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Common knowledge of rationality and market clearing in economies with asymmetric information ☆

Elchanan Ben-Porath [a,*], Aviad Heifetz [b]

[a] The Department of Economics and the Center for the Study of Rationality, The Hebrew University of Jerusalem, Israel
[b] The Economics and Management Department, The Open University of Israel, Israel

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Abstract

Consider an exchange economy with asymmetric information. What is the set of outcomes that are consistent with common knowledge of rationality and market clearing?

To address this question we define an epistemic model for the economy that provides a complete description not only of the beliefs of each agent on the relationship between states of nature and prices but also of the whole system of interactive beliefs. The main result, Theorem 1, provides a characterization of outcomes that are consistent with common knowledge of rationality and market clearing (henceforth, *CKRMC* outcomes) in terms of a solution notion – *Ex-Post Rationalizability* – that is defined directly in terms of the parameters that define the economy. *CKRMC* manifests several intuitive properties that stand in contrast to the full revelation property of *Rational Expectations Equilibrium*. In particular, for a robust class of economies: (1) there is a continuum of prices that are consistent with *CKRMC* in every state of nature, and hence these prices do not reveal the true state, (2) the range of *CKRMC* outcomes is monotonically decreasing as agents become more informed about the economic fundamentals, and (3) trade is consistent with common knowledge of rationality and market clearing even when there is common knowledge that there are no mutual gains from trade.

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* Corresponding author.
E-mail addresses: benporat@math.huji.ac.il (E. Ben-Porath), aviadh@openu.ac.il (A. Heifetz).

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1. Introduction

We study the implications of common knowledge of rationality and market clearing in economies with asymmetric information.

The starting point is the concept of *Rational Expectations Equilibrium* (*REE*). *REE* extends the classical concept of a competitive equilibrium to economies with asymmetric information (i.e., economies in which different agents might have different information). When each agent has only partial information on the value of a commodity or an asset he can deduce additional information from the prices because prices reflect the information that other agents have. *REE* is a solution concept that is based on the assumption that agents make these inferences. However, the concept of *REE* is based on an additional strong assumption that agents know (and therefore agree on) the function that specifies the prices in each state of nature. (A state of nature specifies the real variables of the economy, i.e., preferences and endowments.) As Radner [16] has shown, this strong assumption leads to the strong result that in a generic economy with a finite number of states the only *REE* is a fully revealing equilibrium, i.e., an equilibrium in which each agent can infer from the prices all the information that any other agent has. This conclusion is at odds both with intuition and real-world practice. To take just one example, the daily volume of trade in foreign exchange is significantly larger than the value of international trade, an indication that much of the former is speculative and based on a non-unanimous evaluation of the information embedded in prices.

In the present paper the assumption that players know the price function is relaxed, that is, we consider a situation where each agent may have a different theory on how the vector of prices that is observed materialized and on what would have happened in other states of nature. However, the assumption is maintained that each agent makes inferences from the observed prices and furthermore assumes that other agents are doing likewise. More precisely we ask: what is the set of outcomes that are consistent with *common knowledge of rationality and market clearing*?

To address this question we define an epistemic model of an economy with asymmetric information. This model provides a complete description of the beliefs of each agent not only on the relationship between states of nature and prices but also on the beliefs of other agents. In this model consistency with common knowledge of rationality and market clearing can be defined in a precise way. Our main result, Theorem 1, establishes that under a mild qualification an outcome (s, p), where s is a state of nature and p a vector of prices, is consistent with common knowledge of rationality and market clearing iff it is an *Ex-Post Rationalizable* outcome. Ex-Post Rationalizability is a solution notion that is defined directly in terms of the parameters that define the economy and does not involve type spaces. However, we view it as a derived notion, the fundamental concept being Common Knowledge of Rationality and Market Clearing. (Henceforth, we will often abbreviate and refer to this solution concept as *CKRMC*.)

We use the characterization of Theorem 1 to compute the set of prices that are consistent with common knowledge of rationality and market clearing in a simple example (Example 1) of an economy with two commodities, two states of nature, and a population that consists of two types of agents: informed agents who know the true state and uninformed agents who have no information.[1] We obtain several properties that stand in contrast to the full revelation property of *REE*. Specifically, when the fraction of uninformed agents is large enough (in our example larger than $\frac{1}{2}$) then:

(a) There is a whole range of prices that are consistent with common knowledge of rationality and market clearing in both states of nature and therefore these prices do not reveal any information.

(b) An increase in the fraction of informed agents strictly shrinks the set of *CKRMC* prices.

(c) Trade is consistent with common knowledge of rationality and market clearing despite the fact that it is common knowledge that there are no mutual gains from trade.

The study of the implications of eductive solution concepts (that is, solutions that are based on the assumption that each agent reasons about the reasoning of other agents) in competitive economies was pioneered by Guesnerie. (See Guesnerie [10] for a collection of papers employing an eductive approach.) In particular, Desgranges and Guesnerie [7][2] examine iterative deletion of weakly dominated demand strategies in a simple example that is similar to Example 1 in the present paper. The solution set that they obtain is equal to the set of *CKRMC* and Ex-Post Rationalizable outcomes that is obtained in the present paper. In particular, they show that when the fraction of informed agents increases the set of prices that can be generated by demands that survive iterative deletion of dominated strategies shrinks. Closest to our work is Desgranges [6]. Desgranges defines the notion of Ex-Post Rationalizability,[3] applies it to the example studied in the paper with Guesnerie, and obtains the properties (a) and (b) that are mentioned above.[4] The main difference between Desgranges work and the present paper is that in Desgranges' paper Ex-Post Rationalizability is the starting point while in the present study it is a derived notion that is justified only because it is a useful characterization of the fundamental concept: common knowledge of rationality and market clearing. This is the content of our main result (Theorem 1).

MacAllister [13] and Dutta and Morris [8] propose a solution concept, Belief Equilibrium, which is stronger than *CKRMC* as it assumes that in addition to common knowledge of rationality and market clearing there is also common knowledge of the belief of each player on the joint distribution of prices and states of nature. As we show in Section 3 this additional assumption restricts in a significant way the set of possible outcomes.

Our approach is similar to the one taken in the literature on the epistemic foundations of solution concepts in game theory. The general goal of this literature is to clarify the assumptions that underlie different solution concepts by applying a model of interactive beliefs.[5] In particular, Tan and Werlang [17] use an epistemic model to establish that the set of outcomes that are consistent with common knowledge of rationality in a strategic game is equivalent to the set of rationalizable outcomes defined by Bernheim [4] and Pearce [15]. The present paper has a similar goal in the context of competitive economies with asymmetric information.

Despite the fact that a system of interactive beliefs is at the heart of the eductive approach there is only one other paper – Morris [14] – that we are aware of that applies the epistemic approach to the analysis of competitive economies. Morris shows that if there is a common prior

[1] In Ben-Porath and Heifetz [3] the analysis is extended to a general class of economies with two commodities.

[2] See also Chapter 8 in Guesnerie [10] and Section 7 in Guesnerie [9].

[3] Desgranges uses the term common knowledge equilibrium.

[4] A first draft of Desgranges' paper was written before ours. We developed the concept of Ex-Post Rationalizability before we learned of his work.

[5] The literature that applies the epistemic approach to the analysis of game-theoretic solution concepts is by now fairly extensive. Dekel and Gul [5] and Battigalli and Bonanno [2] provide excellent overviews.

on the set of states of the world, where a state of the world specifies not only the fundamentals of the economy (preferences and endowments) but also the whole system of interactive beliefs, then common knowledge of rationality and market clearing implies that the correspondence between states of the *world* and prices is a Rational Expectations Equilibrium. By contrast we do not assume a common prior on the states of the world and our interest (as in the rest of all the papers that were cited) is in the correspondence between states of *nature* and prices (where a state of nature specifies only the fundamentals).

We now present a simple example that motivates the discussion.

Example 1. There are two commodities in the economy, X and M (money).

The set of states is $S = \{1, 3\}$.

The probability of each state is 0.5.

