Socially Desirable Limits on Individual Choice

by

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"We're thinking maybe it's time you started getting some religious instruction. There's Catholic, Protestant, and Jewish—any of those sound good to you?"

Introduction

- Early in modern psychology it has been observed (Luce and Suppes (1965)) that in choice experiments individuals do not select the same alternative in repetitions of identical situations. To explain these behavioral inconsistencies a probabilistic choice mechanism was introduced. Two alternative approaches have been offered.

Constant Utility Approach

- The decision maker chooses probabilistically, with utilities as parameters. This approach makes specific assumptions about the structure of the probabilities. Let $p_i(S)$ be the probability of choosing alternative i from the set S, $i \in S$, and $\tilde{S}$ is a subset of S, $\tilde{S} \subseteq S$.

The probability of choosing subset $\tilde{S}$ from S, $p_{\tilde{S}}(S)$, is $p_{\tilde{S}}(S) = \sum_{i \in \tilde{S}} p_i(S)$. The *Choice-Axiom* formulated by Luce (1959) makes the following assumption:

A set of choice probabilities defined for all subsets of a finite set S, satisfy the choice axiom provided that for all i, $\tilde{S}$ and S, such that $i \in \tilde{S} \subseteq S$,

$$p_i(S) = p_i(\tilde{S})\, p_{\tilde{S}}(S)$$

In words, the probability of alternative $i \in S$ is the product of the probability of *i* from the subset $\tilde{S}$ that contains *i* and the probability that the choice lies in $\tilde{S}$. This is an *'independence assumption'*, assumed to hold for any subset which contains *i*.

The Choice Axiom implies the property of *independence of irrelevant alternatives* (IIA):

$$\frac{p_i(\tilde{S})}{p_j(\tilde{S})} = \frac{p_i(S)}{p_j(S)}, \;\; i \neq j, \;\; i, j \;\in \tilde{S} \subseteq S$$

Luce argues that this can be viewed as a probabilistic version of the property of transitivity. However, the *"Red bus - Blue bus Paradox"* (Debreu, 1960) demonstrates that the choice-axiom is less appropriate when alternatives are similar and may, in turn, induce manipulations of the choice-set.

Luce (1959) proves that if the choice-axiom holds then there exists positive utility measure, U_i, proportional to the probability of *i*, and these probabilities can be written

$$p_i(S) = \frac{U_i}{\sum_{j \in S} U_i}$$

The utilities U_i are unique up to multiplication by a positive constant. Writing $u_i = \ln U_i$, the MNL model is

$$p_i(S) = \frac{e^{u_i}}{\sum_{j \in S} e^{u_j}}$$

Random Utility Approach

- Manski (1977) made the argument that while individuals always choose the alternative with the highest utility, these utilities are not known to the analyst with certainty and should therefore be treated as random variables.

The probability that alternative *i* will be chosen is equal to the probability that its utility, U_i, is greater than or equal to the utilities of all other alternatives in *S*:

$$p_i(S) = pr\{U_i \geq U_j,\ \text{all}\ \ j \in S\},\ \ i = 1,2,..,n$$

Specific assumptions on the joint distribution of the random utilities $\{U_i, i \in S\}$ are required in order to solve these conditions.

Manski assumes that utilities have a deterministic ("systematic") component, which can in principle be estimated, denoted V_i, and a pure disturbance term, denoted ε_i:

$$U_i = V_i + \varepsilon_i, \ \ i = 1, 2, .., n$$

Hence,

$$p_i(S) = pr\{V_i + \varepsilon_i \geq \max_{\substack{j \in S \\ j \neq i}} (V_j + \varepsilon_j)\}$$

Assumptions are made on the joint distribution of $\{\varepsilon_1, \varepsilon_2,..,\varepsilon_n\}$. Generally, this is quite complex. However, when all the disturbances are *i.i.ds* and follow a *Gumbel distribution* with a common scale parameter q, $q > 0$, then this is equivalent to the MNL model:

$$p_i(S) = \frac{e^{qV_i}}{\sum_{j \in S} e^{qV_j}}$$

The *Gumbel distribution*, $F(\varepsilon)$, is

$$F(\varepsilon) = \exp[-e^{-q\varepsilon}]$$

This distribution has the properties that any linear transformation of the ε's is also Gumbel distributed, the difference between two Gumbel distributed variables, $\varepsilon_1 - \varepsilon_2$, is also Gumbel distributed and, most important, when ε_i are all i.i.d. Gumbel distributed, then the $\max(\varepsilon_1, \varepsilon_2,..,\varepsilon_n)$ is also Gumbel distributed.

For the original derivation see Domencich and McFadden (1975).

A Multinomial Logit Choice Model

Consider a population of heterogeneous individuals, each characterized by a parameter θ ("individual θ"). This parameter represents personal characteristics, such as health, longevity or attitudes towards work, which are regarded as private information. Individuals choose one among a finite number, n, of alternatives, numbered $i = 1, 2,.., n$. They attach a non-negative utility, $u_i(\theta)$, to each alternative. Choice is probabilistic.

The MNL model specifies the probability that individual θ chooses alternative i, denoted $p_i\,(\theta, q)$, as

$$p_i(\theta,q)=\frac{e^{qu_i(\theta)}}{\sum_{j=1}^{n} e^{qu_j(\theta)}} \quad i=1,2,..,n$$

where $q \geq 0$ is a parameter. Clearly, for any (θ, q) and $n > 1$, $0 < p_i\,(\theta, q) < 1$, and

$$\sum_{i=1}^{n} p_i(\theta,q)=1.$$

In case of ties, one of the alternatives with equal probabilities is randomly chosen.

