Performance characteristics of real life heat engines

1) An engine operates at finite power.
2) The engine is limited by heat leaks.
3) The motion is subject to friction.

The engine operates under irreversible conditions

Do quantum engines have the same performance characteristics?
The problem

Tradeoff between power and efficiency

Insight: construct the most simple model showing this effect

Heat engines in finite time governed by master equations

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One more important general point is that the system is operated in finite time. The optimal time allocations or possible minimal cycle time is an important part of the study.

We will be interested to optimize the performance of the refrigerator, its approach to absolute zero, all that by describing the relation between the quantum framework and the thermodynamical observables.
Quantum Otto engine

\[ \omega = g \beta H \]

work = \( S \Delta \omega \)

heat = \( \omega \Delta S \)
Quantum Otto engine

Total work = $\Delta \omega \Delta S$

Efficiency $\eta = \frac{\Delta \omega \Delta S}{\omega_b \Delta S} = 1 - \frac{\omega_b}{\omega_b}$

In equilibrium

$S = -\frac{1}{2} \text{tgh}\{\omega/2kB T\}$

Since $S_1 \approx S_2$  $\eta < 1 - \frac{T_c}{T_h}$
The two-level-system (TLS) Quantum heat engine

Hamiltonian

\[ H = H_{\text{int}} + H_{\text{ext}} \]

\[ H_{\text{ext}} = \omega(t) S_z \]

where \( S_z \) is the polarization

Equations of motion

\[ \dot{A} = i[H, A] + \frac{\partial A}{\partial t} + \mathcal{L}d(A) \]

\[ \mathcal{L}d(A) = \sum_j F_j A F_j^\dagger - \frac{1}{2} \{ F_j F_j^\dagger, A \} \]
The equation of motion for the polarization

On the adiabats:  
\[ \omega_a \rightarrow \omega_b \]  
\[ \omega_a \leftarrow \omega_b \]

\[ \frac{dS_z}{dt} = i[H, S_z] = 0 \]

On the isochores:  
\[ T_A \rightarrow T_B \]  
\[ T_C \rightarrow T_D \]

\[ \frac{dS_z}{dt} = (\kappa \uparrow + \kappa \downarrow)S_z - (\kappa \uparrow - \kappa \downarrow)I \]

Detailed balance:  
\[ \frac{\kappa \uparrow}{\kappa \downarrow} = e^{-\frac{\hbar \omega}{k_B T}} \]

Polarization as a function of time:  
\[ S_z(t) = S_z(\text{eq}) + (S_z(0) - S_z(\text{eq})) \cdot e^{-\Gamma t} \]
Optimizing the power output

Fixed parameters:
\[ \kappa, \omega_a, \omega_b, T_h, T_c \]

\[ P = \frac{W}{\tau} \]

\[ \tau = \tau_a + \tau_b + \tau_c + \tau_h \]

Tova Feldmann, Eitan Geva, Ronnie Kosloff and Peter Salamon,
*Heat Engines in Finite Time Governed by Master Equations*,
The maximum power is finite!

There is no restriction on the time allocation on the adiabats therefore: $\tau_a = \tau_c = 0$

The maximum power is obtained when $\tau_c = \tau_h = 0$

What is needed is a restriction on the adiabatic dynamics

Friction?!
Phenomenological friction

\[ \dot{S} = \left( \frac{\sigma}{t} \right)^2 \]

\[ S(t) = S(0) + \left( \frac{\sigma}{t} \right)^2 t \]

which forces a constant speed of polarization change, \( S \).
Otto refrigerator with friction

First attempt at III law

\[ Q_{F}^{\text{optimum}} \leq \frac{F(x, y)}{4 k_B \tau} \left( \frac{\omega_a}{T_c} \right)^2 T_c. \]
Asymptotic properties of the heat pump when the cold bath temperature approaches absolute zero.

The heat flow:
\[ \frac{Q_c}{T} = \frac{\omega_c (S_2 - S_1)}{T} \]

(No global optimum with respect the fields was found.)