The set of agents is the interval $[0, 1]$. There are two types of agents, I_1 and I_2. Agents in I_1 know the true state; agents in I_2 do not know it. $I_1 = [0, \delta]$ and $I_2 = (\delta, 1]$. All the agents have the same utility function and the same initial bundle. The utility function is

$$u(x, m, s) = s \times \log(x) + m \tag{1.1}$$

where x and m are the quantities of X and M respectively and s is the state.

The initial bundle consists of one unit of X and $\overline{m}$ units of M where $\overline{m} \geqslant 3$.

Let p be the price of a unit of X in units of M. It follows from the definition of the utility function in (1.1) that the demand for X of an agent i who assigns to the state s probability $\gamma(s)$ is

$$x = \frac{\gamma(1) \times 1 + \gamma(3) \times 3}{p} \tag{1.2}$$

In this example for every $\delta > 0$ there is only one *REE*, f^*, where $f^*(s) = s$. To see that we, first, note that if f is an *REE* then $f(1) \neq f(3)$. This follows because if $f(1) = f(3) = p$ then agents in I_2 do not obtain any information about the true state and therefore their demand in both states is the same:

$$x = \frac{0.5 \times 1 + 0.5 \times 3}{p} = \frac{2}{p}$$

However, the demand of agents from I_1 in state 1 is different from their demand in state 3 and therefore the aggregate demands are different as well. Since the aggregate amount of X is fixed this means that the market doesn't clear in at least one of the states and therefore f is not an *REE*. Thus, if f is an *REE* then $f(1) \neq f(3)$. In this case agents in I_2 infer the state from the price and it follows from (1.2) that $f(1) = 1$ and $f(3) = 3$. Thus, the only *REE* is a fully revealing equilibrium in which the price reveals the state. Alternatively put, the only outcomes (p, s) (where p is a price and $s \in S$ is a state) that are consistent with *REE* are $(1, 1)$ and $(3, 3)$.

We now demonstrate in an informal way that with heterogeneous beliefs there are outcomes different from the *REE* outcome that are consistent with common knowledge of rationality and market clearing. In Section 2 we present the formal framework.

Assume that the fraction of informed agents in the economy is $\delta = \frac{1}{6}$. Define two price functions f and g as follows:

$$f(1) = 2 \qquad g(1) = 1$$
$$f(3) = 3 \qquad g(3) = 2$$

We now explain (informally) why f and g are consistent with common knowledge of rationality implying that the outcomes $(2, 1)$ and $(2, 3)$ are *CKRMC* outcomes.

Suppose that a fraction β of the agents in I_2 assign probability $\frac{3}{4}$ to the event that f is the price function and probability $\frac{1}{4}$ to the event that g is the price function; call this belief 'theory A.' Assume that the other agents in I_2 think that g is more likely and assign probability $\frac{1}{4}$ to the event that f is the price function and probability $\frac{3}{4}$ to the event that the price function is g; call this belief 'theory B.'

What are the beliefs of different agents in I_2 about the true state when they observe the price 2?

Since the prior assigns probability 0.5 to each state it is easy to see that agents in I_2 who believe in theory A assign probability $\frac{3}{4}$ to the state 1 and probability $\frac{1}{4}$ to the state 3.[6] Similarly, agents who believe in theory B assign probability $\frac{1}{4}$ to the state 1 and probability $\frac{3}{4}$ to the state 3.

It follows from (1.2) that the demand for X at price 2 of agents who believe in theory A is $(\frac{3}{4} \times 1 + \frac{1}{4} \times 3)/2 = \frac{3}{4}$ while the demand of agents who believe in theory B is $(\frac{3}{4} \times 3 + \frac{1}{4} \times 1)/2 = \frac{5}{4}$.

Let $x(\beta, s, p)$ denote the aggregate demand for X in state s at price p when a proportion β of the agents in I_2 believe in theory A and the rest of I_2 believe in theory B. We have

$$x(\beta, 1, 2) = (1 - \delta) \times \beta \times \frac{3}{4} + (1 - \delta) \times (1 - \beta) \times \frac{5}{4} + \delta \times \frac{1}{2}$$

$$x(\beta, 3, 2) = (1 - \delta) \times \beta \times \frac{3}{4} + (1 - \delta) \times (1 - \beta) \times \frac{5}{4} + \delta \times \frac{3}{2}$$

Let β_f and β_g be the numbers that equate demand and supply at price 2 in the states 1 and 3 respectively, that is, $x(\beta_f, 1, 2) = 1$ and $x(\beta_g, 3, 2) = 1$. For $\delta = \frac{1}{6}$ we obtain $\beta_f = 0.3$ and $\beta_g = 0.7$.

Now we observe that when β_f of the agents in I_2 believe in theory A and $1 - \beta_f$ of them believe in B then the function f specifies prices that clear the market. (We have just seen that the price 2 clears the market in $s = 1$ and when the price is 3 everyone assigns probability 1 to the state 3 and therefore the price 3 clears the market.) Similarly, when β_g of the agents in I_2 believe in theory A (and the rest in B) the function g specifies prices that clear the market.

In Section 2 we present an epistemic model for an exchange economy with asymmetric information and use it to define common knowledge of rationality and market clearing. We then show how to formalize the analysis of Example 1. With this formalization, it will become explicit how there can be common knowledge that each agent in I_2 entertains either of the theories A and B, and that the price $p = 2$ is consistent with common knowledge of rationality and market clearing in both states $s = 1, 3$. In Section 3 we define the concept of Ex-Post Rationalizability and present Theorem 1, which establishes that under a mild qualification an outcome is consistent with common knowledge of rationality and market clearing iff it is an Ex-Post Rationalizable outcome. All the proofs are relegated to Appendix A.

[6] Let $P_A(s|p = 2)$ denote the posterior that an agent who believes in theory A assigns to the state s upon observing the price 2. Then $P_A(1|p = 2) = \frac{0.75 \cdot 0.5}{0.75 \cdot 0.5 + 0.25 \cdot 0.5} = 0.75$.

2. The model

In this section we review the definitions of an exchange economy with asymmetric information and the concept of *Rational Expectations Equilibrium*. We then present an epistemic model of the economy and use it to define the concept of *Consistency with Common Knowledge of Rationality and Market Clearing* (which we will often abbreviate to *CKRMC*). An *outcome* (p, s), where p is a vector of prices and s is a state of nature, is a *CKRMC* outcome if it is consistent with common knowledge of rationality and market clearing. We then demonstrate how the epistemic model can be used to make the analysis of Example 1 complete and precise.

An economy with asymmetric information is defined by:

1. $I = [0, 1]$ – The set of players (consumers).
2. $X_1, \ldots, X_K$ – K commodities.
3. $S = \{s_1, \ldots, s_n\}$ – The set of states of nature.
4. π_i – A partition on S that describes the information of player i.
   $\pi_i(s) \subseteq S$ is the information that player i gets at the state s.
5. $\alpha_i \in \Delta(S)$ – The prior probability of agent i on S.[7]
6. $u_i : R^K \times S \to R$ – A VNM utility function for player i.
   $u_i(x, s)$ is the utility of player i from a bundle
   $x \in R^K$ in the state s.
7. $e_i : S \to R^K$ – $e_i(s)$ is the initial bundle of player i at state s.
   We assume that e_i is measurable w.r.t. π_i and that
   $\forall s \in S$, $\int_i e_i(s)\,\mathrm{d}i$ – the aggregate supply in state s – exists.

A *price* p is a vector $p = (p_1, \ldots, p_{K-1})$ where p_k is the price of X_k. The price of X_K is normalized to be 1.

A *price function* f, $f : S \to R^{K-1}$, assigns to every state s a price $f(s)$.

We let L_i denote the set of signals of agent i. So, $L_i \equiv \{\pi_i(s){:}\ s \in S\}$.

A *demand function* for player i is a function z_i, $z_i : L_i \times R^{K-1} \to R^K$, such that $z_i(l_i, p)$ is in the budget set defined by the price p and the initial endowment $e_i(l_i)$. ($e_i(l_i)$ is well defined because e_i is measurable w.r.t. π_i.)

