f_q\circ\mu)_{\Pi_n}(T) = \phi(f_q\circ\mu)(T)$ which completes the proof of Theorem 1.

PROOF OF THEOREM A. As *ASYMP* is a closed linear subspace of *BV* and $bv'M$ is the closed subspace of *BV* generated by games that have the form $f\circ\mu$ where $f \in bv'$ and μ is a probability measure in M, it is enough to show that each one of these generators is in *ASYMP*. Let μ be a fixed probability measure in M. Any $f \in bv'$ is the sum of two functions f_L and f_R in bv' where f_L is left continuous and f_R is right continuous. If $f = f_L + f_R$, $f_L, f_R \in bv'$ then $f \circ \mu = f_L \circ \mu + f_R \circ \mu$ and therefore it is enough to prove that each one of the summands, $f_L \circ \mu$ and $f_R \circ \mu$, is in $ASYMP$. Also the dual game of $f_L \circ \mu$, $(f_L \circ \mu)^*$ equals $f_L^* \circ \mu$ where $f_L^* \in bv'$ and is right continuous. Therefore it is enough to prove that $f \circ \mu \in ASYMP$ whenever $f \in bv'$ and is right continuous.

Let $f \in bv'$ be right continuous. By Lemma 3.4 of [17],

$$\psi(f \circ \mu)_{\Pi_n}(T) = \int_0^1 \psi(f_q \circ \mu)_{\Pi_n}(T)\, df(q).$$

As for each $q \in (0,1)$, $\psi(f_q \circ \mu)_{\Pi_n}(T) \to_{n \to \infty} \phi(f_q \circ \mu)(T)$, $0 \leqslant \psi(f_q \circ \mu)_{\Pi_n}(T) \leqslant 1$ and $\psi(f_q \circ \mu)_{\Pi_n}(T)$ is measurable (in q), we deduce that $\phi(f_q \circ \mu)(T)$ is measurable and by the bounded convergence theorem we conclude that

$$\psi(f \circ \mu)_{\Pi_n}(T) \xrightarrow[n \to \infty]{} \int_0^1 \phi(f_q \circ \mu)(T)\, df(q),$$

which completes the proof of Theorem A. ∎

5. Weakening of assumption? The existence of an asymptotic value on $bv'M$ is equivalent to the existence of an asymptotic value on all scalar measure games $f \circ \mu$ where μ is a probability measure and f: $[0,1] \to \mathbf{R}$ is continuous at 0 and 1 with $f(0) = 0$ and of bounded variation. We have already seen in [17] that the continuity of f at 0 and 1 can be replaced by the weaker condition that $f(0+) = f(1-) - f(1)$ and that the bounded variation of the function f is essential for the existence of an asymptotic value. Here, we consider weakening the assumptions on the measure μ. Our first concern is the positivity of μ.

We investigate games of the form $f \circ \mu$ where μ is a measure that takes negative as well as positive values and f is a real valued function defined on the range $\mathbf{R}(\mu) = \{\mu(T) \colon T \in \mathscr{C}\}$ of μ with $f(0) = 0$. One case that is (relatively) easy to handle is when the nonatomic part of the measure μ is positive (negative) and there are only finitely many atoms of μ with negative (positive) measure. In that case, for any function f with $f(0) = 0$ that is continuous at 0 and $\mu(I)$ that is of bounded variation on the convex hull of $\mathbf{R}(\mu)$, $f \circ \mu$ has an asymptotic value.

The next result shows that when the nonatomic part of the measure μ takes both negative as well as positive values the situation is more delicate.

THEOREM 5.1. *Let μ be a nonatomic measure with total mass* 1 *and* $0 < q < 1$. *Then $f_q \circ \mu$ has an asymptotic value if and only if μ is positive.*

PROOF. The main theorem asserts that $f_q \circ \mu$ has an asymptotic value whenever μ is positive. Thus, it suffices to show that if $\mu^-(I) > 0$ then $f_q \circ \mu$ does not have an asymptotic value. Assume that $\mu^-(I) > 0$ where μ^- is the negative part of the measure μ. Let I_1, I_2, I_3 be a partition of I, so that $\mu^-(I) = \mu^-(I_3) = \mu^+(I_2)$ and $\mu^+(I_1) = 1$. Let $(\Pi_n)_{n=1}^\infty$ be a sequence of finite subfields of $\mathscr{C}$ with $I_i \in \Pi_n$, $i = 1,2,3$. Denote the set of all atoms of Π_n that are included in I_i, $i = 1,2,3$, by A_n^i. We assume that $\mu^+(a) = n^{-4}\mu^+(I_2)$ if $a \in A_n^2$, and that $\mu(a) = n^{-4}\mu(I_3)$ if $a \in A_n^3$. A_n^1 consists of $2n$ atoms each having measure $1/(2n+1)$, one atom denoted by $\underline{a}$ of size $q - [(2n+1)q]/(2n+1)$, where $[x]$ denotes the integer part of x and one denoted by $\bar{a}$ of size $1 - \mu(\underline{a}) - 2n/(2n+1)$.