Optimum
\[ \frac{Q_c}{T} = \frac{F(x,y)}{4 k_B T} \left( \frac{\omega_c}{T_c} \right)^2 T_c \cosh^2 \left( \frac{\omega_c}{2 k_B T_c} \right) \]

So as \( T_c \to 0 \), the cooling rate vanishes at least linearly with \( T_c \).
Quantum origin of Friction

Control Hamiltonian

\[ \hat{H}(t) = \hat{H}_{\text{int}} + \hat{H}_{\text{cont}}(t) \]

As a result:

\[ [\hat{H}(t), \hat{H}(t')] \neq 0 \]
Seeking a QM origin of friction

Hamiltonian \[ H = H_{\text{int}} + H_{\text{ext}} \]

\[ [H_{\text{ext}}, H_{\text{int}}] \neq 0 \]

Working fluid consisting of interacting particles

\[ H_{\text{ext}} = \omega(t) \left( \sigma^1_z \otimes I^2 + \sigma^2_z \otimes I^1 \right) \]

\[ H_{\text{int}} = J \left( \sigma^1_x \otimes \sigma^2_x - \sigma^2_y \otimes \sigma^1_y \right) \]

Equations of motion for the \( B \) operators

\[ [B_1, B_2] = 4i B_3 \]

\[ [B_i, B_j] = 4i \varepsilon_{ijk} B_k \]

\[ \frac{d}{dt} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4J \\ 0 & 0 & -4\omega(t) \\ -4J & 4\omega(t) & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \]
Equations of motion on the isochores

\[ \dot{A} = i[H,A] + \frac{\partial A}{\partial t} + \mathcal{L} d(A) \]

\[ \mathcal{L} d(A) = \sum_j F_j A F_j^\dagger - \frac{1}{2}\{F_j F_j^\dagger, A\} \]

Equilibration:
\[ \Gamma = k \downarrow + k \uparrow \]

\[ \mathcal{L}_D(\hat{B}_1) = -\Gamma \hat{B}_1 + \frac{2\omega}{\Omega} (k \downarrow - k \uparrow) \hat{I} \]

\[ \mathcal{L}_D(\hat{B}_2) = -\Gamma \hat{B}_2 + \frac{2J}{\Omega} (k \downarrow - k \uparrow) \hat{I} \]

\[ \mathcal{L}_D(\hat{B}_3) = -\Gamma \hat{B}_3 \]

Pure Dephasing
\[ \mathcal{L}_D^d(\hat{X}) = \gamma[H, [H, \hat{X}]] \]
The realization, coupled spin model

For completeness I quote the Pauli matrices.

\[ \hat{\tau}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \hat{\tau}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \hat{\tau}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ \text{The Hamiltonian} \quad (\hat{t} = 1) \]

\[ \hat{H}_{\text{ext}} = \frac{1}{2} \omega(t)(\hat{\tau}_z \otimes I^2 + I \otimes \hat{\tau}_z) = \frac{\omega(t)}{2} \hat{B}_z \quad (Q13) \]

The control field is chosen in the positive \( z \) direction.

\[ \hat{H}_{\text{int}} = \frac{1}{2} \mathbf{J} (\hat{\tau}_x \otimes \hat{\tau}_x - \hat{\tau}_y \otimes \hat{\tau}_y) = \frac{\mathbf{J}}{2} \hat{B}_z \quad (Q14) \]

\( \mathbf{J} \) scales the strength of the interaction, which is non-controllable. \( \hat{H}_{\text{int}} \) is a correlation energy between the two spins in the \( x \) and \( y \) directions.

Therefore \( \hat{H}_{\text{ext}} \) and \( \hat{H}_{\text{int}} \) are not commuting.