It is assumed that the agents do not observe the supply and demand of the commodities, but only the prevailing price vector $p \in R^{K-1}$. The standard solution notion for economies with asymmetric information is *Rational Expectations Equilibrium*, *REE*. A *REE* is a price function f such that for each state s the price $f(s)$ clears the market when every agent i exhibits a demand that is optimal w.r.t. the price $f(s)$ and the information that is revealed by his private signal $\pi_i(s)$ and the fact that the price is $f(s)$. Formally:

Definition. A price function f is an *REE* if there exists a profile of demand functions, $\{z_i\}_{i \in I}$, that satisfies:

1. Rationality, $\forall s \in S$, $z_i(\pi_i(s), f(s))$ is optimal w.r.t. the price $f(s)$ and the posterior $\alpha_i(\cdot|\pi_i(s) \cap f^{-1}(f(s)))$.
2. Market clearing, $\forall s \in S$, $\int_i z_i(\pi_i(s), f(s))\,\mathrm{d}i = \int_i e_i(s)\,\mathrm{d}i$.

A price function f is a *fully revealing REE* if $f(s) \neq f(s')$ when $s \neq s'$.

7 If $\alpha_i = \alpha$ for every $i \in I$ then there is a common prior. However, our results do not require such an assumption.

As we pointed out in the Introduction, the concept of Rational Expectations Equilibrium refers to a situation where all the agents know the price function. In particular, all the agents have the same belief regarding the relationship between the prices and the states of nature. We are interested here in a solution concept that is akin to the concept of Rationalizability in game theory. That is, we ask what is the set of outcomes in the economy when agents may have heterogeneous beliefs concerning the relationship between prices and states and yet there is common knowledge of rationality and market clearing?

To address this question we now define an epistemic model in the spirit of Harsanyi [11] where we represent the choice and belief of each agent i by a *type* t_i in a measurable space T_i and the entire economic situation by a state of the world

$$\omega = \big(s, p, (t_i)_{i\in I}\big) \in \Omega \subseteq S \times R_+^{K-1} \times \prod_{i\in I} T_i$$

Each type $t_i \in T_i$ of agent $i \in I$ is associated with:

1. Its demand function $z_i[t_i]$. That is, for each observed price vector $p' \in R_+^{K-1}$ for which there is a state of the world $(s', p', (t'_j)_{j\in I}) \in \Omega$ with $t'_i = t_i$, $z_i[t_i](\pi_i(s'), p')$ is a bundle that is feasible for agent i given the prices p':

$$z_i[t_i]\big(\pi_i\big(s'\big), p'\big) \cdot p' \leqslant e_i\big(\pi_i\big(s'\big)\big) \cdot p'$$

2. Its ex-ante belief $b_i[t_i] \in \Delta(\Omega)$ about the states of the world, having the property that its marginal on the space of states of nature S is the agent's prior –

$$\mathrm{marg}_S\, b_i[t_i] = \alpha_i$$

and that it knows its own type:

$$b_i[t_i](E) = 1 \quad \text{whenever } \big\{\big(s', p', \big(t'_j\big)_{j\in I}\big) \in \Omega\colon\ t'_i = t_i\big\} \subseteq E$$

We denote by $b_i[t_i|\pi_i(s'), p']$ the interim belief of type t_i for each combination of an observed price $p' \in R_{++}^{n-1}$ and a private signal about the state of nature in a state of the world $(s', p', (t'_j)_{j\in I}) \in \Omega$ with $t'_i = t_i$.

A model $\mathcal{M}$ of the economy is a collection $\mathcal{M} = ((T_i, b_i, z_i)_{i\in I}, \Omega)$. The model $\mathcal{M}$ satisfies Common Knowledge of Market Clearing if for each state of the world $(s, p, (t_i)_{i\in I}) \in \Omega$ the aggregate demand is well defined and equals the aggregate supply, which is well defined as well:

$$\int_I z_i[t_i]\big(\pi_i(s), p\big)\,\mathrm{d}i = \int_I e_i\big(\pi_i(s)\big)\,\mathrm{d}i$$

Indeed, in a model $\mathcal{M}$ with this property not only does the market clear in each state of the world, but also each agent is certain (i.e., assigns probability 1 to the event) that the markets clear, each agent is certain that all agents are certain that the markets clear, and so on *ad infinitum*.

Similarly, the model $\mathcal{M}$ satisfies Common Knowledge of Rationality if for each state of the world $(s, p, (t_i)_{i\in I}) \in \Omega$ the bundle consumed by each agent maximizes the agent's expected utility, i.e.,[8]

[8] Clearly, what is relevant for the maximization problem is the marginal belief on S. So we could equivalently write $z_i[t_i](\pi_i(s'), p') \in \arg\max_{x_i \cdot p \leqslant e_i(\pi_i(s))\cdot p} \sum_{s\in S} u_i(x_i, s)\, \mathrm{marg}_S\, b_i[t_i|\pi_i(s'), p](s)$.

$$z_i[t_i](\pi_i(s), p) \in \arg \max_{x_i \cdot p \leqslant e_i(\pi_i(s)) \cdot p} \int_{\Omega} u_i(x_i, s)\, \mathrm{d}b_i[t_i | \pi_i(s), p]$$

When both properties are satisfied we say that $\mathcal{M}$ satisfies Common Knowledge of Rationality and Market Clearing.

Definition. The price vector p is consistent with common knowledge of rationality and market clearing (CKRMC) at the state of nature $s \in S$, if there exists a model $\mathcal{M}$ satisfying common knowledge of rationality and market clearing which contains a state of the world $(s, p, (t'_j)_{j \in I}) \in \Omega$. In such a case we also say that (p, s) is a CKRMC outcome.

Notice that this definition is *epistemic*, in the sense that it relies on the existence of a *type space* Ω with particular properties. In Section 3 our aim will be to provide a characterization of *CKRMC* prices in terms of properties of the basic economy $(I, S, (\pi_i(\cdot), e_i(\pi_i(\cdot), u_i(\cdot,\cdot)))_{i \in I})$.

It is useful to see how the concepts of *REE* and Belief Equilibrium (Dutta and Morris 1997) can be represented in our framework:

1. An *REE* f, $f : S \to R^{K-1}$, can be represented by a model where each agent i has a single type $\widehat{t_i}$, $\Omega = \{(s, f(s), (\widehat{t_i})_{i \in I}) | s \in S\}$ and $b_i[\widehat{t_i}](s, f(s), (\widehat{t_j})_{j \in I}) = \alpha_i(s)$.

2. A Belief Equilibrium is defined by a profile of functions $(\delta_i)_{i \in I}$, $\delta_i : S \to \Delta(R^{K-1})$, where $\delta_i(s)$ is the conditional probability distribution of agent i on *prices* given s. It is assumed that for $s \in S$ the distributions $(\delta_i)_{i \in I}$ have a common support. The profile $(\delta_i)_{i \in I}$ is a Belief Equilibrium if for every $s \in S$ and $p \in R^{K-1}$ such that $\delta_i(s)(p) > 0$ for every $i \in I$, the price p clears the market at the state s (when each agent i chooses an optimal bundle w.r.t. his conditional probability on S given his private signal and the price p). Thus, a Belief Equilibrium is a more permissive solution concept than *REE* because it allows different agents to have different beliefs on the relationship between prices and states. On the other hand, Belief Equilibrium is more restrictive than *CKRMC* because it (implicitly) assumes that the beliefs of each agent are common knowledge. In terms of our framework a Belief Equilibrium corresponds to a model where each agent has just one type. To see that, we now define a model that corresponds to a Belief Equilibrium, $(\delta_i)_{i \in I}$. Let $Q(s) = \{p \mid \delta_i(s)(p) > 0,\ \forall i \in I\}$. Define now $T_i = \{\widehat{t_i}\}$,

$$\Omega = \{(s, p, (\widehat{t_i})_{i \in I}) \mid p \in Q(s)\} \quad \text{and} \quad b_i[\widehat{t_i}](s, p, (\widehat{t_j})_{j \in I}) = \alpha_i(s) \cdot \delta_i(s)(p)$$

Thus, the definition of *CKRMC* is more general than the definition of Belief Equilibrium because it allows different types of a given agent to have different beliefs on the types of the other agents. In particular, agent i is uncertain about the beliefs of the other agents. This is in line with our motivation to define the most permissive model that is consistent with common knowledge of rationality and market clearing. Our approach is consistent with the concept of Rationalizability that is defined in the game-theoretic literature (Bernheim [4] and Pearce [15]). In particular, Rationalizability does not assume that the beliefs of each player are known to the other players.