Let $(X_a)_{a \in A_n}$, $A_n = A_n^1 \cup A_n^2 \cup A_n^3$, be i.i.d random variables that are uniformly distributed on $(0,1)$. Let $X_1, \ldots, X_{2n}$ be defined by:

$$X_1 = \min\{X_a \colon a \in A_n^1 \setminus \{\underline{a}, \bar{a}\}\}, \quad X_{k+1} = \min\{X_a \colon a \in A_n^1 \setminus \{\underline{a}, \bar{a}\}, X_a > X_k\}.$$

Let b be an element of A_n^2. We will show below that $\psi(f_q \circ \mu)_{\Pi_n}(b)$ is at least of the order of magnitude of n^{-3}. Therefore, $\lim_{n\to\infty}\psi(f_q \circ \mu)_{\Pi_n}(A_n^2) = \infty$. As it is possible to generate such a sequence of finite subfields for which there is a subsequence which is increasing and generating the σ-field $\mathscr{C}$, it follows that $f_q \circ \mu$ does not have an asymptotic value. Thus, all that we have to prove is that $\psi(f_q \circ \mu)_{\Pi_n}(b)$ is $\geqslant cn^{-3}$ for some constant c (that might depend on q and $\mu^-(I)$). Denote $\alpha = \mu^-(I) = \mu^+(I_2) = \mu^-(I_3)$ and $k = k(n,q) = [(2n+1)q]$. Then $\mu(b) = \alpha n^{-4}$ and

$$\psi(f_q \circ \mu)_{\Pi_n}(b) = \int_0^1 E\left(I\left(\sum_{\substack{a\in A_n\\ a\neq b}} \mu(a) I(X_a \leqslant t) \in \left[q - \alpha n^{-4}, q\right)\right)\right) \cdot dt.$$

$$E\left(I\left(\sum_{\substack{a\in A_n\\ a\neq b}} \mu(a) I(X_a \leqslant t) \in \left[q - \alpha n^{-4}, q\right)\right)\right)$$

$$= E\left(E\left(I\left(\sum_{\substack{a\in A_n\\ a\neq b}} \mu(a) I(X_a \leqslant t) \in \left[q - \alpha n^{-4}, q\right)\right) \middle| X_{\underline{a}}, X_{\bar{a}}, X_1, \dots, X_{2n}\right)\right)$$

$$\geqslant E\left(E\left(I\left(\sum_{\substack{a\in A_n^3\cup A_n^2\\ a\neq b}} \mu(a) I(X_a \leqslant t) \in \left[-\alpha n^{-4}, 0\right)\right) \middle| X_{\underline{a}}, X_{\bar{a}}, X_1, \dots, X_{2n}\right)\right.$$

$$\left. I(X_{\underline{a}} \leqslant X_k < t < X_{k+1} < X_{\bar{a}})\right)$$

$$= E\left(I\left(\sum_{\substack{a\in A_n^3\cup A_n^2\\ a\neq b}} \mu(a) I(X_a \leqslant t) \in \left[-\alpha n^{-4}, 0\right)\right)\right.$$

$$\left. \cdot I(X_a \leqslant X_k < t < X_{k+1} < X_{\bar{a}})\right).$$

For each fixed $n^{-4} \leqslant t \leqslant 1$,

$$(*) \qquad E\left(I\left(\sum_{\substack{a\in A_n^3\cup A_n^2\\ a\neq b}} \mu(a) I(X_a \leqslant t) \in \left[-\alpha n^{-4}, 0\right)\right)\right) \geqslant c \cdot n^{-2}$$

for some universal constant c, and X_a, $a \in A_n^3 \cup A_n^2$ are independent of X_a, $a \in A_n^1$

and, therefore,

$$cn^{-6} + \psi(f_q \circ \mu)_{\Pi_n}(b) \geqslant \int_0^1 cn^{-2} E(I(X_{\underline{a}} < X_k < t < X_{k+1} < X_{\bar{a}})) \cdot dt$$
$$= E((X_{k+1} - X_k) I(X_{\underline{a}} < X_k < X_{k+1} < X_{\bar{a}})) cn^{-2}$$
$$\geqslant c^1 n^{-1} cn^{-2}$$

which completes the proof that $f_q \circ \mu$ does not have an asymptotic value. ∎