Two limiting cases are immediate:

Case 1

$$p_i(\theta,0) = \lim_{q\to 0} p_i(\theta,q) = \frac{1}{n} \quad i = 1,2,..,n.$$

All alternatives are equally likely to be chosen;

Case 2

$$p_i(\theta,\infty) = \lim_{q\to\infty} p_i(\theta,q) = \begin{cases} 1 \text{ if } u_i(\theta) > \max\limits_{\text{all } j\neq i} u_j(\theta) \\ 0 \text{ if } u_i(\theta) < \max\limits_{\text{all } j\neq i} u_j(\theta) \end{cases} \quad i = 1,2,..,n.$$

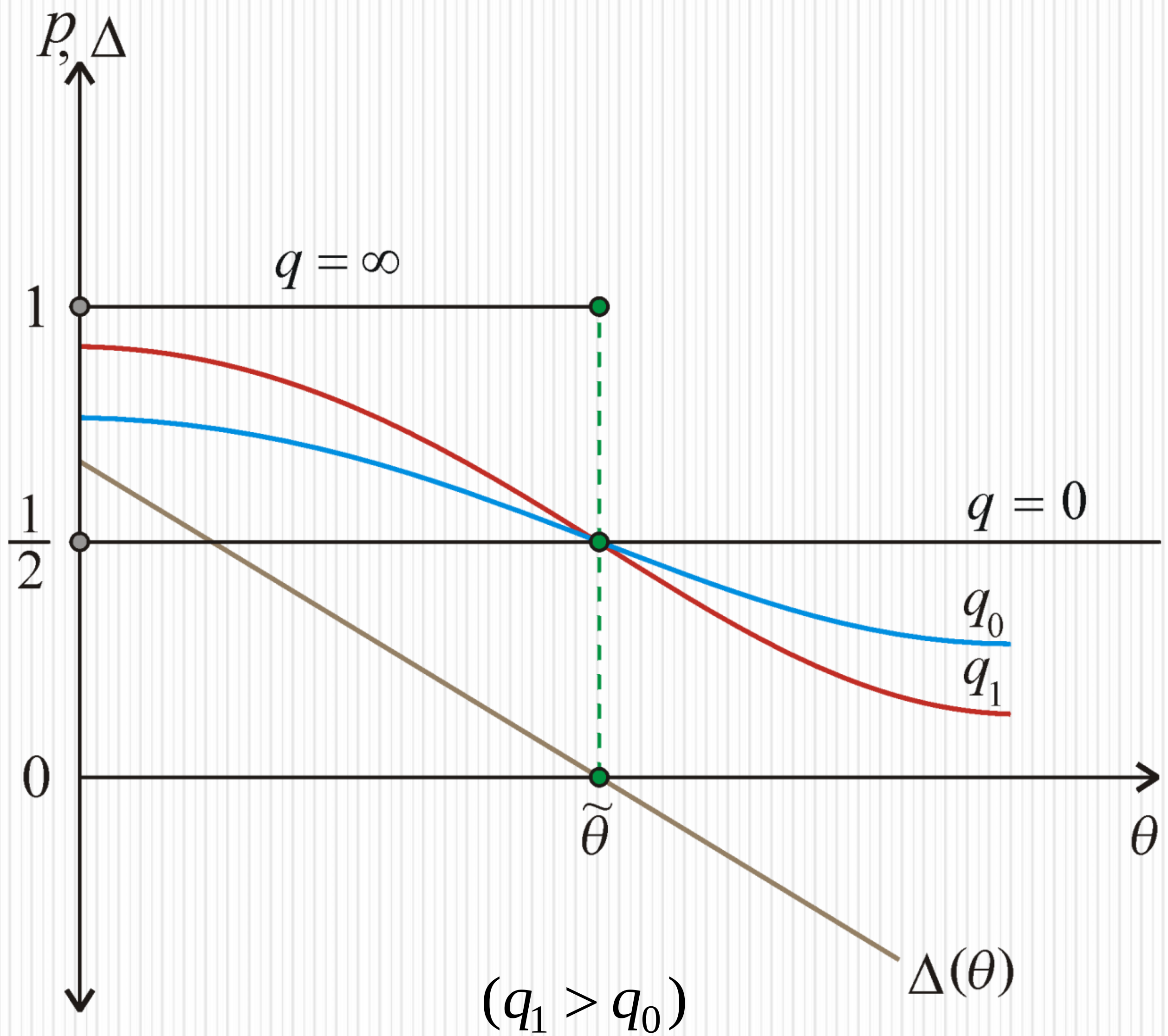
[Suppose that there are ties among n^*, $1 < n^* < n$, alternatives with the highest utilities.

Then, the limit in case 2 is $P_i(\theta,\infty) = \dfrac{1}{n^*}$ for i among these alternatives and the probabilities for $q = \infty$ are zero for the remaining $n - n^*$ alternatives. For simplicity, we shall disregard below the possibility of ties].

The parameter q can be viewed as representing the *precision of choice*. We refer to q as the *degree of rationality* (with $q = \infty$ called *'perfect rationality'*).

A *binary example* ($i = 1, 2$, and $\theta \geq 0$):

Suppose that $\Delta(\theta) = u_1(\theta) - u_2(\theta) \gtreqless 0$ as $\theta > \tilde{\theta}$ for some $\tilde{\theta} > 0$.



p, Δ
$q = \infty$
1
$\frac{1}{2}$
0
$q = 0$
q_0
q_1
$\tilde{\theta}$
θ
$\Delta(\theta)$
$(q_1 > q_0)$

Social Welfare and Optimum Choice-Sets

Individuals' welfare is represented by expected utility, $V(\theta, q)$,

$$V(\theta,q)=\sum_{i=1}^{n} p_i(\theta,q)\,u_i(\theta)$$

A change in the parameter q affects V through the probabilities $p_i(\theta, q)$:

$$\frac{\partial p_i(\theta,q)}{\partial q}=p_i(\theta,q)(u_i(\theta)-V(\theta,q)) \quad i=1,2,..,n.$$

An increase in q raises the probabilities of alternatives whose utility is higher than expected utility and vice-versa.

Consequently, a higher q raises expected utility:

$$\frac{\partial V}{\partial q} = \sum_{i=1}^{n} p_i (u_i - V)^2 > 0,$$

(assuming that not all alternatives have the same utility).

The level of V in the limiting cases is:

$$V(\theta,0)=\lim_{q\to 0}V(\theta,q)=\frac{1}{n}\sum_{i=1}^{n}u_i$$

and

$$V(\theta,\infty)=\lim_{q\to\infty}V(\theta,q)=\overline{V}(\theta)$$

where $\overline{V}(\theta)=u_i(\theta),\ u_i(\theta)>\max_{j\neq i}u_j(\theta).$

When choice is purely random, expected utility is the (arithmetic) average of utilities. Under perfect rationality, individuals choose the alternative with the highest utility, denoted $\bar{V}$.

Social welfare, W, is assumed to be utilitarian:

$$W(q) = \int_{\underline{\theta}}^{\bar{\theta}} V(\theta, q)\, dF(\theta)$$

where $F(\theta)$ is the distribution function of θ in the population. It is assumed that $F(\theta)$ is defined ever a finite, non-empty, interval, $(\underline{\theta}, \bar{\theta})$.

A higher q raises W: $\dfrac{dW(q)}{\partial q} > 0.$

(a) The optimum social choice sets for high and low *q*'s.

Individual utilities are private information, but the government knows the distribution of utilities in the population.