2.1. A formalization of Example 1

We now describe Example 1 in terms of our epistemic model.

The set of agents who are informed about the state of nature is $I_1 = [0, \delta]$, where $\delta = \frac{1}{6}$. The demand of an agent who knows that state of nature is determined by the price and is not affected by his beliefs on the behavior of other agents. Thus, we can omit the definition of these beliefs from the description of our formal model and focus only on the beliefs and demands of the uninformed agents.

The set of uninformed agents is $I_2 = (\delta, 1]$. The agents in I_2 are divided into 10 cohorts $(\delta, \delta + 0.1(1-\delta)]$, $(\delta + 0.1(1-\delta), \delta + 0.2(1-\delta)], \ldots, (\delta + 0.9(1-\delta), 1]$. In each state of the world the types of all the agents within a given cohort are identical and completely correlated, so we can effectively speak of 10 (representative) agents $j = 1, \ldots, 10$, where each agent $r \in (\delta, 1]$ is represented by a representative agent from her cohort.

Each agent $j = 1, \ldots, 10$ has two possible types

$$t_j^A, t_j^B$$

Before we proceed with the formal definitions we make some general comments on the structure of the model. The beliefs of type t_j^A and t_j^B correspond respectively to theories A and B that were described in the Introduction. That is, type t_j^A assigns probability $\frac{3}{4}$ to the event that the profile of types induces the price function f and a probability of $\frac{1}{4}$ to the event that the profile of types induces the function g; type t_j^B assigns probability $\frac{3}{4}$ to the event that the profile of types induces the price function g and a probability of $\frac{1}{4}$ to the event that the profile of types induces the function f. In addition, each type t_j^x, $x \in \{A, B\}$, of each player assigns a probability $\frac{1}{2}$ to each one of the two states of nature, and the beliefs of t_j^x on the state of nature are independent of his beliefs on the possible profile of types. In particular, at the price $p = 2$,

- the (conditional) belief of type t_j^A assigns marginal probability $\frac{3}{4}$ to the state of nature $s = 1$ and marginal probability $\frac{1}{4}$ to the state of nature $s = 3$;
- the (conditional) belief of type t_j^B assigns marginal probability $\frac{3}{4}$ to the state of nature $s = 3$ and marginal probability $\frac{1}{4}$ to the state of nature $s = 1$.

In contrast, at the price $p = s$ (for each of the two states of nature $s = 1, 3$) each type is certain that the state of the world is s.

The demand $z_j[t_j^x](p)$[9] of type t_j^x maximizes his expected utility w.r.t. his conditional beliefs. Thus, following the calculation in the Introduction we define the demand of each type for the commodity X at the prices $p = 1$ and $p = 3$ to be one unit. The demand of a type t_j^A for X at the price $p = 2$ is $\frac{3}{4}$ while type t_j^B demands $\frac{5}{4}$ units of X.

We now explain how to embed these types in a model that satisfies common knowledge of rationality and market clearing.

In the model there are altogether 20 type profiles denoted $\bar{t}^{k,f}, \bar{t}^{k,g}$ for $k = 1, \ldots, 10$. (As will soon become clear, the profile $\bar{t}^{k,f}$ ($\bar{t}^{k,g}$) induces the price function $f(g)$.) Each type profile contains one type for each agent $j = 1, \ldots, 10$, in a way that will be specified below. In the state of nature $s = 1$ the possible prices are $p = 1, 2$; in the state of nature $s = 3$ the possible prices are $p = 2, 3$. The set of states of the world in the model is

$$\Omega = \{(1, 2, \bar{t}^{k,f}), (1, 1, \bar{t}^{k,g}), (3, 2, \bar{t}^{k,g}), (3, 3, \bar{t}^{k,f})\}_{k=1,\ldots,10}$$

Denote by $\oplus$ and $\ominus$ addition and subtraction modulo 10, respectively.

9 Since the agents have no information about the state of nature we omit reference to the informational signal in the definition of the demand function.

For each agent j the type t_j^A is a member of 10 out of the 20 type profiles, and the other type t_j^B of agent j is a member of the remaining 10 profiles:

$$t_j^A \quad \text{is in the type profile } \bar{t}^{k,f} \quad \text{iff} \quad 0 \leqslant |k \ominus j| \leqslant 1$$
$$t_j^A \quad \text{is in the type profile } \bar{t}^{k,g} \quad \text{iff} \quad 2 \leqslant |k \ominus j| \leqslant 8$$

and similarly

$$t_j^B \quad \text{is in the type profile } \bar{t}^{k,g} \quad \text{iff} \quad 0 \leqslant |k \ominus j| \leqslant 1$$
$$t_j^B \quad \text{is in the type profile } \bar{t}^{k,f} \quad \text{iff} \quad 2 \leqslant |k \ominus j| \leqslant 8$$

In other words, for $k = 1, \dots, 10$,

$$\bar{t}^{k,f} = \{t_{k\ominus 1}^A, t_k^A, t_{k\oplus 1}^A, t_{k\oplus 2}^B, \dots, t_{k\oplus 8}^B\} \qquad \bar{t}^{k,g} = \{t_{k\ominus 1}^B, t_k^B, t_{k\oplus 1}^B, t_{k\oplus 2}^A, \dots, t_{k\oplus 8}^A\}$$

Thus, in a profile $\bar{t}^{k,f}$ ($\bar{t}^{k,g}$) 30 percent of the population have beliefs that correspond to theory $A(B)$ and the other 70 percent have beliefs that correspond to theory $B(A)$. As we saw in the Introduction the profile of demands that corresponds to a profile $\bar{t}^{k,f}$ induces the function f and the profile of demands that correspond to a profile $\bar{t}^{k,g}$ induces the function g.

We now turn to the definition of the beliefs. Recall that we want a type $t_j^A(t_j^B)$ to: (1) assign a probability $\frac{3}{4}$ to the event that the profile of types induces the function $f(g)$, (2) assign a probability $\frac{1}{2}$ to each state of nature, and (3) let the two beliefs (1) and (2) are to be independent of one another. Consider a type t_j^A; a simple way of defining his beliefs so that he assigns a probability $\frac{3}{4}$ to profiles of types that induce f and a probability $\frac{1}{4}$ to profiles of types that induce g is to have him assign to each one of the three profiles that induce f to which he belongs a probability of $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$ and similarly to have him assign to each one of the seven profiles that induce g to which he belongs a probability $\frac{1}{4} \times \frac{1}{7} = \frac{1}{28}$. The beliefs of a type t_j^B will be defined in a similar way. With this in mind we now define the beliefs as follows:

$$b_j[t_j^A](1, 2, \bar{t}^{k,f}) = b_j[t_j^A](3, 3, \bar{t}^{k,f}) = \frac{1}{2} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{8} \quad \text{for } 0 \leqslant |k \ominus j| \leqslant 1$$
$$b_j[t_j^A](3, 2, \bar{t}^{k,g}) = b_j[t_j^A](1, 1, \bar{t}^{k,g}) = \frac{1}{2} \times \frac{1}{4} \times \frac{1}{7} = \frac{1}{56} \quad \text{for } 2 \leqslant |k \ominus j| \leqslant 8$$

and

$$b_j[t_j^B](3, 2, \bar{t}^{k,g}) = b_j[t_j^B](1, 1, \bar{t}^{k,g}) = \frac{1}{2} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{8} \quad \text{for } 0 \leqslant |k \ominus j| \leqslant 1$$
$$b_j[t_j^B](1, 2, \bar{t}^{k,f}) = b_j[t_j^B](3, 3, \bar{t}^{k,f}) = \frac{1}{2} \times \frac{1}{4} \times \frac{1}{7} = \frac{1}{56} \quad \text{for } 2 \leqslant |k \ominus j| \leqslant 8$$

This completes the definition of the beliefs of the agents' types and hence the definition of Ω. In every state of the world $\omega \in \Omega$ markets clear and each agent is choosing a bundle which is optimal w.r.t. its conditional beliefs. It follows that the model $\mathcal{M}$ satisfies common knowledge of rationality and market clearing and therefore the price 2 is a *CKRMC* price in both states of nature.