To close the algebra one computes

\[ i \hat{B}_z = \frac{1}{2} [\hat{B}_y, \hat{B}_z] = \frac{1}{2} (\hat{\tau}_y \otimes \hat{\tau}_z + \hat{\tau}_z \otimes \hat{\tau}_y) \quad (Q15) \]

\[ \Rightarrow \text{SU}(2) \]
\[ H = H_{\text{ext}} + H_{\text{int}} \]  \hspace{1cm} (Q16)

and in matrix representation

\[ H = \begin{pmatrix}
\omega & 0 & 0 & J \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
J & 0 & 0 & -\omega
\end{pmatrix} \]  \hspace{1cm} (Q17)

with eigenvalues \(-\Omega, 0, 0, \Omega\)

where \(\Omega = \sqrt{\omega^2 + j^2}\)

The equation of motion on the dissipative part. As we said we identified the \(F\) operators with the raising and lowering operators in the energy frame. (Dephasing was left out)  \hspace{1cm} (Q18)

\[
\frac{d}{dt} \begin{pmatrix}
\langle \hat{B}_1 \rangle \\
\langle \hat{B}_2 \rangle \\
\langle \hat{B}_3 \rangle
\end{pmatrix} = \begin{pmatrix}
-\Gamma & 0 & J \\
0 & -\Gamma & 0 \\
J & 0 & -\omega
\end{pmatrix} \begin{pmatrix}
\langle \hat{B}_1 \rangle \\
\langle \hat{B}_2 \rangle \\
\langle \hat{B}_3 \rangle
\end{pmatrix} - \frac{\omega}{2} \begin{pmatrix}
(k_1 - k_2) \\
k_2 \\
0
\end{pmatrix}
On the isochors $\omega$ is time independent, therefore there is an analytical solution for the whole time on those branches. We get the propagator:

$$U_{iso}(T_{iso}) = e^{-i\Phi} \begin{pmatrix}
\frac{c J^2}{\omega^2} & -\omega J c & J s \\
-\omega J c & \frac{c^2 J^2}{\omega^2} & -\omega s \\
-J s & \omega s & c
\end{pmatrix} \tag{Q19}$$

where $c = \cos(\omega T)$, $s = \sin(\omega T)$

The solution of (Q18) then:

$$\vec{b}(t+\Delta t) = U_{iso}(\Delta t)(\vec{b}(t) - \vec{b}_0) + \vec{b}_0$$
On the adiabats ...

We get the global propagator which is the product of the individual propagators on the branches. One can choose any point on the cycle.

\[ U_{\text{global}} = U_{\text{cyc}} = U_{\text{he}} \cdot U_{\text{h}} \cdot U_{\text{heh}} \cdot U_{\text{e}} \]

The limit cycle is the cycle with eigenvalue one of the global propagator. The corresponding eigenvector are the expectation values of the cycle.
The main points of the proof use the concept of the distance between states and the theorem of Lindblad on conditional entropy, namely that the conditional entropy decreases if a completely positive map $\mathcal{T}$ is applied to both the state $\rho$ and the reference state $\rho_{\text{ref}}$:

$$S(\rho | \rho_{\text{ref}}) \geq S(\mathcal{T}\rho | \mathcal{T}\rho_{\text{ref}}) \quad (1)$$

The mapping imposed by the cycle of operation of a heat engine or a heat pump is a product of the individual evolution steps, $U_k$ (in our case $U_k = \exp(-iH_k\tau)$), where each one of these steps is a completely positive map.

We get:

$$\mathcal{T} = U_{\text{eye}} U_{\text{ch}} U_{\text{he}} U_{\text{hc}}$$

If:

$$U_{\text{eye}} S_{\text{lc}} = S_{\text{lc}}, \text{ or } U_{\text{eye}}(i) = (i)$$

Then $S_{\text{lc}}$ is the limit cycle state (NN).

Unfortunately the conditional entropy is not symmetric in $\rho$ and $\rho_{\text{ref}}$, therefore different symmetrical measures are defined which can serve as distance; for they have a metric.

see Fig 2b
Cycle in the entropy field plane

\[ \omega_b \rightarrow T_h \rightarrow \omega_b \]

\[ S_h \]

\[ \omega \rightarrow T_c \rightarrow \omega \]

\[ S_c \]
The optimal cycle trajectory

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Optimal time allocation:
Comparison with a model with phenomenological friction.

\[ \dot{Q}_f = \sigma^2(\dot{\omega}) \]
Intermediate conclusions ......

1) Maximum power $P$ is obtained at the expense of efficiency $\eta$.