The government determines the set of alternatives from which individuals make choices, called the *'choice-set'*.

Let $S \subseteq \{1, 2, ..., n\}$ and denote by $p_i^S(\theta, q) = \dfrac{e^{qu_i}}{\sum_{i \in S} e^{qu_i}}, \; i \in S.$

With choice-set S, expected utility and social welfare are, respectively, $V^S(\theta, q) = \sum_{i \in S} p_i^S(\theta, q) u_i(\theta)$ and

$$W^S(q) = \int_{\underline{\theta}}^{\overline{\theta}} V^S(\theta, q) dF(\theta).$$

Given *q*, optimum policy is the choice-set, $S^*(q)$, which maximizes social welfare:

$$S^*(q) = \left\{ S \,\middle|\, W^S \geq W^{S'}, \text{for all } S, S' \subseteq (1,2,..,n) \right\}$$

It is natural to assume that no alternative is dominated by other alternatives *for all* θ's because such an alternative will never be chosen. Denoting

$$\Omega_i = \left\{ \theta \mid u_i(\theta) > \max_{j \neq i} u_j(\theta) \right\} \;\; i = 1, 2, .., n.$$

Assumption *All* Ω_i, $i = 1, 2, .., n$ *are non-empty.*

Optimum policy for the two limiting cases discussed above is straightforward. Maximum social welfare, denoted $\overline{W}$, is attained when all individuals choose their most preferred alternative thereby attaining their maximum utility, $\overline{V}(\theta)$, for all θ. Under perfect rationality and the assumption above, this can be attained when the choice-set includes *all* alternatives. Denote this set by $\overline{S} = \{1, 2, \ldots, n\}$. Thus,

$$\overline{V}(\theta) = \lim_{q \to \infty} V^{\overline{S}}(\theta, q) \text{ and } \overline{W} = \lim_{q \to \infty} W^{\overline{S}}(q) = \int_{\underline{\theta}}^{\overline{\theta}} \overline{V}(\theta) dF(\theta).$$

At the other extreme, with pure random choice by individuals,

$$W(0) = \lim_{q \to 0} W(q) = \frac{1}{n} \sum_{i=1}^{n} W^i$$

where W^i is social welfare when all individuals choose alternative *i*,

$$W^i = \int_{\underline{\theta}}^{\overline{\theta}} u_i(\theta) dF(\theta)$$

Since $W(q)$ is continuous in q, we can now state the following

Proposition *For large and for small q's, the optimum social choice-sets are as follows:*

(a) *When q is large, the choice-set includes all alternatives;*

(b) *When q is small, the choice-set is a singleton, that is, it contains one alternative, say alternative m, where* $W^m > \max_{j \neq m} W^j$.

(b) The optimum social choice sets for intermediate values of q's.

As stated in the Proposition, continuity of W in q implies that there exists a q_0, $q_0 > 0$, such that for all $q < q_0$, the optimum choice-set includes a single alternative while for $q \geq q_0$ the choice-set includes two or more alternatives. We wish to investigate how $S^*(q)$ changes as q rises above q_0.

Characterization of $S^*(q)$ for intermediate values is rather complex. This can best be demonstrated by a numerical example.

Let there be three alternatives, each identified by a number x_i, $i = 1, 2, 3$, and three individuals θ_j, $j = 1, 2, 3$.
Individual θ_j's utility of alternative i is $u_i(\theta_j) = -(x_i - \theta_j)^2$.

This formulation resembles Hotelling's (1929) well-known model: individuals and stores (firms) are located on a line.

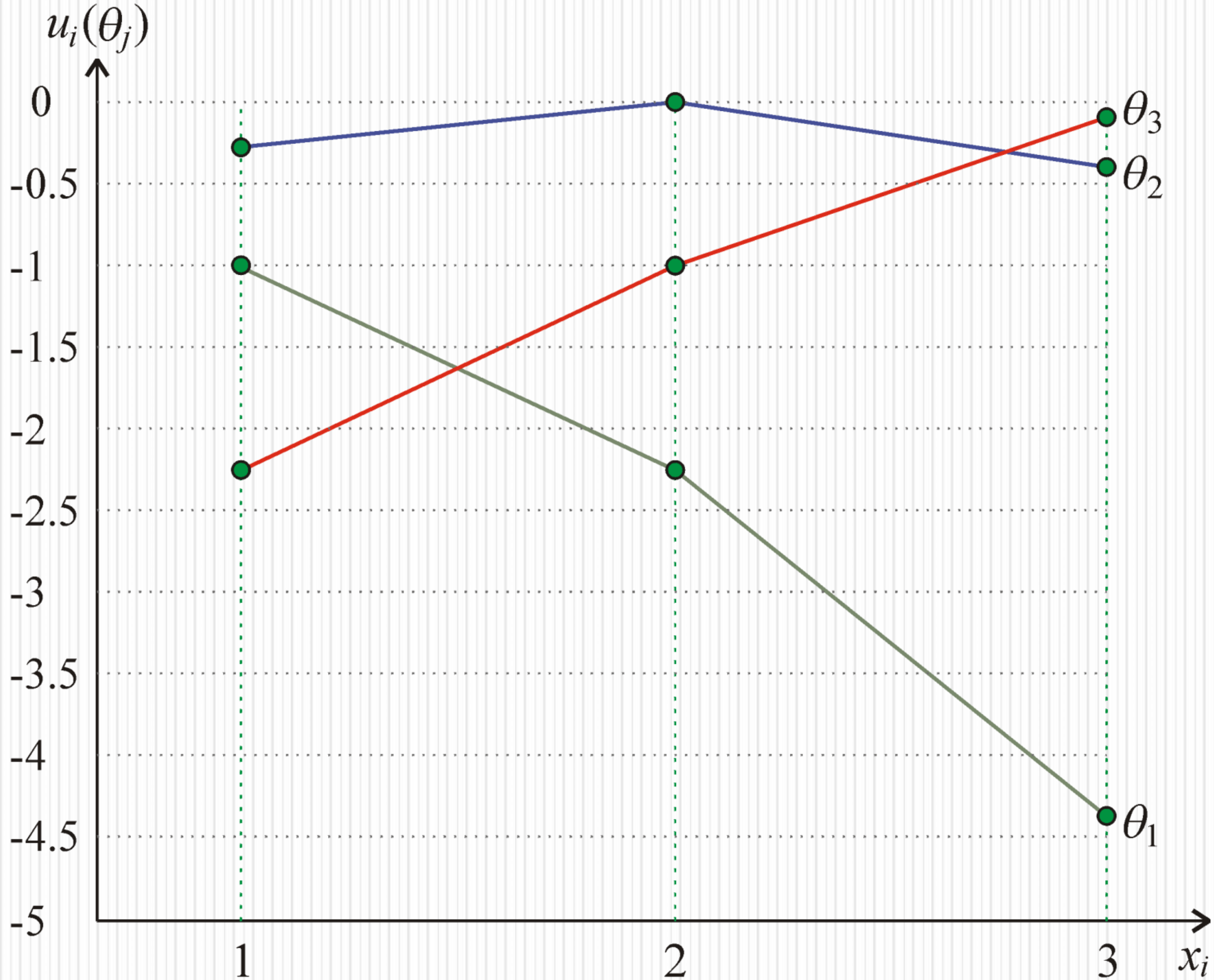
Due to transportation costs, utility decreases with the distance of individual *j*'s location, θ_j, from store x_i.