3. A characterization

In this section we provide a characterization of *CKRMC* outcomes in terms of properties of the basic economy. Specifically we present the concept of *Ex-Post Rationalizability*[10] and then show (Theorem 1) that every *CKRMC* outcome is an Ex-Post Rationalizable outcome and that under a mild qualification the opposite implication is also true. We then use this characterization to compute the set of *CKRMC* outcomes in Example 1.

Definition. A price p is *Ex-Post Rationalizable* w.r.t. a set of states $\widehat{S} \subseteq S$ if for every $s \in \widehat{S}$ there exists a profile of probabilities $\{\gamma_i^s\}_{i\in I}$ on $\widehat{S}$, $\gamma_i^s \in \Delta(\widehat{S} \cap \pi_i(s))$, and a profile of demands $\{x_i^s\}_{i\in I}$, $x_i^s \in R^K$, such that:

1. For every $i \in I$, x_i^s is an optimal bundle at the price p w.r.t. γ_i^s.
2. Markets clear, that is, $\int_i x_i^s \,\mathrm{d}i = \int_i e_i^s \,\mathrm{d}i$.

We will say that the price p can be *supported* in the state s by the beliefs $\gamma^s = \{\gamma_i^s\}_{i\in I}$ on $\widehat{S}$ if there exists a profile of demands $x^s = \{x_i^s\}_{i\in I}$ such that conditions 1 and 2 above are satisfied.

The idea that underlies the concept of Ex-Post Rationalizability is that if p is Ex-Post Rationalizable w.r.t. $\widehat{S}$ then $\widehat{S}$ is a set of states in which p could be a clearing price because for every $s \in \widehat{S}$ there is a profile of beliefs on $\widehat{S}$, $\{\gamma_i^s\}_{i\in I}$, which is consistent with the private information of the players and which rationalizes demands that clear the markets at p. (The belief γ_i^s, in turn, is possible for player i because p can be a clearing price in every $s \in \widehat{S}$.)

Definition. An outcome (p, s) is *Ex-Post Rationalizable* (alternatively, p is *Ex-Post Rationalizable* in s) if there exists a set of states $\widehat{S}$ such that $s \in \widehat{S}$ and p is *Ex-Post Rationalizable* w.r.t. $\widehat{S}$.

Let $S(p)$ be the set of all states such that (p, s) is Ex-Post Rationalizable. It is easy to see that p is Ex-Post Rationalizable w.r.t. $S(p)$ and that $S(p)$ is the maximal set of states w.r.t. which p is Ex-Post Rationalizable.

The concept of Ex-Post Rationalizability does not specify a complete description of the beliefs of the agents. In particular, it does not specify (as *REE* and Belief Equilibrium do) the joint probability distribution of an agent on the state of nature and prices. It also does not specify the interactive beliefs, in particular, what agent i believes about the beliefs of other agents. Thus, one cannot tell whether and under what conditions an Ex-Post Rationalizable outcome is an outcome that is consistent with common knowledge of rationality and market clearing. Indeed, as we will see (Example 2), there are economies where there exist Ex-Post Rationalizable outcomes that are not *CKRMC* outcomes. Despite these fundamental differences in the definitions of the two concepts Theorem 1 establishes that under a mild qualification an outcome is a *CKRMC* outcome iff it is an Ex-Post Rationalizable outcome.

Theorem 1.

a. *If (p, s) is a CKRMC outcome then (p, s) is Ex-Post Rationalizable.*

[10] As we pointed out in the Introduction, the concept of Ex-Post Rationalizability was first studied in Desgranges [6].

b. *Let E be an economy in which there is a fully revealing Rational Expectations Equilibrium, $\widetilde{f}$. Let p be a price such that $p \notin \bigcup_{s\in S} \widetilde{f}(s)$. Then (p,s) is a CKRMC outcome iff (p,s) is Ex-Post Rationalizable.*

Remarks. 1. The set of economies in which there exists a fully revealing Rational Expectations Equilibrium is generic. For this set, Theorem 1 provides a characterization of *CKRMC* outcomes modulo outcomes that involve prices that are in the range of every fully revealing Rational Expectation Equilibrium.[11]

2. The concept of Ex-Post Rationalizability is defined in a way that is independent of the subjective priors – $(\alpha_i)_{i\in I}$ – on S. Similarly, if a price function f is a fully revealing Rational Expectations Equilibrium for some profile of subjective priors on S it is a fully revealing Rational Expectations Equilibrium for every profile of priors. It follows that under the condition specified in part b of Theorem 1 if (p,s) is a *CKRMC* outcome for some profile of subjective priors with full support on S then it is a *CKRMC* outcome for *every* such profile of priors. In particular, if (p,s) is a *CKRMC* outcome for some profile of subjective priors with full support then for every $\alpha \in \Delta(S)$ with full support (p,s) is a *CKRMC* outcome for the economy where α is a common prior.

3. There can be *CKRMC* outcomes that do not satisfy the condition formulated in part b of Theorem 1. (See Example 3 in Ben-Porath and Heifetz [3].) We do not have a general necessary and sufficient condition for an Ex-Post Rationalizable outcome to be a *CKRMC* outcome. (It is clear that such a condition would be cumbersome.)

We now use Theorem 1 to compute the set of *CKRMC* outcomes in Example 1. Let P_s, $s = 1, 3$, denote the set of prices that are Ex-Post Rationalizable in s. We will compute P_s and conclude, using Theorem 1, that P_s is also the set of *CKRMC* prices in the state s. Let $P(\widehat{S})$ denote the set of prices that are Ex-Post Rationalizable w.r.t. the set of states $\widehat{S}$, $\widehat{S} \subseteq S$. It follows from the definitions that $P_s = \bigcup_{\widehat{S}, s\in\widehat{S}} P(\widehat{S})$. In our example:

$$P_1 = P\big(\{1\}\big) \cup P\big(\{1,3\}\big) \tag{3.1}$$

$$P_3 = P\big(\{3\}\big) \cup P\big(\{1,3\}\big) \tag{3.2}$$

$P(\{1\}) = 1$ and $P(\{3\}) = 3$ because 1 and 3 are the prices that clear the markets in the states 1 and 3 respectively when everyone knows the state. We now compute $P(\{1,3\})$. Let $P_s(\{1,3\})$ denote the set of prices that can clear the markets in state s, $s = 1, 3$, when players in I_2 may have any profile of beliefs on $\{1,3\}$. It follows from the definition of $P(\{1,3\})$ that

$$P\big(\{1,3\}\big) = P_1\big(\{1,3\}\big) \cap P_3\big(\{1,3\}\big) \tag{3.3}$$

We claim that $P_1(\{1,3\}) = [1, 3 - 2\delta]$. This follows because the price 1 clears the market when every agent in I_2 assigns probability 1 to the state 1 (every agent in I_1 *knows* that the state is 1). Clearly, the aggregate demand for X and therefore its price are minimal when everyone assigns probability 1 to the state 1. Similarly, the price $3-2\delta$ clears the market when every agent in I_2 assigns probability 1 to the state 3 and therefore the maximal point in $P_1(\{1,3\})$ is $3-2\delta$. It is easy to see that for every $1 \leqslant p \leqslant 3-2\delta$ there is a probability $\gamma(p)$ such that if every agent in I_2 assigns probability $\gamma(p)$ to the state 3 then p clears the market. The set $P_3(\{1,3\})$ is computed in a similar way. When each agent in I_2 assigns probability 1 to the state 1 the clearing price is

[11] There are economies in which the set of fully revealing Rational Expectations Equilibrium is not a singleton.