2) The same conclusion are obtained irrespective of the cycle of operation and the dynamics of the working fluid, for example:

$$H_{ext} = \omega(t)S_z \quad H_{ext} = \omega(t)a^+a$$

3) Friction like behavior is obtained whenever: $[H_{ext}, H_{int}] \neq 0$, This causes an additional penalty on fast motion on the adiabatic branches.

4) Where is the heat leak?
Quantum lubrication: Suppression of friction in a first-principles four-stroke heat engine

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\[ \mathcal{L}_{Na}(\hat{A}) = -\gamma_a[\hat{H}, [\hat{H}, \hat{A}]] \]
Scaling of the optimal cooling rate when $\Omega_{\text{min}} = J$

$$\frac{Q_c}{\tau} \propto \frac{J^2}{C} e^{-\frac{hJ}{k_B T_c}}$$
Quantization of the adiabats

Thermodynamical observables

The role of noise on the controls
Minimal temperature of quantum refrigerators
Quantization of the adiabats

The frictionless conditions define a quantization condition for the adiabatic parameter $\mu$:

$$\mu = \left( \frac{2\pi l}{\Phi_{hc}} \right)^2 - 1 \right)^{-\frac{1}{2}}. \quad (12)$$
The quantum Otto refrigeration cycle

Cycle Propagator: \( U_{\text{cycle}} = U_h U_{hc} U_c U_{ch} \)

\( U_h \)  Hot Isochore (isomagnetic) \( A \rightarrow B \) \( \omega = \omega_h \). The working medium is in contact with the hot bath of temperature \( T_h \).  

\( U_{hc} \)  Expansion adiabat (demagnetization) \( B \rightarrow C \) The field changes from \( \omega_h \) to \( \omega_c \)

\( U_c \)  Cold Isochore (isomagnetic) \( C \rightarrow D \) \( \omega = \omega_c \). The working medium is in contact with the hot bath of temperature \( T_c \).  

\( U_{ch} \)  Compression adiabat (magnetization) \( D \rightarrow A \) The field changes from \( \omega_c \) to \( \omega_h \).

\[ [U_h U_{hc}, U_c U_{ch}] \neq 0 \]

\( Y = Y \) limit cycle
Energy Balance

\[ \frac{dE}{dt} = \langle \mathcal{L}^*(\hat{H}) \rangle + \left\langle \frac{\partial \hat{H}}{\partial t} \right\rangle. \]

\[ \hat{H} = \hat{H}_{\text{ext}} + \hat{H}_{\text{int}}. \]

Power

Only \( H_{\text{int}} \) is time dependent

\[ H_{\text{int}} = \omega \sum H_i \]

\[ \mathcal{P} = \omega \sum_i \langle \hat{H}_i \rangle, \]

Heat flow

\[ \dot{Q} = \langle \mathcal{L}_D^*(\hat{H}) \rangle = \langle \mathcal{L}_D^*(\hat{H}_{\text{ext}} + \hat{H}_{\text{int}}) \rangle, \]

[Note: \( \mathcal{L}^*(\hat{H}) = \mathcal{L}_D^*(\hat{H}) \) since \( \mathcal{L}_H^*(\hat{H}) = 0 \)]

First law of thermodynamics

\[ \frac{dE}{dt} = \mathcal{P} + \dot{Q}. \]
Typical Refrigerator Cycle
The generic model of a the refrigerator working fluid:

\[ H = H_{\text{int}} + H_{\text{ext}}(\omega) \]

The time dependent control field \( \omega = \omega(t) \)

Typically: \([H_{\text{int}}, H_{\text{ext}}] \neq 0\) leading to: \([H(0), H(t)] \neq 0\)

Two examples of a working fluid:

- Interacting spins in an external magnetic field.
- Adiabatic demagnetization refrigerator ADR

- Harmonic oscillator with time dependent potential
- Adiabatic expansion of a BEC
Quantum thermodynamical observables and their dynamics.

The analysis of the performance requires a quantum dynamical description of the changes in the thermodynamical observables during the engine's cycle of operation. The thermodynamical observables are associated with the expectation values of operators of the working medium.

