Queueing Systems Vol. 100
Special Issue: 100 views on queues

Martingales associated with functions of Markov and finite variation processes

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February 9, 2022

1 Introduction

Suppose that $X = \{X_t \,|\, t \geq 0\}$ is a càdlàg Markov process taking values in some metric space (where the notion of càdlàg is well defined) with respect to some filtration satisfying the usual conditions and that $Y = \{Y_t \,|\, t \geq 0\}$ is a $\mathbb{R}^r$ valued càdlàg, adapted process of finite variation on finite intervals. Assume that X has an (extended) generator $\mathscr{A}$ such that for any continuous ξ in its domain (so that $\xi(X_t)$ is also càdlàg and adapted) we have that

$$\xi(X_t) - \xi(X_0) - \int_0^t \mathscr{A}\xi(X_s)ds \tag{1}$$

is a local martingale (Dynkin's formula). If η is a continuously differentiable function, then the Lebesgue-Stieltjes integration formula reads

$$\eta(Y_t) = \eta(Y_0) + \int_0^t \nabla\eta(Y_s)^T dY_s^c + \sum_{0<s\leq t} \Delta\eta(Y_s) \tag{2}$$

where $\Delta Y_s = Y_s - Y_{s-}$, $Y_t^c = Y_t - \sum_{0<s\leq t} \Delta Y_s$ and ∇ is the gradient operator. The superscript T is for transposition.

In [14] it was shown that for sufficiently nice functions and sufficiently nice Markov processes we have that

$$\begin{aligned} M_t = {} & f(X_t,Y_t) - f(X_0,Y_0) - \int_0^t \mathscr{A}f(X_s,Y_s)ds \\ & - \int_0^t \nabla_y f(X_s,Y_s)^T dY_s^c - \sum_{0<s\leq t} (f(X_s,Y_s) - f(X_s,Y_{s-})) \end{aligned} \tag{3}$$

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is a local martingale, where $\mathscr{A} f(x,y)$ means that we fix y and then take the generator of $f_y(\cdot) = f(\cdot,y)$ and $\nabla_y f(x,y)$ means that we fix x and take the gradient of $f_x(y) = f(x,y)$. It is emphasized that X_s (not X_{s-}) appears in both $f(X_s,Y_s)$ and $f(X_s,Y_{s-})$ in the sum on the right hand side.

Note that when f does not depend on Y or Y is constant, then the second line of (3) is zero and (3) reduces to Dynkin's formula. When f does not depend on X or X is constant, then $\int_0^t \mathscr{A} f(X_s,Y_s)ds = M_t = 0$ and then (3) reduces to the Lebesgue-Stieltjes integration formula. When X is a Brownian motion and Y is continuous then (3) becomes Itô's formula.

Under some further (usually) easily verifiable conditions it was shown that in fact M_t is a square integrable martingale which behaves nicely as a function of t both in the almost sure sense and in the L^2 sense.

It turns out that special cases of the results of this paper are the results reported [13, 5] and it is shown that for these cases the "easily verifiable conditions" referred to in the previous paragraph are automatically satisfied (see Corollary 3 of [14]). The results from [13,5] have been used in quite a few theoretical and applied studies (mainly in queueing, risk and finance) and extensively cited over the years. I find it unnecessary to give an exhaustive summary, as this can easily be found via a simple web search. Some book references may be found in [2,3,20,15,16,1,10,17]. A very small random sample of applications is, (*e.g.*): [4] (finance), [11,19] (risk), [9,8] (queueing), [18,7] (theoretical).

My own involvement with the theoretical development of this type of results may be found in [13,5,6,12,14]. I was also associated with around 30 other publications applying these results to various queueing and stochastic storage processes and networks but I see no point in listing these here.

2 The problem

There are two separate conditions under which the local martingale in (3) was established in [14]. Most of the effort in this paper was directed at the case where the Markov process X is a (possibly multivariate) real valued jump diffusion process of which Lévy processes, Markov additive processes, continuous time Markov chains and piecewise deterministic Markov processes are special cases and thus this probably covers all the Markov processes which we encounter in application and applied probability and in particular queueing and storage theory. For this case the results were established for functions $f \in C^{2,1}$ which are twice differentiable in x, differentiable in y and all derivatives and mixed derivatives are continuous. There are further natural restrictions on f and some more restrictions (Assumption 3 in [14]) if we would like M_t to be a (L^2) martingale which behaves nicely in t.

The other case is when X is some general Markov process obeying whatever appears in the first paragraph of the introduction (and no other assumptions) and $f(x,y) = \xi(x)\eta(y)$, where ξ is in the domain of the generator $\mathscr{A}$ and η is continuously differentiable. This of course implies that (3) is a local martingale for any function which is a linear combination of such products so that it is possible that a Stone-Weierstrass type result would lead to a generalization to more general functions.

The problem that would be nice to have an answer to is, therefore, the following.

Problem: *Let $f(x,y)$ be some function such that, for every y, $f(\cdot,y)$ (as a function on the metric space containing X) is in the domain of $\mathscr{A}$ and, for every x, $f(x,\cdot)$ (a real valued multivariate function) is continuously differentiable. What are the weakest additional conditions on f and/or X needed so that (3) is a local martingale and under what further conditions is it a martingale?*

Possibly this is a question more in the areas of Markov processes/functional analysis (semigroups)/stochastic analysis than in queueing or applied probability.

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