$1 + 2\delta$. When agents in I_2 assign probability 1 to the state 3 the clearing price is 3. It follows that $P_3(\{1, 3\}) = [1 + 2\delta, 3]$. From (3.3) we obtain that for $\delta \leqslant 0.5$, $P(\{1, 3\}) = [1 + 2\delta, 3 - 2\delta]$. For $\delta > 0.5$, $P(\{1, 3\}) = \emptyset$. From (3.1) and (3.2) we have that for $\delta \leqslant 0.5$, $P_1 = \{1\} \cup [1+2\delta, 3-2\delta]$ and $P_3 = \{3\} \cup [1+2\delta, 3-2\delta]$ and for $\delta > 0.5$, $P_1 = \{1\}$ and $P_3 = \{3\}$. It follows from Theorem 1 that the difference between the set P_s and the set of *CKRMC* prices in s, $s = 1, 3$, is at most the price s. Now, s is the Rational Expectations Equilibrium price in the state s and therefore s is a *CKRMC* price at s. It follows that the sets P_s, $s = 1, 3$, that we have computed are the sets of *CKRMC* prices in the respective states.

The solution of the example is interesting in several ways. First, when δ is smaller than 0.5 there is a whole range of prices that are *CKRMC* prices in both states. Second, the set of *CKRMC* prices (i.e., P_1 and P_3) depends on δ (the fraction of agents who know the true state) in an intuitive way. As δ increases the set P_s shrinks and when more than 0.5 of the population is informed ($\delta > 0.5$) the only *CKRMC* price at a state s is the Rational Expectations Equilibrium price. Thus, when $\delta > 0.5$ the assumption of rationality and knowledge of rationality is sufficient to select the Rational Expectations Equilibrium[12] (without assuming a priori that the price function is known).

Consider now the case where all the agents have the same initial endowment and $\delta < 0.5$. In this case consistency with common knowledge of rationality and market clearing allows for trade despite the fact that it is common knowledge that there are no gains from trade (all the agents have the same utility function and the same initial endowment) and furthermore it is common knowledge that trade benefits agents in I_1 at the expense of some of the agents in I_2. The point is that when agents may have different beliefs and when the fraction of agents who are uninformed is high enough, common knowledge of rationality does not preclude the possibility that each uninformed agent is optimistic and believes that he is making a profit at the expense of other uniformed agents. The result that speculative trade is consistent with common knowledge of rationality and market clearing hinges on the following two properties of *CKRMC* : (1) Different agents may have different beliefs on the set of price functions. (2) Each agent does *not* know the beliefs of the other agents. Property (2) distinguishes *CKRMC* from the solution concept that is studied by MacAllister [13] and Dutta and Morris [8] and that is based on the assumption that the beliefs of the players are common knowledge. To appreciate the importance of property (2) we note that when $\delta > 0$ it is impossible to obtain trade, even with different beliefs, if these beliefs are common knowledge. The reason for this impossibility is that the beliefs of the uninformed agents determine their demands. So if an agent i in I_2 knows these beliefs he knows the aggregate demand of the uninformed agents. Since the aggregate amount of X is known, agent i can infer the aggregate demand of the informed agents. However, the aggregate demand of the informed agents reveals the state. Thus, if an uninformed agent i knows the beliefs of the other agents and observes the price p he can infer the true state and if everyone infers the true state there is no trade. Indeed for every $\delta > 0$ the unique *REE* is the only belief equilibrium in the models of MacAllister and Dutta and Morris.

We now turn to an example demonstrating two issues: first, the possibility of non-existence of a *CKRMC* outcome; second, the possibility of a difference between the set of *CKRMC* outcomes

[12] Since there are just two states in our example, the set of outcomes that is consistent with (just) rationality and knowledge of rationality equals the set of outcomes that are consistent with common knowledge of rationality. In particular, $P_s(\{1, 3\})$ is the set of prices that are consistent with rational behavior in state s. When $\delta > 0.5$ $P_1(\{1, 3\})$ and $P_3(\{1, 3\})$ are disjoint and therefore an agent who knows that all the other agents are behaving rationally can infer the state from the price.

and the set of Ex-Post Rationalizable outcomes. The example is similar to examples of non-existence of Rational Expectations Equilibrium that were given by Kreps [12] and Allen [1]. However, the argument that establishes non-existence of a *CKRMC* outcome is somewhat more involved.[13]

Example 2. (The example is a variation on Example 1.) There are two states, $S = \{1, 2\}$. The probability of each state is 0.5. The set of agents is $I = [0, 1]$ where agents in $I_1 = [0, \delta]$ know the true state and agents in $I_2 = (\delta, 1]$ don't know it. The utility of an agent in I_1 is $u_1(x, m, s) = a_s \cdot \log(x) + m$. The utility of an agent in I_2 is $u_2(x, m, s) = b_s \cdot \log(x) + m$. The aggregate amount of X is 1 and the number of units of M that each agent has exceeds $\text{Max}\{a_s, b_s\colon\ s = 1, 2\}$. All this implies that if p is the price of X in units of M then the demand for X of an agent in I_1 in state s is $\frac{a_s}{p}$ and the demand of an agent $i \in I_2$ who assigns probability $\gamma_i(s)$ to the state s is $\frac{\gamma_i(1)\cdot b_1 + \gamma_i(2)\cdot b_2}{p}$.

We make the following assumptions:

$$a_1 > a_2 \quad \text{and} \quad b_1 < b_2 \tag{3.4}$$

There exists a number $\widehat{p}$ such that

$$a_1 \cdot \delta + b_1(1 - \delta) = a_2 \cdot \delta + b_2(1 - \delta) = \widehat{p} \tag{3.5}$$

We claim that under these assumptions the set of *CKRMC* outcomes is empty.

To prove this we compute, first, the set of outcomes that are Ex-Post Rationalizable outcomes. Let $\gamma = \{\gamma_i\}_{i \in I_2}$ be a profile of probabilities on S (agents in I_1 assign probability 1 to the true state), and let $x_s^p(\gamma)$ denote the aggregate demand for X in the state s at the price p when the profile is γ. Since $b_1 < b_2$ the demand of each agent in I_2 is increasing in the probability that he assigns to the state 2. It follows that for every profile of probabilities γ, the aggregate demand in state $s = 1$ satisfies

$$x_1^p(\gamma) \geqslant \frac{a_1 \cdot \delta + b_1(1 - \delta)}{p} \tag{3.6}$$

and the aggregate demand in state $s = 2$ satisfies

$$x_2^p(\gamma) \leqslant \frac{a_2 \cdot \delta + b_2(1 - \delta)}{p} \tag{3.7}$$

Now we claim that (3.5)–(3.7) and the fact that the aggregate supply of X is 1 imply that the only outcomes that are Ex-Post Rationalizable are $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$. To see that we, first, observe that $\widehat{p}$ is the clearing price in state s when every agent in I_2 assigns probability 1 to the state s and therefore $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$ are Ex-Post Rationalizable. Now, assume by contradiction that there exists $p \neq \widehat{p}$ such that $(p, 1)$ is Ex-Post Rationalizable. It follows from (3.6) that if $p < \widehat{p}$ then for every profile of probabilities γ it is the case that $x_1^p(\gamma) > 1$, but this is impossible because the aggregate amount of X is 1. If $p > \widehat{p}$ then (3.7) implies that for every profile γ it is the case that $x_2^p(\gamma) < 1$ which means that p cannot be a clearing price in state 2. It follows that p cannot be Ex-Post Rationalizable w.r.t. the set S. Clearly, p cannot be Ex-Post Rationalizable w.r.t. $\{1\}$

[13] In Ben-Porath and Heifetz [3] we present an example (Example 3) that demonstrates that non-existence of a Rational Expectations Equilibrium does not imply non-existence of *CKRMC* outcomes.

and therefore we have obtained a contradiction. A similar argument establishes that $\widehat{p}$ is the only price that is Ex-Post Rationalizable in the state 2. It follows from part a of Theorem 1 that the only possible *CKRMC* outcomes are $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$.