In the calculations below:

$$\begin{array}{ccc} \theta_1 & \theta_2 & \theta_3 \\ \hline -.5 & 1 & 2 \end{array}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline .5 & 1 & 1.6 \end{array}$$

By construction, preferences are *single-peaked*:



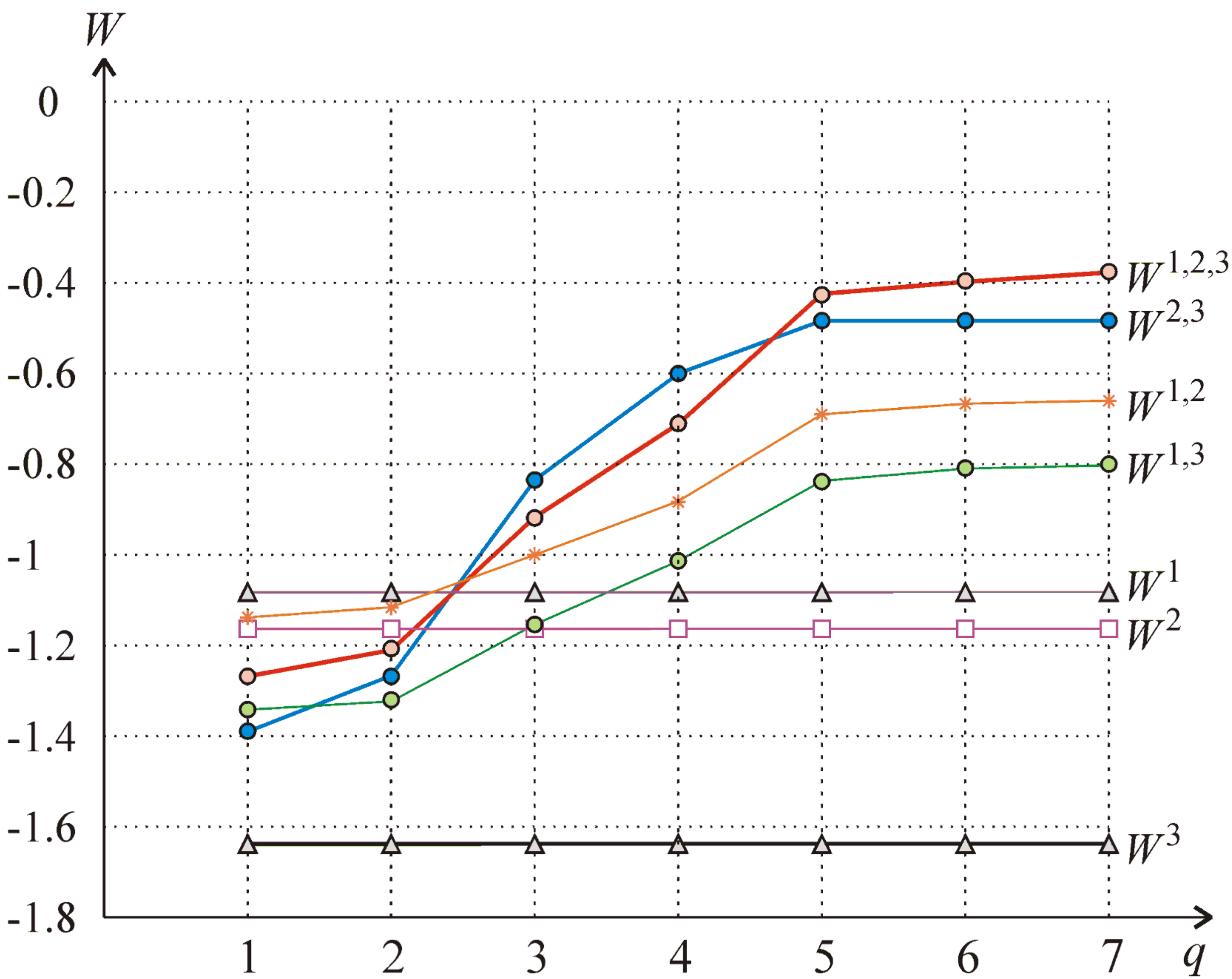
$u_i(\theta_j)$
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
1
2
3
x_i
θ_3
θ_2
θ_1

Possible choice-sets are {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, and {1, 2, 3}. The corresponding social welfare functions are denoted W^1, W^2, W^3, $W^{1,2}$, $\mathrm{W}^{2,3}$, $\mathrm{W}^{1,3}$, and $\mathrm{W}^{1,2,3}$, respectively.

In the Figure below, each of these functions is plotted against different levels of q. For each q, the optimum choice-set corresponds to *the outer envelope* of these curves.

The figure demonstrates the above Proposition: at low q's, the optimum social set has a single alternative (W^1) and at high q's the optimum set includes all alternatives ($\mathrm{W}^{1,2,3}$).

Of particular interest is the fact that the optimum social choice-sets are not nested and reswitching is possible as q rises. For example, the choice set {1, 2} is optimum for certain values of q between 2 and 3 while the set {2, 3} is optimum for still higher levels of q.


W
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
1
2
3
4
5
6
7
q
$W^{1,2,3}$
$W^{2,3}$
$W^{1,2}$
$W^{1,3}$
W^{1}
W^{2}
W^{3}

The above example also demonstrates that a subset of a certain set which is socially superior to the subset at some q, may become socially superior at higher q's.

To further understand the factors which determine the optimum choice sets consider a case with three alternatives {1, 2, 3}. Let the welfare ranking of the single alternative sets be $W^1 > W^2 > W^3$, so at low q's only alternative 1 is offered. As q increases, it becomes desirable to expand the choice sets.