It is of interest to characterize all games of the form $f \circ \mu$ where μ is a signed measure, that have an asymptotic value. Let μ be a nonatomic measure with range $\{\mu(S) | S \in C\} = [-a, b]$ with $-b < -a < 0 < b$, and let f be a monotonic function on $[-a, b]$ with $f(0) = 0$. Our conjecture is that a necessary condition for $f \circ \mu$ to have an asymptotic value is that $f \circ \mu$ has bounded variation, i.e., (see [3, Proposition 9.1]) that the function $g_f(x, y) = (x + a)(b - y)(f(y) - f(x))/(y - x)$ is bounded in the domain $-a < x < y < b$. Moreover, if f is piecewise continuously differentiable we conjecture that this condition is sufficient for $f \circ \mu$ to have an asymptotic value. Another sufficient condition for $f \circ \mu$ to have an asymptotic value is concavity of f.

In the following example we demonstrate an example of a signed purely atomic measure μ with $\mu(I) = 1$ for which $f_{1/2} \circ \mu$ does not have an asymptotic value.

EXAMPLE 5.2. Let $(n_k)_{k=1}^{\infty}$ be an increasing sequence of positive integers with

$$\sqrt{n_k}\, 4^{-4\Sigma_{i<k} n_i} \xrightarrow[n \to \infty]{} \infty. \tag{5.3}$$

Let μ be a purely atomic signed measure on $(I, \mathscr{C})$ having countably many atoms: two atoms, $y(1)$ and $y(2)$ of measure $1/2$ each; for each $k \geqslant 1$, $2n_k$ atoms, $y^+(k, j)$, $1 \leqslant j \leqslant 2n_k$ each having measure $2^{-k}/n_k$ and $2n_k$ atoms, $y^-(k, j)$, $1 \leqslant j \leqslant 2n_k$ each one of measure $-2^{-k}/n_k$. Let I^+, I^- be the Hahn decomposition of I with respect to μ. We will prove that $f_{1/2} \circ \mu$ does not have an asymptotic value, by demonstrating an increasing sequence of finite subfields $(\Pi_k)_{k=1}^{\infty}$ with $I^+ \in \Pi_1$ and $\bigcup_{k \geqslant 1} \Pi_k$ generates $\mathscr{C}$ for which

$$\lim_{k \to \infty} \psi(f_{1/2} \circ \mu)_{\Pi_k}(A_k^+) = \infty$$

where A_k^+ are all atoms of Π_k that are subsets of I^+. Let $(\Pi_k)_{k=1}^{\infty}$ be an increasing sequence of finite subfields with $\bigcup_{k \geqslant 1} \Pi_k$ generating $\mathscr{C}$, $I^+ \in \Pi_1$ and the set of atoms A_k of the finite field Π_k satisfies:

$$A_k \supseteq \{a(1), a(2)\} \cup \bigcup_{j=1}^{k} A_k^j(+) \cup A_k^j(-) \cup \{b_1(+), b_2(+), b_1(-), b_2(-)\}$$

where $a(1) \neq a(2)$, $y(1) \in a(1)$, $y(2) \in a(2)$, $|A_k^j(+)| = 2n_j = |A_k^j(-)|$ and

$$a \in A_k^j(+) \Rightarrow \mu(a) = 2^{-j}/n_j$$

while $a \in A_k^j(-) \Rightarrow \mu(a) = -2^{-j}/n_j$,

$$\mu(b_1(+)) = \mu(b_2(+)) = \sum_{i>k} (2^{-i}/n_i) n_i = 2^{-k} = -\mu(b_1(-)) = -\mu(b_2(-)).$$

$$\psi(f_{1/2} \circ \mu)_{\Pi_k}(A_k^+) \geqslant \psi(f_{1/2} \circ \mu)_{\Pi_k}(A_k^k(+)) = 2n_k \psi(f_{1/2} \circ \mu)_{\Pi_k}(a)$$

where $a \in A_k^k(+)$. Let X_b, $b \in A_k$ be i.i.d. random variables that are uniformly distributed on $(0,1)$. Set $A_k^j = A_k^j(+) \cup A_k^j(-)$.