We now show that $(\widehat{p}, 1)$ is not a *CKRMC* outcome. (The proof that $(\widehat{p}, 2)$ is not a *CKRMC* outcome is identical.) Assume by contradiction that there exists a model $\mathcal{M}$ that is consistent with common knowledge of rationality and market clearing and a state $\omega \in \Omega$ such that $\omega = (1, \widehat{p}, (t_i)_{i\in I})$. It follows that for almost every[14] $i \in I_2$, $\operatorname{marg}_S b_i[t_i]$ assigns probability 1 to the state 1. (To see that recall that the demand of an agent $i \in I_2$ equals $\frac{\gamma_i(1)\cdot b_1+\gamma_i(2)\cdot b_2}{\widehat{p}} \geqslant \frac{b_1}{\widehat{p}}$ where $\gamma_i(s)$ is the posterior probability that i assigns to the state s. If $\operatorname{marg}_S b_i[t_i]$ assigns a positive probability to state 2 then the demand of agent i for X at $\widehat{p}$ is greater than $\frac{b_1}{\widehat{p}}$. If there is a positive measure of such agents then it follows from (3.6) and (3.5) that the aggregate demand for X is higher than one unit, but that is impossible because the aggregate amount of X is one unit.) Let $\omega' \in \Omega$ be another state of the world such that $\omega' = (2, p, (t_i')_{i\in I})$. Since $\widehat{p}$ is the only *CKRMC* price we must have $p = \widehat{p}$ but now again we obtain that for almost every $i \in I_2$ $\operatorname{marg}_S b_i[t_i']$ assigns probability 1 to the state 2. (Otherwise, an argument that is similar to the one we just gave implies that at the state $s = 2$ the aggregate demand for X at the price $\widehat{p}$ is smaller than one unit.) However, this implies that for almost every $i \in I_2$ $\operatorname{marg}_S b_i[t_i] \neq \operatorname{marg}_S b_i[t_i']$, but this is impossible because we must have

$$\operatorname{marg}_S b_i[t_i] = \operatorname{marg}_S b_i\left[t_i'\right] = \alpha_i$$

Appendix A

A.1. *The proof of Theorem 1*

We start with part a.

Let $(\widehat{p}, \widehat{s})$ be a *CKRMC* outcome. Let $\mathcal{M}$ be a model that satisfies common knowledge of rationality and market clearing and let $\widehat{\omega} \in \Omega$ be a state of the world such that $\widehat{\omega} = (\widehat{s}, \widehat{p}, (\widehat{t_i})_{i\in I})$. We have to show that $(\widehat{p}, \widehat{s})$ is an Ex-Post Rationalizable outcome. Define

$$\widehat{S} = \left\{s \;\middle|\; \exists \omega \in \Omega \text{ s.t. } \omega = \left(s, \widehat{p}, (t_i)_{i\in I}\right)\right\}$$

We claim that $\widehat{p}$ is Ex-Post Rationalizable w.r.t. the set $\widehat{S}$. The proof is simple: let $\omega \in \Omega$ be a state of the world such that $\omega = (s, \widehat{p}, (t_i)_{i\in I})$. Clearly, for every $i \in I$ the support of the probability distribution $\operatorname{marg}_S b_i[t_i|\pi_i(s), \widehat{p}]$ is contained in $\widehat{S}$. Also, since $\mathcal{M}$ satisfies common knowledge of rationality and market clearing the demand $z_i[t_i](\pi_i(s), \widehat{p})$ of each t_i is optimal w.r.t. $\operatorname{marg}_S b_i[t_i|\pi_i(s), \widehat{p}]$ and the aggregate demand equals the aggregate supply. It follows that the profile of probabilities $\{\operatorname{marg}_S b_i[t_i|\pi_i(s), \widehat{p}]\}_{i\in I}$ supports the price $\widehat{p}$ at the state s. Since this holds for every $s \in \widehat{S}$, $\widehat{p}$ is Ex-Post Rationalizable w.r.t. $\widehat{S}$. Since $\widehat{s} \in \widehat{S}$ we obtain that $\widehat{p}$ is an Ex-Post Rationalizable price at $\widehat{s}$.

We now turn to part b.

Let $(\widehat{p}, \widehat{s})$ be an Ex-Post Rationalizable outcome. For $s \in S(\widehat{p})$ we let $\gamma^s = \{\gamma_i^s\}_{i\in I}$ and $x^s = \{x_i^s\}_{i\in I}$ denote respectively the profiles of beliefs and demands that support the price $\widehat{p}$ in the state s w.r.t. $S(\widehat{p})$. For $s \in S$ we let p^s and $y^s = (y_i^s)_{i\in I}$ denote respectively the price

[14] That is, for every $i \in I_2$ except possibly for a set of agents of measure zero.

vector $\widetilde{f}(s)$ and a profile of demands for the agents that constitute a Walrasian equilibrium in the complete information economy where the state s is common knowledge. For $\bar{s} \in S(\widehat{p})$ and $i \in I$ we define a type $t_i^{\bar{s}}$ for agent i that has a demand function that is defined as follows:

$$z_i\left[t_i^{\bar{s}}\right](\pi_i, p) = \begin{cases} x_i^{\bar{s}} & p = \widehat{p} \\ y_i^s & p = p^s \text{ for some } s \in S \end{cases}$$

We note that the profile of demand functions $(z_i[t_i^{\bar{s}}])_{i \in I}$ induces the price function $f^{\bar{s}}$ defined by

$$f^{\bar{s}}(s) = \begin{cases} \widehat{p} & s = \bar{s} \\ p^s & s \neq \bar{s} \end{cases}$$

That is, for every $s \in S$, $\int_i z_i[t_i^{\bar{s}}](\pi_i(s), f^{\bar{s}}(s))\,\mathrm{d}i = \int_i e_i(\pi_i(s))\,\mathrm{d}i$.

Thus, a state of the world $(\bar{s}, \widehat{p}, (t_i^{\bar{s}})_{i \in I})$ is a state of the world where the state of nature is $\bar{s}$ and the demands of the agents clear the market at the price $\widehat{p}$. We now construct Ω so that we can assign to each type $t_i^{\bar{s}}$, $\bar{s} \in S(\widehat{p})$, $i \in I$, beliefs that rationalize his demand function, $z_i[t_i^{\bar{s}}]$. To do this we need the following definitions and Lemma 1.

Define: $F = \{f^{\bar{s}} \mid \bar{s} \in S(\widehat{p})\}$. Let α be a probability distribution on S and let μ be a probability distribution on F. We let $\alpha \times \mu$ denote the product probability distribution over $S \times F$. For $S' \subseteq S$ and a price p such that the event $(S', p) = \{(s, f) \mid s \in S',\ f \in F,\ f(s) = p\}$ has a positive $\alpha \times \mu$ probability we let $\alpha \times \mu(\cdot|S', p)$ denote the conditional of $\alpha \times \mu$ on (S', p). Finally, we recall that α_i denotes the prior probability of agent i on S and that α_i has a full support.

Lemma 1. *For every $\bar{s} \in S$ and $i \in I$ there exists a probability distribution on F, $\mu_i^{f^{\bar{s}}}$, such that*:

(a) *For every $s \in S$,* $\operatorname{marg}_S \alpha_i \times \mu_i^{f^{\bar{s}}}(\cdot|(\pi_i(s), p^s))$ *assigns probability 1 to the state s and* $\operatorname{marg}_S \alpha_i \times \mu_i^{f^{\bar{s}}}(\cdot|(\pi_i(\bar{s}), \widehat{p})) = \gamma_i^{\bar{s}}$.

(b) *For every $s \notin \pi_i(\bar{s})$, $\mu_i^{f^{\bar{s}}}(f^s) = 0$.*

The proof of Lemma 1 is given at the end of the section.

Note that property (a) implies that the beliefs $\mu_i^{f^{\bar{s}}}$ rationalize the demand function $z_i[t_i^{\bar{s}}]$. That is, for every event $(\pi_i(s), p)$ that has a positive $\alpha_i \times \mu_i^{f^{\bar{s}}}$ probability the bundle $z_i[t_i^{\bar{s}}](\pi_i(s), p)$ is optimal w.r.t. $\operatorname{marg}_S \alpha_i \times \mu_i^{f^{\bar{s}}}(\cdot|(\pi_i, p))$. We now construct Ω so that we can assign to each type $t_i^{\bar{s}}$ beliefs that are induced by the beliefs $\alpha_i \times \mu_i^{f^{\bar{s}}}$ that satisfy the properties defined in Lemma 1. These beliefs rationalize his demand $z_i[t_i^{\bar{s}}]$.