Taking, for example, the set {1, 2} (omitting the θ's and q's in the respective functions)

$$W^{1,2} - W^{1} = \int_{\underline{\theta}}^{\overline{\theta}} P_2^{1,2}(u_2 - u_1)dF$$

where $P_2^{1,2} = e^{q(u_2 - u_1)} \big/ (1 + e^{q(u_2 - u_1)})$. By assumption, $u_2 - u_1$ is both negative and positive over $(\underline{\theta}, \overline{\theta})$.

At $q = 0$, $W^{1,2} - W^{1} = ½\,(W^{2} - W^{1}) < 0$. For positive q's, $P_2^{1,2}$ increases for θ's which have $u_2 - u_1 > 0$ and decreases for θ's with $u_2 - u_1 < 0$.

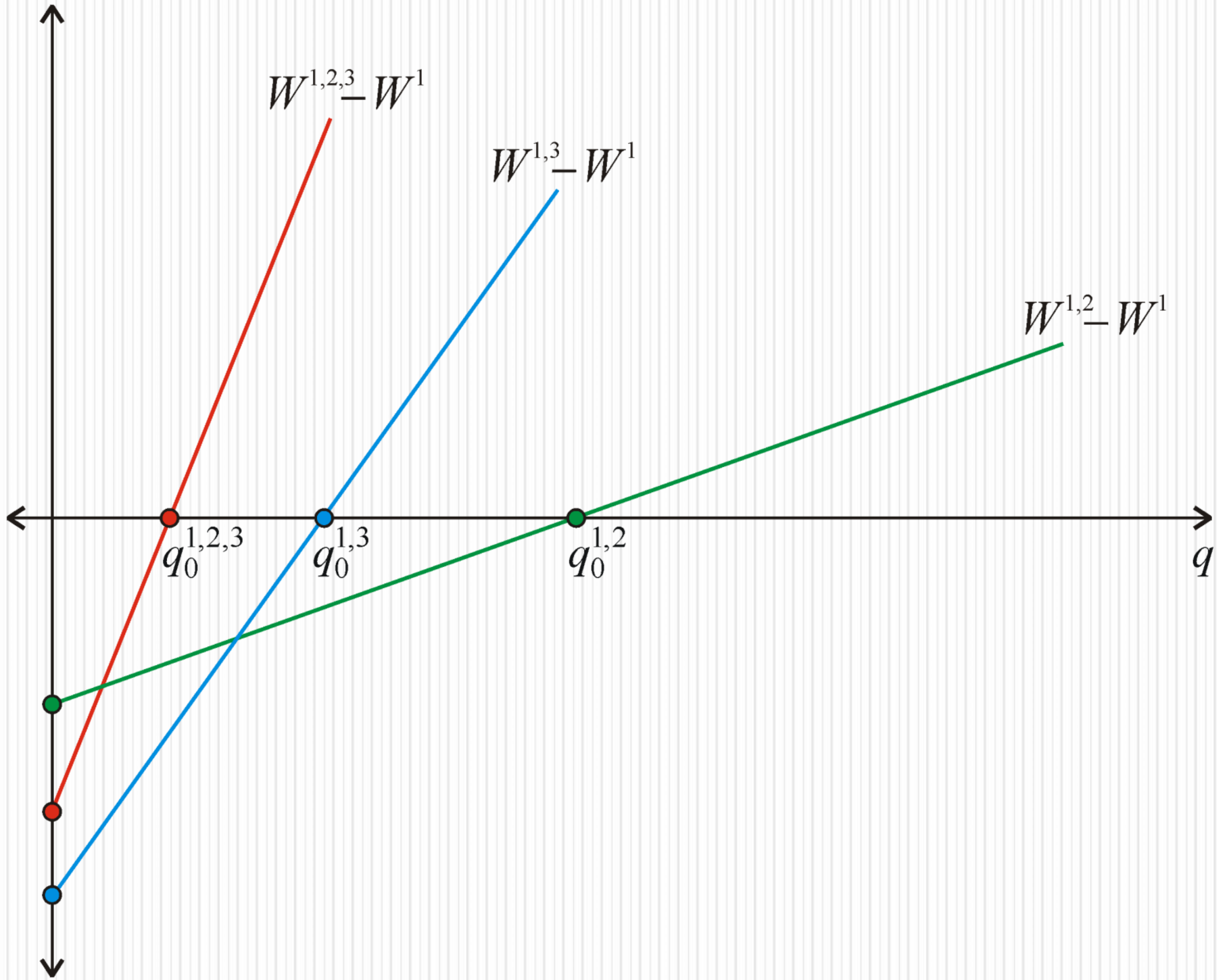
For high q's, $W^{1,2} - W^{1}$ converges to $\int (u_2 - u_1)\,dF > 0$ where the integral is only over the θ's for which $u_2 - u_1 > 0$.

It follows that there is a critical positive q, say $q_0^{1,2}$, such that $W^{1,2} - W^{1} \gtreqless 0$ as $q \gtreqless q_0^{1,2}$. The same argument applies to the sets {1, 3}, {2, 3} and {1, 2, 3}. There exist positive values, $q_0^{1,3}$, $q_0^{2,3}$ and $q_0^{1,2,3}$, such that $W^{1,3} - W^{1} \gtreqless 0$ as $q \gtreqless q_0^{1,3}$, $W^{2,3} - W^{1} \gtreqless 0$ as $q \gtreqless q_0^{2,3}$ and $W^{1,2,3} - W^{1} \gtreqless 0$ as $q \gtreqless q_0^{1,2,3}$.

As in the example above and numerous simulations which we conducted, there is no necessary order for $q_0^{1,2}$, $q_0^{1,3}$, $q_0^{2,3}$, and $q_0^{1,2,3}$.

In the figure below, $q_0^{1,3} < q_0^{1,2}$, implying that though $W^3 < W^2$, the choice set {1, 3}, may supersede {1, 2}, in replacing {1}.

It is also possible that the set which includes all alternatives, {1, 2, 3}, will supersede any of the binary sets when expanding {1}.