$$\psi(f_{1/2} \circ \mu)_{\Pi_k}(a)$$

$$= E\left(I\left(\sum_{b \in A_k} u(b) I(X_b \leqslant X_a) \in [1/2, 1/2 + \mu(a)) \right) \right)$$

$$\geqslant E\left(I(1/4 < X_a < 3/4) \prod_{j<k} I\left(\sum_{b \in A_k^j(+)} I(X_b \leqslant 1/4) = n_j \right) \right.$$

$$\cdot I\left(\sum_{b \in A_k^j(+)} I(X_b \geqslant 3/4) = n_i \right)$$

$$\cdot I\left(\sum_{b \in A_k^j(-)} I(X_b \leqslant 1/4) = n_i \right) I\left(\sum_{b \in A_k^j(-)} I(X_b \geqslant 3/4) = n_i \right)$$

$$\cdot I\Big(X_{a(1)} < 1/4,\ X_{a(2)} > 3/4,\ X_{b_{2(+)}} > 3/4,\ X_{b_1(+)} < 1/4,$$

$$\left. X_{b_1(-)} < 1/4,\ X_{b_2(-)} > 3/4 \Big) I\left(\sum_{b \in A_k^k} I(X_b \leqslant X_a) \in [0, \mu(a)) \right) \right).$$

As $(X_a)_{a \in A_k^k}$ are independent of $(X_b)_{b \notin A_k^k}$, we deduce that for $a \in A_k^k(+)$,

$$\psi(f_{1/2} \circ \mu)_{\Pi_k}(a) \geqslant E\left(\left[\prod_{j<k} I\left(\sum_{b \in A_k^j(+)} I(X_b \leqslant 1/4) = n_i \right) \right. \right.$$

$$\cdot I\left(\sum_{b \in A_k^j(+)} I(X_b \geqslant 3/4) = n_i \right)$$

$$\left. \cdot I\left(\sum_{b \in A_k^j(-)} I(X_b \leqslant 1/4) = n_i \right) I\left(\sum_{b \in A_k^j(-)} I(X_b \geqslant 3/4) = n_i \right) \right]$$

$$\cdot I\Big(X_{a(1)} < 1/4,\ X_{a(2)} > 3/4,$$

$$\left. X_{b_1(+)} < 1/4,\ X_{b_1(-)} > 1/4,\ X_{b_2(+)} > 3/4,\ X_{b_2(-)} > 3/4 \Big) \right)$$

$$\cdot E\left(I\left(\sum_{b \in A_k^k} I(X_b \leqslant X_a) \mu(b) \in [0, \mu(a)) \right) \cdot I(1/4 < X_a < 3/4) \right)$$

$$\geqslant 4^{-4\Sigma_{i<k} n_i} \cdot 4^{-6} \cdot \int_{1/4}^{3/4} E\left(I\left(\sum_{\substack{b \in A_k^k \\ b \neq a}} I(X_b \leqslant t) \mu(b) \in [-\mu(a), 0) \right) \right) dt$$

$$\geqslant c \cdot 4^{-4\Sigma_{i<k} n_i} / \sqrt{n_k}$$

for some positive constant c. Therefore

$$\psi(f_{1/2}\circ\mu)_{\Pi_k}(A_k^+) \geqslant \psi(f_{1/2}\circ\mu)_{\Pi_k}(A_k^k(+)) \geqslant 2\sqrt{n_k}\,c\cdot 4^{-4\Sigma_{i<k}n_i} \xrightarrow[n\to\infty]{} \infty.$$

It is of interest to characterize those games of the form $f\circ\mu$ where μ is a purely atomic signed measure that have an asymptotic value. Examples of specific questions in this direction are: (1) for fixed $-a<0<b$, characterize the monotonic functions $f:[-a,b]\to R$ for which any game of the form $f\circ\mu$ where μ is purely atomic with $\{\mu(S)|S\in\mathscr{C}\}\subset[-a,b]$, has an asymptotic value; (2) for a fixed purely atomic measure μ, characterize all monotonic functions f for which $f\circ\mu$ has an asymptotic value.