Using the formalism of von Neumann, an expectation value of an observable \( \langle \hat{A} \rangle \) is defined by the scalar product of the operator \( \hat{A} \) representing the observable and the density operator \( \hat{\varrho} \) representing the state of the working medium:

\[
\langle \hat{A} \rangle = (\hat{A} \cdot \hat{\varrho}) = \text{Tr} \{ \hat{A} \hat{\varrho} \} \quad (Q1)
\]

The dynamics of the system is as before influenced by the external change of variables in addition to the heat transport from or into the heat baths.
In the formulation of quantum open systems, the dynamics is generated by the Liouville superoperator \( \mathcal{L} \) either as an equation of motion for the state \( \hat{\gamma} \) (Schrödinger picture)

\[
\dot{\gamma} = \mathcal{L}(\gamma)
\]  

(Q2)

or as an equation of motion for the operator (Heisenberg picture).

\[
\dot{\hat{A}} = \mathcal{L}^*(\hat{A}) + \frac{2\dot{\hat{A}}}{\hbar}
\]  

(Q3)

we will adopt the Heisenberg picture with the following additional conditions:

(a) The actual operators form an orthogonal set \( \hat{B}_i \), or:

\[
(\hat{B}_i \cdot \hat{B}_j) = \delta_{ij}
\]  

(Q4)

\( \hat{B}_i \) will serve as the basis of the Hilbert space.

(b) The set is closed under the operation \( \mathcal{L}^* \), or:

\[
\hat{B}_i = \mathcal{L}^*(\hat{B}_i) = \sum_j B_{ij} \hat{B}_j
\]  

(Q5)

(c) The equilibrium density operator is a linear combination of the set

\[
\hat{\gamma}^{eq} = \frac{1}{N} + \sum_k \langle \hat{B}_k^{eq} \rangle \hat{B}_k
\]
The Heisenberg eqn (Q3) enables one to solve the equation of motion by diagonalizing $\mathcal{L}$, and to perform the exponentiation. That way one creates the propagator $U$

$$ U = e^{\mathcal{L} \Delta t} \quad (Q6) $$

which relates every observable at time $t$, to time $t+\Delta t$. Finally, having the time dependent expectation values $b_k(t) = \langle B_k \rangle(t)$ one can reconstruct the density operator

$$ \hat{\rho} = \frac{1}{N} + \sum_k b_k \hat{B}_k \quad (Q7) $$

The Liouville super-operator $\mathcal{L}$ for an open quantum system, can be partitioned into a unitary part, $\mathcal{L}_H$, and a dissipative part, $\mathcal{L}_D$.

$$ \mathcal{L} = \mathcal{L}_H + \mathcal{L}_D \quad (Q8) $$
The unitary part is represented by the Hamiltonian $\hat{A}$:

$$\mathcal{L}_H^*(\hat{A}) = i\left[\hat{A}, \hat{A}\right] \quad (Q9)$$

The meaning that the set $\hat{B}_k$ is closed under $\mathcal{L}_H^*$ is that if the Hamiltonian can be decomposed as a linear combination of $\hat{B}_k$'s, and if $B_k$ form a Lie algebra, namely:

$$[\hat{B}_l, \hat{B}_j] = \sum_k C_{ij}^k \hat{B}_k \quad \text{for every } (i,j)$$

then the set is closed under $\mathcal{L}_H^*$. For the dissipative part, the Lindblad operator is used.

$$\mathcal{L}_D^*(\hat{A}) = \sum_i \left\{\hat{F}_i \hat{A} \hat{F}_i^+ - \frac{1}{2} (\hat{F}_i \hat{F}_i^+ \hat{A} + \hat{A} \hat{F}_i \hat{F}_i^+)\right\} \quad (Q10)$$

where $\hat{F}_i$ is chosen from the Hilbert space. In our case they are the transition operators between levels of the working fluid. For our system the set $\{\hat{B}_k\}$ is closed also under $\mathcal{L}_D^*$.

$$\dot{Q} = \langle \mathcal{L}_D^*(\hat{H}) \rangle$$
Entropy is a measure of the dispersion of the measurement of an observable \( \hat{A} \). It becomes

\[
S_{\hat{A}} = \sum_i p_i \ln p_i \quad \text{(Q11)}
\]

when the probabilities in (Q11) can be obtained from the diagonal elements of the density operator \( \hat{\rho} \) in the eigenrepresentation of \( \hat{A} \).