$W^{1,2,3} - W^{1}$
$W^{1,3} - W^{1}$
$W^{1,2} - W^{1}$
$q_0^{1,2,3}$
$q_0^{1,3}$
$q_0^{1,2}$
q

To understand why nesting is not inevitable, consider any subset $\tilde{S}$ of S, $\tilde{S} \subseteq S$:

$$W^S(q) - W^{\tilde{S}}(q) = \int_{\underline{\theta}}^{\overline{\theta}} (V^S(\theta, q) - V^{\tilde{S}}(\theta, q) dF(\theta) =$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} \sum_{i \in S - \tilde{S}} p_i^S(\theta, q)(u_i(\theta) - V^{\tilde{S}}(\theta, q)) dF(\theta) =$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} \left(\sum_{i \in S - \tilde{S}} p_i^S \right) (V^{S - \tilde{S}}(\theta, q) - V^{\tilde{S}}(\theta, q)) dF(\theta)$$

where $S-\tilde{S}$ is the set which contains all alternatives in S but not in $\tilde{S}$, and $V^{S-\tilde{S}} = \sum_{i \in S-\tilde{S}} \left(\frac{p_i^S}{\sum_{i \in S-\tilde{S}} p_i^S} \right) u_i$ is expected utility over the set of alternatives in $S-\tilde{S}$. The integrand is the probability of choosing an alternative in $S-\tilde{S}$ times net expected utility of the inclusion of $S-\tilde{S}$. The negative term $V^{\tilde{S}}$ is the lower expected utility obtained from the alternatives in $\tilde{S}$, because the inclusion of those in $S-\tilde{S}$ reduces the probability of choosing alternatives in $\tilde{S}$. Suppose that $W^S - W^{\tilde{S}} > 0$ for some q.

Higher q's raise both $V^{S-\tilde{S}}$ and $V^{\tilde{S}}$ for all θ, reflecting the higher weight given to the most preferred alternative in each set. The probability of choosing any alternative in $S-\tilde{S}$ increases for the highest utility in $S-\tilde{S}$ while all other probabilities decrease. This lends higher weight to the increase in $V^{S-\tilde{S}}$, but still $V^{\tilde{S}}$ may rise more than $V^{S-\tilde{S}}$, even reversing the sign of $W^{S}-S^{\tilde{S}}$. In the limit, though, the integrand above is equal, for each θ, to the utility of the alternative in $S-\tilde{S}$ with the highest utility, and this is clearly positive, consistent with the Proposition stated above.

Policy Affecting Choice Probabilities

The government has various ways to influence choice probabilities. The well documented tendency to choose default alternatives (e.g. Johnson *et-al* (1993) or Caroll *et-al* (2009)) is one example. Many studies show that "framing" issues (such as "opting-out" and "opting-in" design affects individual choices (e.g. Choi *et-al* (2003)).

These studies demonstrate that control over the method of choice enables the designer, whether the government or private firms, to affect choice probabilities.

Governments may also use fiscal instruments to shift individuals' choice. Consider the imposition of a tax/subsidy, t_i, on alternative i.

The policy $\underline{t} = (t_1, t_2,.., t_n)$ affects the choice probabilities, which are now rewritten

$$p_i(\theta, q, \underline{g}) = \frac{e^{q(u_i - t_i)}}{\sum_{j=1}^{n} e^{q(u_j - t_j)}} = \frac{e^{qu_i} g_i}{\sum_{j=1}^{n} e^{qu_j} g_j}$$

where $g_i = e^{-qt_i}$, $g_i \geq 0$, $i = 1, 2,.., n$ and $\underline{g} = (g_1, g_2,.., g_n)$.

The vector $\underline{g}$ are weights given to choice probabilities. The government's objective is to choose the vector $\underline{g}$ that maximizes social welfare. Of particular interest are cases when the optimum weight is zero, that is, when an alternative is excluded.

Since $p_i(\theta, q, \underline{g})$ is homogeneous of degree zero in $\underline{g}$, normalize

$$\sum_{i=1}^{n} g_i = 1.$$

The limiting cases now become:

$$p_i(\theta, 0, \underline{g}) = \lim_{q \to 0} p_i(\theta, q, \underline{g}) = g_i \quad \text{and}$$

$$p_i(\theta, \infty, \underline{g}) = \lim_{q \to \infty} p_i(\theta, q, \underline{g}) = \begin{cases} 1 & \text{if } u_i(\theta) > \max\limits_{j \neq i} u_j(\theta) \\ 0 & \text{if } u_i(\theta) < \max\limits_{j \neq i} u_j(\theta) \end{cases}$$

For any *q*, the F.O.C. are

$$\frac{\partial W(q,\underline{g}^*)}{\partial g_i} = \frac{1}{g_i^*}\int_{\underline{\theta}}^{\overline{\theta}} p_i(\theta,q,\underline{g}^*)(u_i(\theta)-V(\theta,q,\underline{g}^*)) \le 0 \quad i = 1,2,..,n$$

where $\frac{1}{g_i^*} p_i(\theta,q,\underline{g}^*) = \frac{e^{qu_i(\theta)}}{\sum_{j=1}^{n} e^{qu_j(\theta)} g_j^*}$.

(Sufficient second-order conditions can be derived).

A Binary Example

Let $i = 1, 2,\ \ p = \dfrac{e^{qu_1} g}{e^{qu_1} g + e^{qu_2}(1-g)}$ is the probability of choosing

alternative 1 and g, $0 \le g \le 1$, the weight given to this alternative. The F.O.C. condition for the optimum g, g^* is

$$\frac{\partial W(q, g^*)}{\partial g} = \int_{\underline{\theta}}^{\overline{\theta}} \frac{e^{q\Delta(\theta)}\Delta(\theta)}{(e^{q\Delta(\theta)} g^* + 1 - g^*)^2} dF(\theta) \le 0$$

where $\Delta(\theta) = u_1(\theta) - u_2(\theta)$.

By assumption, each of the two alternatives is ranked first by some individuals, hence Δ changes sign at least once over $(\underline{\theta}, \overline{\theta})$.

Assume that for some $\tilde{\theta}$, $\underline{\theta} < \tilde{\theta} < \overline{\theta}$,

$$\Delta(\theta) \gtreqless 0 \text{ as } \theta \gtreqless \tilde{\theta}$$

Then, for any $0 < g < 1$, $p(\theta, \infty, g) = 1$ for $\underline{\theta} < \theta < \tilde{\theta}$

and $p(\theta, \infty, g) = 0$ for $\tilde{\theta} < \theta < \overline{\theta}$.

Maximum social welfare, $\bar{W}$, is

$$\overline{W} = W(\infty, g) = \int_{\underline{\theta}}^{\tilde{\theta}} u_1(\theta)dF(\theta) + \int_{\tilde{\theta}}^{\bar{\theta}} u_2(\theta)dF(\theta)$$

The other limiting case is

$$W(0, g) = gW_1 + (1-g)W_2.$$

Assume that $W_1 > W_2$, that is, when the choice-set has only one alternative, alternative 1 yields higher social welfare. Hence, when $q = 0$, the optimum policy is $g^* = 1$.