For $\bar{s} \in S(\widehat{p})$ and $s' \in \pi_i(\bar{s})$ we let $(t_{-i}^{s'}, t_i^{\bar{s}})$ denote the profile of types where the type of agent i is $t_i^{\bar{s}}$ and the type of an agent $j \neq i$ is $t_j^{s'}$. We note that the profile $(t_{-i}^{s'}, t_i^{\bar{s}})$ induces the demand function $f^{s'}$. To see that observe that

(a) Since there is a continuum of agents the aggregate demand in the economy is not affected by the demand of the single agent i.

(b) The profile of demand functions $(z_j[t_j^{s'}])_{j \neq i}$ clears the markets at the prices specified by $f^{s'}$.

With this in mind we turn to the definition of Ω. Define

$$\Omega_1 = \{(s', \widehat{p}, (t^{s'}_{-i}, t^{\bar{s}}_i)) \mid \bar{s}, s' \in S(\widehat{p}),\ s' \in \pi_i(\bar{s}),\ i \in I\}$$
$$\Omega_2 = \{(s, p^s, (t^{s'}_{-i}, t^{\bar{s}}_i)) \mid \bar{s}, s' \in S(\widehat{p}),\ s' \in \pi_i(\bar{s}),\ s \neq s',\ i \in I\}$$

Now, define

$$\Omega = \Omega_1 \cup \Omega_2$$

We note that Ω_1 is a set of states of the world in which the price is $\widehat{p}$ while Ω_2 is a set of states of the world in which the price fully reveals the state of nature.

The demand function of type $t^{\bar{s}}_i - z_i[t^{\bar{s}}_i]$ – has already been defined; to define his beliefs first define

$$\Omega^{\bar{s},i}_1 = \{(s', \widehat{p}, (t^{s'}_{-i}, t^{\bar{s}}_i)) \mid s' \in S(\widehat{p}) \text{ and } s' \in \pi_i(\bar{s})\}$$
$$\Omega^{\bar{s},i}_2 = \{(s, p^s, (t^{s'}_{-i}, t^{\bar{s}}_i)) \mid s' \in S(\widehat{p}),\ s' \in \pi_i(\bar{s}) \text{ and } s \neq s'\} \quad \text{and}$$
$$\Omega^{\bar{s},i} = \Omega^{\bar{s},i}_1 \cup \Omega^{\bar{s},i}_2$$

We note that $\Omega^{\bar{s},i}_1 \subseteq \Omega_1$ and $\Omega^{\bar{s},i}_2 \subseteq \Omega_2$. Thus, $\Omega^{\bar{s},i}_1$ is a set of states of the world in which the price is $\widehat{p}$ while $\Omega^{\bar{s},i}_2$ is a set of states of the world in which the price fully reveals the state of nature.

Now define

$$b_i[t^{\bar{s}}_i](s, p, (t^{s'}_{-i}, t^{\bar{s}}_i)) = \begin{cases} \alpha_i(s) \times \mu^{f^{\bar{s}}}_i(f^{s'}) & \text{if } (s, p, (t^{s'}_{-i}, t^{\bar{s}}_i)) \in \Omega^{\bar{s},i} \\ 0 & \text{otherwise} \end{cases}$$

To understand this construction we observe that the belief of $t^{\bar{s}}_i$ on S is α_i and his beliefs on the types of the other players is defined by $\mu^{f^{\bar{s}}}_i$. Specifically, $t^{\bar{s}}_i$ assigns probability $\mu^{f^{\bar{s}}}_i(f^{s'})$ to the event that the profile of types of the other players is $t^{s'}_{-i}$. Now, since the profile of types $(t^{s'}_{-i}, t^{\bar{s}}_i)$ induces the price function $f^{s'}$ the posterior probability of $t^{\bar{s}}_i$ on S at an event $(\pi_i(s), p)$ that is consistent with Ω is $\text{marg}_S\, \alpha_i \times \mu^{f^{\bar{s}}}_i(\cdot|(\pi_i(s), p))$ and therefore the beliefs of $t^{\bar{s}}_i$ rationalize his demand function. We have thus both explained the construction of $\mathcal{M}$ and proved that $\mathcal{M}$ satisfies common knowledge of rationality and market clearing. If follows that $(\widehat{p}, \widehat{s})$ is a *CKRMC* outcome.

We have completed the proof of Theorem 1 given Lemma 1. The proof of Lemma 1 relies on Lemma 2.

Lemma 2. *Let $\beta_1, \ldots, \beta_m$ be m positive numbers and let $\gamma = (\gamma_1, \ldots, \gamma_m)$ be a probability vector. There exists a probability vector $\delta = (\delta_1, \ldots, \delta_m)$ that solves the following system of equations*

$$\gamma_k = \frac{\beta_k \cdot \delta_k}{\sum_{j=1}^m \beta_j \cdot \delta_j}, \quad k = 1, \ldots, m$$

Proof of Lemma 2. First, we assume w.l.o.g. that $\gamma_k > 0$ for every k because otherwise we would define $\delta_j = 0$ if $\gamma_j = 0$ and proceed to prove the lemma for the set $\{k\colon \gamma_k > 0\}$.

Second, multiplying the equations by the denominator and subtracting the RHS from the LHS gives a system of m homogeneous linear equations in $\delta_1, \ldots, \delta_m$ that are linearly dependent (the sum of all the equations is zero). Therefore there exists a solution to this system, $\overline{\delta} = (\overline{\delta_1}, \ldots, \overline{\delta_m})$ that is different from zero.

Third, if $\overline{\delta}$ is a solution and c is a constant then $c \cdot \overline{\delta}$ is also a solution.

Finally, since $\gamma_k > 0$ for all $k = 1, \ldots, m$, then if $\overline{\delta}$ is a solution then $\overline{\delta_1}, \ldots, \overline{\delta_m}$ all have the same sign which is the sign of the denominator.

It follows from all this that there is a solution $\widehat{\delta}$ to the system that is a probability vector because if $\overline{\delta}$ is some solution there is a constant c such that $c \cdot \overline{\delta}$ is a probability vector. □

Proof of Lemma 1. We map Lemma 2 to the proof of Lemma 1 as follows. Suppose that $S(\widehat{p}) \cap \pi_i(\bar{s})$ is the set $\{1, \ldots, m\}$. For $s \in \{1, \ldots, m\}$ define $\beta_s \equiv \alpha_i(s)$, the prior probability that agent i assigns to the state s, and $\gamma_s \equiv \gamma_i^{\bar{s}}(s)$, the probability of the state s according to $\gamma_i^{\bar{s}}$. We claim that if for $s \in \{1, \ldots, m\}$ we define $\mu_i^{f^{\bar{s}}}(f^s)$ to be δ_s so that the equations in the statement of Lemma 2 are satisfied then property (a) is satisfied (property (b) is satisfied as well because the probability of a price function different from $f^1, \ldots, f^m$ is zero). To see that (a) is satisfied we observe that $\mathrm{marg}_S\, \alpha_i \times \mu_i^{f^{\bar{s}}}(\cdot|\pi_i(s), p^s)$ assigns probability 1 to the state s[15] and that Bayesian updating implies that for $s = 1, \ldots, m$,

$$\mathrm{marg}_S\, \alpha_i \times \mu_i^{f^{\bar{s}}}\big(s|\pi_i(s), \widehat{p}\big) = \frac{\alpha_i(s) \cdot \mu_i^{f^{\bar{s}}}(f^s)}{\sum_{s'=1}^{m} \alpha_i(s') \cdot \mu_i^{f^{\bar{s}}}(f^{s'})}$$

and therefore[16] by Lemma 2 $\mathrm{marg}_S\, \alpha_i \times \mu_i^{f^{\bar{s}}}(s|\pi_i(s), \widehat{p}) = \gamma_i^{\bar{s}}(s)$. □

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[15] Because (s, p^s) is the only element in the set $\{(s', f^{s^*}(s')) \mid s' \in S,\ s^* \in \{1, \ldots, m\}\}$ with the price p^s.

[16] The fact that Bayesian updating is given by the equation above relies on the assumption that $\widehat{p} \neq p^s$ for every $s \in S$. This is the only point where this assumption is used.

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