Our coupled spin system originally will be diagonal in the polarization representation. Therefore to get the entropy in the energy representation, we have to diagonalize the Hamiltonian, and transform accordingly all the set.

The entropy of the operator \( \hat{A} \), which leads to minimum dispersion, defines an invariant of the system, termed the von Neumann entropy

\[
S_{\text{VN}} = -\mathfrak{h} \cdot \{ \ln p \} \quad \text{(Q12)}
\]

and always \( S_{\hat{A}} \geq S_{\text{VN}} \).
Thermodynamical quantum variables

Coupled spin system

\[ H = \omega(t)B_1 + JB_2 \]
\[ L = -J B_1 + \omega(t) B_2 \]
\[ C = \Omega(t) B_3 \quad \Omega = \sqrt{\omega^2 + J^2} \]

SU(2) Algebra

\[ B_1 = (\sigma_z^1 \otimes I^2 + \sigma_z^2 \otimes I^1) \]
\[ B_2 = \frac{1}{2}(\sigma_x^1 \otimes \sigma_x^2 - \sigma_y^1 \otimes \sigma_y^2) \]
\[ B_3 = \frac{1}{2}(\sigma_y^1 \otimes \sigma_x^2 + \sigma_x^1 \otimes \sigma_y^2) \]

Quantum oscillator

\[ H(t) = \frac{1}{2m} P^2 + \frac{1}{2} m \omega(t)^2 Q^2 \]
\[ L(t) = \frac{1}{2m} P^2 - \frac{1}{2} m \omega(t)^2 Q^2 \]
\[ C(t) = \frac{1}{2} \omega(t) (PQ + QP) \]

Heisenberg Weil Algebra

\[ P^2 \]
\[ Q^2 \]
\[ \frac{1}{2}(PQ + QP) \]
Thermodynamical quantum variables (state vectors)
Reconstructing the state:

**Coupled spin system**

\[
\rho = f(\langle H \rangle, \langle L \rangle, \langle C \rangle, \langle V \rangle, \langle D \rangle)
\]

\[
\rho = \frac{1}{4} I + \frac{1}{2\Omega} (\langle H \rangle H + \langle L \rangle L + \langle C \rangle C + \langle V \rangle V + \langle D \rangle D)
\]

\[\hat{\rho}_e = \frac{1}{4} \begin{pmatrix}
1 + \frac{1}{\hbar \Omega} (D - 2E) & 0 & 0 & \frac{2}{\hbar \Omega} (L + iC) \\
0 & 1 - \frac{1}{\hbar \Omega} D & 0 & 0 \\
0 & 0 & 1 - \frac{1}{\hbar \Omega} D & 0 \\
\frac{2}{\hbar \Omega} (L - iC) & 0 & 0 & 1 + \frac{1}{\hbar \Omega} (D + 2E)
\end{pmatrix}
\]

**Quantum oscillator**

\[
\rho = f(\langle H \rangle, \langle L \rangle, \langle C \rangle)
\]

\[
\rho = \frac{1}{Z} e^{\gamma a^2} e^{-\beta H} e^{\gamma a^2}
\]

If \(\langle L \rangle = 0\), and \(\langle C \rangle = 0\) then \(\rho\) is diagonal in \(H\)
Equation of motion on the isochores (oscillator)

\[
\dot{X} = \frac{i}{\hbar} [H, X] + L_D(X)
\]

\[
\frac{d}{dt}\begin{pmatrix} H \\ L \\ C \\ I \end{pmatrix} = \begin{pmatrix} -\Gamma & 0 & 0 & \Gamma \langle H \rangle_{\text{eq}} \\ 0 & -\Gamma & -2\omega & 0 \\ 0 & 2\omega & -\Gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} H \\ L \\ C \\ I \end{pmatrix}
\]

\[
H(t) = e^{-\Gamma t} (H(0) - i \langle H \rangle_{\text{eq}})
\]

\[
\begin{pmatrix} L \\ C \end{pmatrix}(t) = e^{-\Gamma t} \begin{pmatrix} \cos(2\omega t) & -\sin(2\omega t) \\ \sin(2\omega t) & \cos(2\omega t) \end{pmatrix} \begin{pmatrix} L \\ C \end{pmatrix}(0)
\]

\[
\langle L \rangle_{\text{eq}} = 0 \text{ and } \langle C \rangle_{\text{eq}} = 0
\]