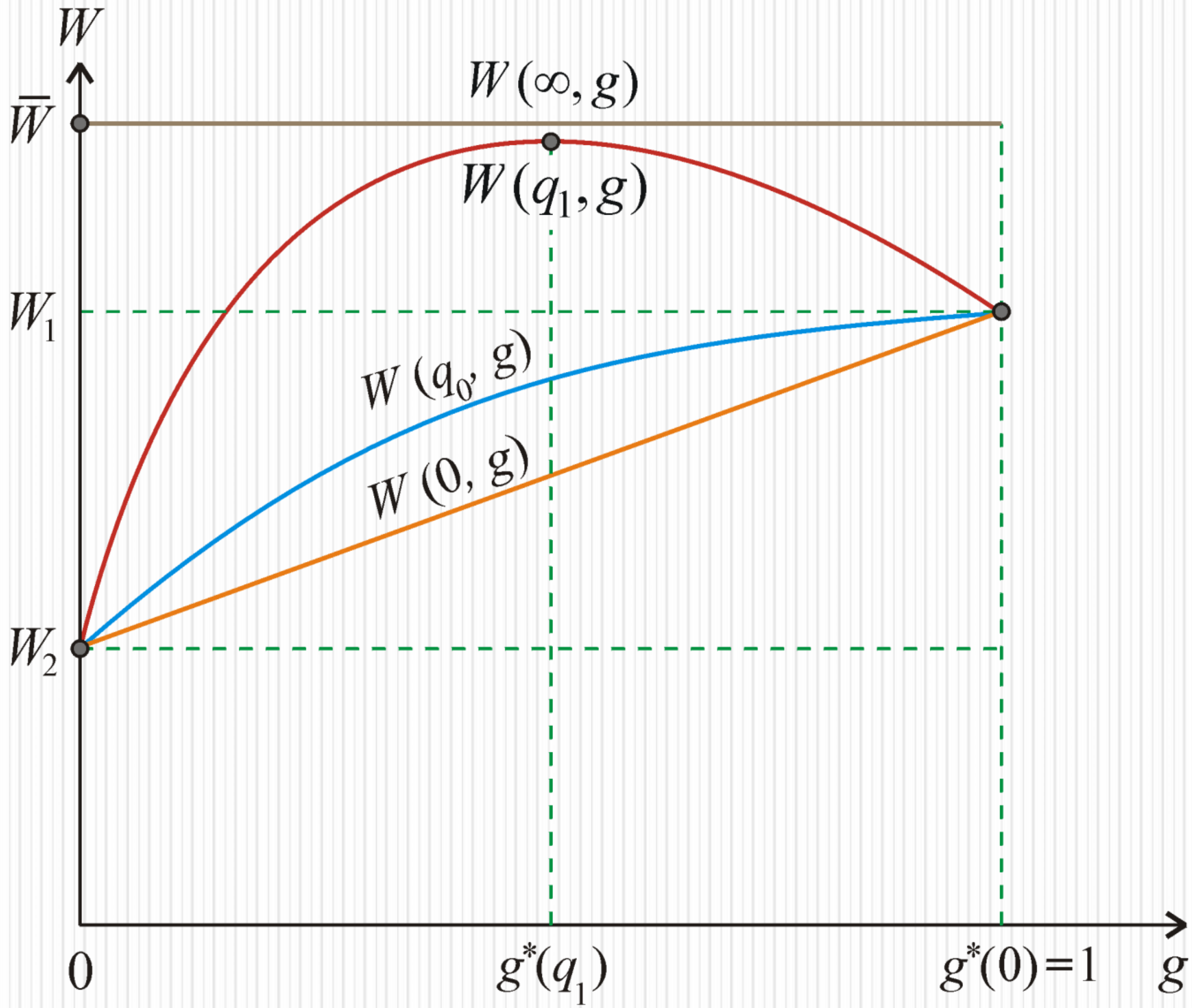
A higher q raises W for all $0 < g < 1$. At $g = 1$, the slope

$\frac{\partial W}{\partial g}$ decreases as q rises, while at $g = 0$ the slope increases

with q.

At q_0, $\frac{\partial W(q_0,1)}{\partial g} = 0$ and at any $q > q_0$, $\frac{\partial W(q,1)}{\partial g} < 0$, implying

that there is a unique interior solution, $0 < g^*(q) < 1$.



$(q_1 > q_0)$

The binary example can be used to further understand what factors affect q_0. Taking a linear approximation for

$$e^{-q\Delta},\ \text{solve}\,\frac{\partial W(q_0,1)}{\partial g}=0\ \text{ for}\, q_0:$$

$$q_0=\frac{W^1-W^2}{\sigma_1^2+\sigma_2^2-Cov_{12}}$$

where $\sigma_i^2=\int_{\underline{\theta}}^{\tilde{\theta}}(u_i(\theta)-W^i)^2\,dF(\theta)$, is the variance of u_i, $i=1,2$, and

$$Cov_{12}=\int_{\underline{\theta}}^{\tilde{\theta}}(u_1(\theta)-W^1)(u_2(\theta)-W^2)\,dF(\theta).$$

This is intuitively clear: the larger the dispersion and the lower the covariance of tastes in the population the lower is the critical level of q when multiple alternatives become socially desirable.

Application to Early Eligibility for Retirement Benefits

Public social security systems (and private pensions) have an early eligibility age at which a person can start receiving a pension. This age differs widely across countries.

In the US it is age 62 and full benefits were reached at 65, moving gradually to 67. In the UK, early eligibility and full benefits are both at age 65.

Imposing a constraint of earliest age for claiming benefits hurts workers who would 'sensibly' stop working before this age due to health and other personal circumstances.

On the other hand, it prevents people from retiring too early (inadequate savings or shortsightedness). The early eligibility age is supposed to strike a balance between these considerations.

Let $u(c_a) - \theta$ be workers' utility where c_a is their consumption and θ is disutility from work, $v(c_b)$ is the utility of non-workers, where c_b is their consumption (pension benefits). Individuals differ in their labor disutility, θ, whose distribution in the population is $F(\theta)$. Take the range of θ to be $(0, \overline{\theta})$.

Under perfect rationality, individuals work or retire as $u(c_a) - \theta \gtreqless v(c_b)$. Define $\hat{\theta}$,

$$\hat{\theta} = \max\left(u(c_a) - v(c_b), 0\right)$$

Individuals with $\theta < \hat{\theta}$ work and those with $\theta > \hat{\theta}$ do not work (retire). Assume that $0 < \hat{\theta} < \overline{\theta}$, so that in the First-Best some work and some do not work.