Next we would like to weaken the assumption of countable additivity of the measure μ. Under the standard definition of an asymptotic value, even pFA (i.e., the space generated by powers of finitely additive measures) is not contained in ASYMP. The essential reason is that for a finitely additive measure μ, the measures μ on (I,Π_n) where Π_n is an increasing sequence of finite fields that generates $\mathscr{C}$ do not disclose enough information regarding μ.

PROPOSITION. $pFA \not\subset$ ASYMP.

PROOF. Let μ be a nonatomic finitely additive probability measure. If the finitely additive measure μ is not countably additive there exists a decreasing sequence $(S_i)_{i=1}^{\infty}$ of coalitions with $S_i\searrow\varnothing$ and $\mu(S_i)\to_{i\to\infty}\alpha>0$ and $\alpha\neq 1$. We show that the game $v=\mu^3$ does not have an asymptotic value. First note that by the nonatomicity of μ (a finitely additive probability measure μ is nonatomic iff for every measurable set S with $\mu(S)>0$ there is a partition of S into measurable sets S_1 and S_2 such that $\mu(S_i)>\mu(S)/3$), there exists an increasing sequence of finite subfields Π_i with S_i an atom of Π_i and $\max\{\mu(A)|A$ an atom of Π_i with $A\neq S_i\}\to 0$ as $i\to\infty$. Then, for the game $v=\mu^3$, we have

$$\lim_{i\to\infty}\psi v_{\Pi_i}(S_i)=\int_0^1\left((t(1-\alpha)+\alpha)^3-(t(1-\alpha))^3\right)dt=\alpha-\alpha^2/2+\alpha^3/2$$

and therefore for each fixed k,

$$\lim_{i\to\infty}\psi v_{\Pi_i}(S_k)=\alpha-\alpha^2/2+\alpha^3/2+\left(1-\alpha+\alpha^2/2-\alpha^3/3\right)\left(\mu(S_k)-\alpha\right)/(1-\alpha).$$

As the right-hand side converges as $k\to\infty$ to $\alpha-\alpha^2/2+\alpha^3/3$ which differs from $\alpha=\lim_{k\to\infty}\mu(S_k)$, it follows that there is k for which $\lim_{i\to\infty}\psi v_{\Pi_i}(S_k)\neq\mu(S_k)$. There exists an increasing sequence of finite subfields Π_i^1 with $S_k\in\Pi_i^1$ and $\bigcup_{i=1}^{\infty}\Pi_i^1$ generates $\mathscr{C}$, for which $\max\{\mu(A)|A$ an atom of $\Pi_i^1\}\to 0$ as $i\to\infty$. For such a sequence, $\lim_{i\to\infty}\psi v_{\Pi_i^1}(S_k)=\mu(S_k)$. Altogether we deduce that $v=\mu^3$ does not have an asymptotic value.

The arguments show actually that if μ is a finitely additive nonatomic probability measure then μ^3 has an asymptotic value if and only if μ is countably additive.

We introduce now a slight modification of the definition of an asymptotic value.

DEFINITION. Let $v:\mathscr{C}\to R$ with $v(\varnothing)=0$ be a game. A game ϕv is said to be the weak asymptotic value of v if: for every $S\in\mathscr{C}$ and every $\epsilon>0$, there is a finite subfield Π with $S\in\Pi$ such that for any finite subfield Π' with $\Pi'\supset\Pi$, $|\psi v_{\Pi'}(S)-\phi v(S)|<\epsilon$.

REMARKS. (1) Let $v: \mathscr{C} \to R$ with $v(\varnothing)$ be a game. The game v has a weak asymptotic value if and only if for every S in $\mathscr{C}$ and $\epsilon > 0$ there exists a finite subfield Π with $S \in \Pi$, such that for any finite subfield Π^1 with $\Pi^1 \supset \Pi$, $|(\psi v_{\Pi^1})(S) - (\psi v_{\Pi})(S)| < \epsilon$.

(2) A game v has at most one weak asymptotic value.

(3) The weak asymptotic value is a finitely additive game.

(4) If v is a finitely additive game then v has a weak asymptotic value $\phi v = v$.

(5) If v has an asymptotic value ϕv then v has a weak asymptotic value $(= \phi v)$.

