Two notions of causal sufficiency
Formal machinery

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Definition 1 (Causal Structure): A causal structure of a set of proposition letters $\mathcal{P}$ is a directed acyclic graph (DAG) in which each node corresponds to a distinct element of $\mathcal{P}$, and each link represents direct functional relationship among the corresponding propositions.

Definition 2 (Causal Model): A causal model is a pair $M =< D, \Theta_D >$ consisting of a causal structure D and a set of parameters Θ_D compatible with D. The parameters Θ_D assign a function $\psi_i = f_i(\Sigma)$ to each $\psi_i \in \mathcal{P}$, where Σ is the set of all nodes that ψ_i causally depends on in D.

Definition 3 (Truth values generated by a causal model): Let $\mathcal{P}$ be a set of proposition letters and $\mathcal{L}$ the closure of $\mathcal{P}$ under conjunction and negation. Furthermore, let $M =< D, \Theta_D >$ be a causal model for $\mathcal{P}$, and $I : \Sigma \rightarrow 0, 1, u$ an interpretation of a set of variables Σ of M. For arbitrary $\psi \in \mathcal{L}$ we define the interpretation of ψ with respect to M and I, $[[\psi]]^{M,I}$ recursively as follows:
$[[\psi]]^{M,I}$ = I(ψ), if $\psi \in \Sigma$,
$[[\psi]]^{M,I}$ = $[[F(\psi)]]^{M,I}$, if $\psi \in \mathcal{P} - \Sigma$
$[[\neg\psi]]^{M,I}$ = 1, iff $[[\psi]]^{M,I}$ = 0 and
$[[\psi \wedge \phi]]^{M,I}$ = 1, iff $[[\psi]]^{M,I}$ = 1 and $[[\phi]]^{M,I}$ = 1.

Definition 4 (Situation: A set of pairs of propositional variable Σ in $\mathcal{P}$ and their values is a situation.

Definition 5 (Causal relevance): A situation s (consisting of a set of pairs of propositional variables Σ in $\mathcal{P}$ and their values) of M is causally relevant for a certain value of the variable ψ, when the interpretation of ψ with respect to M and I is defined (1 or 0), when $I : \Sigma^c$ (the complementary set of Σ in $\mathcal{P}$) assigns u for all its members.

Definition 6 (Causal necessity): χ is causally necessary for a certain value of ψ in a situation s, i.e. the variable and its value in s ($S\chi$) is necessary for a certain value of ψ (0 or 1) if: There is a set of propositional variables Σ and there are two situations s and s', which are two situations of Σ, such that

i. the situation s is causally relevant for a propositional variable ψ
ii. $I : \Sigma \rightarrow 0, 1$ is an interpretation of the set of variables Σ of M, in situation s and $I' : \Sigma \rightarrow 0, 1$ an interpretation of the set of variables Σ of M, in situation s' and
iii. $s(\chi) \neq s'(\chi)$.
iv. The cardinality of the complementary set J, such that $J = s - s'$ is 2, and the two members of J are the pair of χ and its value, different in each pair, and
v. $[[\psi]]^{M,I} \neq [[\psi]]^{M,I'}, and$
vi. There is no interpretation I'' of the set of variables Σ, in which $s'(\chi) = s''(\chi)$ and $[[\psi]]^{M,I'} \neq [[\psi]]^{M,I''}$.

Definition 7 (Sufficient set): A situation s is defined as a sufficient set for a certain interpretation of ψ, if the set of propositions Σ whose value are defined in s, is causally relevant for the proposition ψ, and all members of S are causally necessary for ψ.
$\{X \in s|$ such that x is causally necessary for $\psi\}$

Definition 8 (A completed sufficient set): A situation s is the completed sufficient for a certain interpretation of ψ, which is the superset of all comparable sufficient sets.

(1) **Overt *cause***
$\exists Q \exists R \exists e \exists \Sigma$:$\textsc{suff}(\Sigma, R)^M = 1$ & $(Q \in \Sigma^M)$ & $\Sigma(e)$ & $\forall \Omega$ [$(\Omega \neq \Sigma)$ & $\textsc{suff}(\Omega, R)^M = 1 \rightarrow \neg\Omega\ (e)]$

(2) **Lexical causative - Take 1**
$\exists Q \exists R \exists e \exists t \exists \Sigma$:$\textsc{suff}(\Sigma, R)^M = 1$ & $(Q \in \Sigma^M)$ & $\Sigma(e)$ & $\tau(e) \subseteq t$ & $\forall t' < t \forall e' : \tau(e') \subseteq t' \rightarrow [\neg Q(e')]$ & $\forall \Omega$ [$(\Omega \neq \Sigma)$ & $\textsc{suff}(\Omega, R)^M = 1 \rightarrow \neg\Omega\ (e)]$

(3) **Lexical causative - Take 2 (Completed sufficient set)**
$\exists Q \exists R \exists e \exists t \exists \Sigma \exists \Upsilon$:$\textsc{suff}(\Sigma, R)^M = 1$ & $\textsc{compsuff}(\Upsilon, R)^M = 1$&$(\mathrm{Q} \in \Sigma^M)$ & $(Q \in \Upsilon^M)$ & $\Sigma(e)$ & $\tau(e) \subseteq t$ & $\forall t' < t \forall e' : \tau(e') \subseteq t' \rightarrow [\neg Q(e')]$ & $\forall \Omega$ [$(\Omega \neq Y)$ & $\textsc{compsuff}(\Omega, R)^M = 1 \rightarrow \neg\Omega(e)]$