With bounded rationality, the probability that a θ-individual works is

$$p(\theta, q) = \frac{e^{q(\hat{\theta}-\theta)}}{e^{q(\hat{\theta}-\theta)} + 1}$$

Social welfare when everyone works is $W_a = u(c_a) - E(\theta)$,

where $E(\theta) = \int_0^{\bar{\theta}} \theta dF(\theta)$ is expected labor disutility.

Social welfare when nobody works is $v(c_b)$. Assume that everybody working is socially preferred to nobody working: $W_a > v(c_b)$.

The relevant comparison is between social welfare with a retirement option, $W(q)$, and without the retirement option, W_a :

$$W(q)=\int_0^{\overline{\theta}}[p(\theta,q)(u(c_a)-\theta)+(1-p(\theta,q))u(c_b)]dF(\theta)$$

$$W_a=\int_0^{\overline{\theta}}(u(c_a)-\theta)dF(\theta)$$

$$W(q)-W_a \quad =-\int_0^{\overline{\theta}}(1-p(\theta,q))(\hat{\theta}-\theta)dF(\theta)$$

$$=-\int_0^{\overline{\theta}}\left(\frac{\hat{\theta}-\theta}{e^{q(\hat{\theta}-\theta)}+1}\right)dF(\theta)$$

By assumption $W(0) - W_a < 0$.

There exists a $q_0 > 0$, defined by $\int_0^{\overline{\theta}} \left(\frac{\hat{\theta} - \theta}{e^{q_0(\hat{\theta}-\theta)} + 1} \right) dF(\theta) = 0$

such that for all $q > q_0$ a retirement option is desirable.

The calculations below are for $u(c) = v(c) = \ln c$ and $F(\theta)$ a uniform distribution over $(0, \frac{1}{3})$.

$\hat{\theta} = \dfrac{u(c_a)}{v(c_b)} = \ln(\dfrac{c_a}{c_b})$ and we chose values for $\dfrac{c_a}{c_b}$, the ratio of pre-retirement to post-retirement consumption (the inverse of the *'replacement rate'*), in a commonly observed range: 1.2, 1.25, and 1.3.

$\frac{c_a}{c_b}$	$\hat{\theta}$	q_0	*Percent working*	E_1	E_2
1.2	.18	3.29	54	.24	.21
1.25	.22	13.10	67	.15	.12
1.3	.26	26.53	79	.09	.06

For each $\frac{c_a}{c_b}$, calculate $\hat{\theta}\left(=\ln\frac{c_a}{c_b}\right)$, q_0 and the percent of the population working in the First-Best $(=3\hat{\theta})$.

Most insightful are the *'type one'* and *'type two'* errors at q_0 (the level of q at which choice is introduced), $E_1(q_0)$ and $E_2\,(q_0)$.

That is, the percent of those who work in the First-Best but choose to retire under bounded rationality, and the percent of those who are non-workers in the First-Best but choose to work under bounded rationality:

$$E_1(q_0) = \int_0^{\hat{\theta}} (1 - p(\theta, q_0))\, dF(\theta); \quad \text{and} \quad E_2(q_0) = \int_{\hat{\theta}}^{\bar{\theta}} p(\theta, q_0)\, dF(\theta)$$

The size of these errors decreases significantly as $\frac{c_a}{c_b}$ increases. This is not surprising. A rise in this ratio raises the preference for work, decreasing the value of the non-work option.

Varying Degrees of Rationality Among Individuals

- We assumed that individuals have a common degree of rationality, q. Relaxing this assumption requires modification of certain conclusions. Suppose that individuals are identified by two parameters, θ and q.

These parameters are assumed to be jointly distributed in the population. When the support of the (marginal) distribution of q is a narrow interval then the results in Proposition 1 are still applicable.

Specifically, with small q's, the optimum choice-set is a singleton, and with large q's, all alternatives are contained in the optimum choice-set.

However, when the support of the distribution of q is wide, that is, individuals have widely varying degrees of rationality, then some questions explored earlier have to be rephrased and conclusions modified.

Consider, for example, the following question:

is there a fraction of individuals with high levels of q that warrants the inclusion of all alternatives in the optimum choice-set?

A binary choice model demonstrates that the answer to this question is negative. Let choice be between two alternatives, 1 and 2. There are two types of individuals, each identified by the pair (θ_i, q_i), $i = 1, 2$.

Let the fraction of type 1 individuals is f, $0 < f < 1$. Denote by $u_j^i = u_j(\theta_i)$, $i, j = 1, 2$, and $p_i(\theta_i, q_i)$ is the probability of type i individuals choosing alternative 1. Expected utilities, V^i, are

$V^i = p^i u_1^i + (1 - p^i) u_2^i$, $i = 1, 2$, and social welfare, W, is

$$W(q_1, q_2) = V^1 f + V^2 (1 - f).$$

To have a meaningful problem, assume that the two types have opposite preferences: $\Delta^1 > 0$ and $\Delta^2 < 0$, where

$\Delta^i = u_1^i - u_2^i,\ i = 1, 2.$

If $W^1 > W^2$, then alternative 1 is included in the choice-set for any (q_1, q_2). Starting with W^1, consider whether the inclusion of alternative 2 is desirable:

$$W(q_1, q_2) - W^1 = -\left(\frac{\Delta^1 f}{e^{q_1 \Delta^1} + 1} + \frac{\Delta^2 (1 - f)}{e^{q_2 \Delta^2} + 1} \right)$$

By assumption $W(0, 0) - W^1 < 0$. To see the effect of large differences in the q's, take $q_2 = 0$:

$$W(q_1, 0) - W^1 = -\left(\frac{\Delta^1 f}{e^{q_1 \Delta^1} + 1} + \frac{\Delta^2 (1 - f)}{2} \right)$$

It is seen that $W(\infty, 0) - W^1 = \lim_{q_1 \to \infty} W(q_1, 0) - W^1 = -\frac{\Delta^2 (1 - f)}{2} > 0.$

For large q_1, the choice-set includes both alternatives.

Alternatively, let $q_1 = 0$. Then

$$W(0, q_2) - W^1 = -\left(\frac{\Delta^1 f}{2} + \frac{\Delta^2 (1 - f)}{e^{q_2 \Delta^2} + 1} \right)$$

$$W(0, \infty) - W^1 = \lim_{q_2 \to \infty} W(0,\ q_2) - W^1 = -\left(\frac{\Delta^1 f}{2} + \Delta^2 (1 - f) \right) \text{☹} 0.$$

When all type 2 individuals choose perfectly, alternative 2 is included in the optimum choice-set provided the fraction of type 1 individuals is small.