The heat transport

\[
\dot{Q} = \langle L_D(H) \rangle
\]

\[
\dot{Q} = -\Gamma (\langle H \rangle - \langle H \rangle_{\text{eq}})
\]
Equation of motion on the isochores (spins)

\[ \dot{\mathbf{X}} = \frac{i}{\hbar} [\mathbf{H}, \mathbf{X}] + \mathbf{L}_D(\mathbf{X}) \]

The heat transport

\[ \dot{Q} = -\Gamma (\langle \mathbf{H} \rangle - \langle \mathbf{H} \rangle_{\text{eq}}) \]
Equation of motion on the adiabats (oscillator)

\[ \dot{X} = \frac{i}{\hbar} [H(t), X] + \frac{\partial X}{\partial t} \]

\[
\begin{pmatrix}
\frac{d}{\omega dt} \\
\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
H \\
L \\
C \\
I
\end{pmatrix}
= \begin{pmatrix}
\mu & -\mu & 0 & 0 \\
-\mu & -\mu & -2 & 0 \\
0 & 2 & \mu & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
H \\
L \\
C \\
I
\end{pmatrix}
\]

The external power

\[ P = \langle \frac{\partial H}{\partial t} \rangle \]

\[ P = \omega \mu (\langle H \rangle - \langle L \rangle) \]

\[ \begin{cases} 
P_{\text{ext}} = \omega \mu \langle H \rangle \quad \text{useful work} \\
P_{\text{fric}} = \omega \mu \langle L \rangle \quad \text{work against friction} \end{cases} \]
Equation of motion on the adiabats (spins)

\[ \dot{X} = \frac{i}{\hbar} [H(t), X] + \frac{\partial X}{\partial t} \]

\[ \frac{\partial}{\partial \Omega} \begin{pmatrix} H \\ L \\ C \\ I \end{pmatrix} = \begin{pmatrix} \mu & -\mu & 0 & 0 \\ \mu & -\mu & -1 & 0 \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} H \\ L \\ C \\ I \end{pmatrix} \]

The external power

\[ P = \langle \frac{\partial H}{\partial t} \rangle \]

\[ P = \Omega \mu \langle H \rangle - \langle L \rangle \]

\[ \{ P_{\text{ext}} = \Omega \mu \langle H \rangle \text{ useful work} \]

\[ P_{\text{fric}} = \Omega \mu \langle L \rangle \text{ work against friction} \]
Propagator on the adiabats \( U_{hc} \) (oscillator) \( \omega_h \) to \( \omega_c \)

For constant \( \mu \)

\[
\mu = \frac{\omega}{\omega^2}
\]

\[
\frac{1}{\omega(t)} = f \quad f \text{ linear in time}
\]

\[
U_{hc} = U_1 U_2
\]

\[
U_1 = \frac{\omega_c}{\omega_h} 1
\]

\[
U_2 = \frac{1}{q^2} \begin{pmatrix} \mu^2 c - 4 & \mu s q & 2\mu(c-1) \\ \mu s q & q^2 c & 2s q \\ -2\mu(c-1) & -2s q & \mu^2 - 4c \end{pmatrix}
\]

\[
q = \sqrt{\mu^2 - 4} \quad c = \cosh(q\theta) \quad s = \sinh(q\theta)
\]

when \( \mu < 2 \) then \( q \) imaginary

Quantization of the adiabat

when \( iq\theta = 2\pi l \) \( l = 0,1,2,... \)

\[
U_2 (1,1) = 1
\]

\[
E_c = E_h \frac{\omega_c}{\omega_h}
\]