Remarks (1)–(4) are obvious. For completeness we present a proof of Remark (5): assume that a game v does not have a weak asymptotic value. Then, there exists S in $\mathscr{C}$ and $\epsilon > 0$ such that for any finite subfield Π with $S \in \Pi$ there exists a finite subfield Π^1 with $\Pi^1 \supset \Pi$ and $|\psi v_{\Pi^1}(S) - \psi v_{\Pi}(S)| > \epsilon$. Let $(\Pi_k)_{k=1}^{\infty}$ be an increasing sequence of finite subfields such that $S \in \Pi_1$ and $\bigcup_{i=1}^{\infty}\Pi_i$ generates $\mathscr{C}$. Define an increasing sequence of finite subfields $(\Pi_k^1)_{k=1}^{\infty}$ inductively by: $\Pi_k^1 \supset \Pi_k \cup \Pi_{k-1}^1$ is such that $|\psi v_{\Pi_k^1}(S) - \psi v_{\Pi_k^*}(S)| > \epsilon$ where Π_k^* is the finite field generated by $\Pi_k \cup \Pi_{k-1}^1$. Note that both $(\Pi_k^1)_{k=1}^{\infty}$ and $(\Pi_k^*)_{k=1}^{\infty}$ are increasing sequences of finite subfields and as $\Pi_k^* \supset \Pi_k$ and $\Pi_k^1 \supset \Pi_k$ both $\bigcup_{i=1}^{\infty}\Pi_i^*$ and $\bigcup_{i=1}^{\infty}\Pi_i^1$ generate $\mathscr{C}$. As $\underline{\lim}_{i\to\infty}|\psi v_{\Pi_k^1}(S) - \psi v_{\Pi_k^*}(S)| \geqslant \epsilon > 0$ it follows that the game v does not have an asymptotic value.

The following is an almost direct consequence of this definition and our main result.

THEOREM. *The set of all games having a weak asymptotic value is a linear symmetric space of games and the operator mapping each game to its weak asymptotic value is a value on that space. If $ASYMP^*$ denotes all games with bounded variation having a weak asymptotic value then $ASYMP^*$ is a closed subspace of BV with $bv'\mathrm{FA} \subset ASYMP^*$.*

PROOF. The only part that might need some explanation is the one that asserts that $bv'\mathrm{FA} \subset \mathrm{ASYMP}^*$. For that, it is enough to verify that for every $\mu \in \mathrm{FA}_1^+$ (where FA_1^+ denotes all positive finitely additive games v with total mass 1, i.e., with $v(I) = 1$) and $f \in bv'$, $f \circ \mu$ has a weak asymptotic value. For that it suffices to show that for every S in C, there is a sequence of finite subfields $(\Pi_k)_{k=1}^{\infty}$ with $S \in \Pi_k$, such that for every sequence of finite subfields $(\Pi_k^*)_{k=1}^{\infty}$ with $\Pi_k^* \supset \Pi_k$.

$$(*) \qquad \lim_{k\to\infty} |\psi(f\circ\mu)_{\Pi_k}(S) - \psi(f\circ\mu)_{\Pi_k^*}(S)| = 0.$$

Any μ in FA_+^1 has a decomposition as a countable sum $\mu = \sum_{i\geqslant 0}\alpha_i\mu_i$ where $\sum_{i\geqslant 0}\alpha_i = 1$, $\alpha_i \geqslant 0$ and μ_0 is a nonatomic element of FA_1^+ and for every $i \geqslant 1$, μ_i is a $\{0,1\}$-valued measure in FA_1^+ with $\mu_i \neq \mu_j$ wherever $i \neq j$. Let Π_k be a sequence of finite subfield with $S \in \Pi_k$, such that for any atom A of Π_k and $1 \leqslant i < j \leqslant k$, $\mu_i(A)\mu_j(A) = 0$ and $\max\{\mu_0(B)|B$ an atom of $\Pi_k\} < 1/k$ ($\to 0$ as $k \to \infty$) (one of the equivalent definitions of a nonatomic finitely additive positive measure μ_0 is that for every $\epsilon > 0$ there is a finite subfield Π with $\max\{\mu_0(B)|B$ an atom of $\Pi\} < \epsilon$). Our proof showed that in this case for every sequence of finite fields Π_k^* with $\Pi_k^* \supset \Pi_k$ both limits $\lim_{k\to\infty}\psi(f\circ\mu)_{\Pi_k}(S)$ and $\lim_{k\to\infty}\psi(f\circ\mu)_{\Pi_k^*}(S)$ exist and are equal and therefore $(*)$ follows which proves that $f \circ \mu \in \mathrm{ASYMP}^*$.

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