Perfect adiabatic solution

Frictionless!

equivalent to \( \mu \rightarrow 0 \)
Propagator on the adiabats $U_{hc}$ (spins)  \( \omega_h \) to \( \omega_c \)

For constant \( \mu \)

\[
\mu = J \dot{\omega}/\Omega^3 \quad \omega(t) = Jf/\sqrt{1-f^2} \quad f \text{ linear in time}
\]

\[
U_{hc} = U_1 U_2
\]

\[
U_1 = \frac{\Omega_c}{\Omega_h} 1
\]

\[
U_2 = \frac{1}{q^2} \begin{pmatrix}
1 + \mu^2 c & -\mu s q & \mu(1-c) \\
\mu s q & q^2 c & -s q \\
\mu (1-c) & s q & \mu^2 + c
\end{pmatrix}
\]

\[
q = \sqrt{1 + \mu^2} \quad c = \cos(q\theta) \quad s = \sin(q\theta)
\]

Quantization of the adiabat

when \( q\theta = 2\pi l \) \( l=0,1,2,... \)

\[
U_2 (1,1) = 1
\]

\[
E_c = E_h \frac{\Omega_c}{\Omega_h}
\]

Perfect adiabatic solution

Frictionless!

equivalent to \( \mu \to 0 \)
The different scheduling constant vs linear
The quest to cool to the absolute zero temperature

The minimum temperature $T_c$

$Q_c = \hbar \omega_c \Delta N$

$0 < \Delta N < e^{-\frac{\hbar \omega_c}{kT_c}} - N_c$

If $N_c$ is not zero there is a minimum temperature $T_c$

$N_c$ can be zero only for frictionless solutions $U_2(1,1)=1$ and $\omega_h \to \infty$

If $\omega_c$ has a lower bound then the cooling rate will vanish exponentially with $T_c$

Otherwise $\omega_c \propto T_c$
Noise and control

\[ H = H_0 + \sum_{j=1}^{L} (u_j(t) + \xi_j(t)) X_j \]

where \( \xi \) represent a delta correlated noise

\[ \langle \xi_j(t) \xi_j(t') \rangle = 2 \eta_j |u_j(t)| \delta(t-t') \]

Then the Master equation becomes:

\[ \frac{d}{dt} \rho = -i[H,\rho] + \sum \eta_j |u_j(t)| [X_j,[X_j,\rho]] \]

The same Master equation is obtained for a system subject to a continuous measurement of the observables associated with \( X_j \).

Any control field has to involve noise!
Noise in the controls

Phase noise

\[ \mathcal{L}_{Na}(\hat{A}) = -\gamma_a [\hat{H}, [\hat{H}, \hat{A}]] \]

\[ H = H_{int} + H_{ext}(\omega) \]

Amplitude noise

\[ \mathcal{L}_\omega \hat{X} = -\gamma_b [\hat{B}_1, [\hat{B}_1, \hat{X}]] \]

Error in updating

Error in amplitude
The minimum temperature
Quantum refrigerator

\[ T_c \geq \frac{\hbar J}{-k_B \log(\delta/2)} \]
The quest to cool to the absolute zero temperature

Quantum friction is the inability to stay on the energy Shell

A typical control Hamiltonian

$$H = H_{\text{int}} + H_{\text{ext}}(\omega)$$

$$[H_{\text{int}}, H_{\text{ext}}] \neq 0$$ leading to: $$[H(0), H(t)] \neq 0$$

Any friction will lead to finite $T_c$

Frictionless adiabatic expansion can be found!

Noise on the controls can cancel frictionless solutions
The quantum refrigerator: The quest for absolute zero

Y. Rezer\textsuperscript{1(a)}, P. Salamon\textsuperscript{2}, K. H. Hoffmann\textsuperscript{3} and R. Kosloff\textsuperscript{1}
The sudden limit

\[ \tau_{hc} = \tau_{ch} \]
Decreasing cycle time
Quantum signature cooling power at zero cycle time
Tova's legacy

Quantum friction

Quantum thermodynamical observables

Constant adiabatic speed

Noise on the controls

Exceptional points