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Minimal temperature of quantum refrigerators

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Abstract – A first-principle reciprocating quantum refrigerator is investigated to determine the limitations of cooling to absolute zero. If the energy spectrum of the working medium possesses an uncontrollable gap, then there is a minimum achievable temperature above zero. Such a gap, combined with a negligible amount of noise, prevents adiabatic following during the expansion stage which is the necessary condition for reaching $T_c \to 0$.

Introduction. – It is well accepted that cooling to the absolute zero becomes prohibitively difficult [1,2]. More precisely, is the absolute-zero state connected to possible physical states or cooling stops before this point? This problem has been termed the untenability principle [3,4]. In the present study we want to show that there is quantum twist to this issue. In the tradition of thermodynamics we try to generalize from a specific example. Our previous analysis of quantum models have shown that the cooling rate vanishes as T_c^δ where $\delta > 1$ [5–7], *i.e.* $T_c = 0$ is reachable. The present study analyzes a reciprocating quantum refrigerator which is subject to noise on the external controls of the adiabatic expansion. We find that such noise is sufficient to make $T_c = 0$ unattainable.

Reciprocating refrigerators operate by a working medium shuttling heat from the cold to the hot reservoir. This requires external control of the temperature of the working medium. Upon contact with the cold side the working-medium temperature has to become lower than T_c —the cold-bath temperature.

A generic working medium possesses a Hamiltonian that is only partially controlled externally:

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_{int} + \hat{\mathbf{H}}_{ext}(\omega), \tag{1}$$

where $\omega = \omega(t)$ is the time-dependent external control field. Typically, $[\hat{\mathbf{H}}_{int}, \hat{\mathbf{H}}_{ext}] \neq 0$, therefore, $[\hat{\mathbf{H}}(t), \hat{\mathbf{H}}(t')] \neq 0$ as a result a state diagonal in the temporary energy eigenstates cannot follow adiabatically. This fact, which is the source of quantum friction, has a profound effect on the performance of the heat pump [8,9]. Almost perfect adiabaticity is the key to low-temperature refrigeration. Typically, the internal interaction leads to an uncontrollable finite gap $\hbar J$ in the energy level spectrum between the ground and first-excited state. We will show that this gap combined with unavoidable quantum friction leads to a finite minimal temperature.

The cycle of operation, the quantum heat pump. – The working medium in the present study is composed of an interacting spin system. Equation (1) is modeled by the $SU(2)$ algebra of operators. We can realize the model by a system of two coupled spins $\hat{\mathbf{H}}_{int} = \frac{1}{2}\hbar J(\hat{\boldsymbol{\sigma}}_x^1 \otimes \hat{\boldsymbol{\sigma}}_x^2 - \hat{\boldsymbol{\sigma}}_y^1 \otimes \hat{\boldsymbol{\sigma}}_y^2) \equiv \hbar J \hat{\mathbf{B}}_2$, where $\hat{\boldsymbol{\sigma}}$ represents the spin-Pauli operators, and J scales the strength of the interparticle interaction. For $J \to 0$, the system approaches a working medium with non-interacting atoms [5]. The external Hamiltonian represents the interaction of spins with an external magnetic field:

$$\hat{\mathbf{H}}_{ext} = \frac{1}{2}\hbar\omega(t)(\hat{\boldsymbol{\sigma}}_z^1 \otimes \hat{\mathbf{I}}^2 + \hat{\mathbf{I}}^1 \otimes \boldsymbol{\sigma}_z^2) \equiv \hbar\omega(t)\hat{\mathbf{B}}_1.$$

The $SU(2)$ is closed with $\hat{\mathbf{B}}_3 = \frac{1}{2}(\hat{\boldsymbol{\sigma}}_y^1 \otimes \hat{\boldsymbol{\sigma}}_x^2 + \hat{\boldsymbol{\sigma}}_x^1 \otimes \hat{\boldsymbol{\sigma}}_y^2)$ and $[\hat{\mathbf{B}}_1, \hat{\mathbf{B}}_2] \equiv 2i\hat{\mathbf{B}}_3$.

The total Hamiltonian then becomes

$$\hat{\mathbf{H}} = \hbar\left(\omega(t)\hat{\mathbf{B}}_1 + \mathrm{J}\hat{\mathbf{B}}_2\right). \tag{2}$$

The temporary energy levels, the eigenvalues of $\hat{\mathbf{H}}$ are $\epsilon_1 = -\hbar\Omega, \epsilon_{2/3} = 0, \epsilon_4 = \hbar\Omega$, where $\Omega = \sqrt{\omega^2 + J^2}$. For $J \neq 0$ there is a zero-field splitting. Equation (2) contains the essential features of the Hamiltonian of magnetic materials [10].

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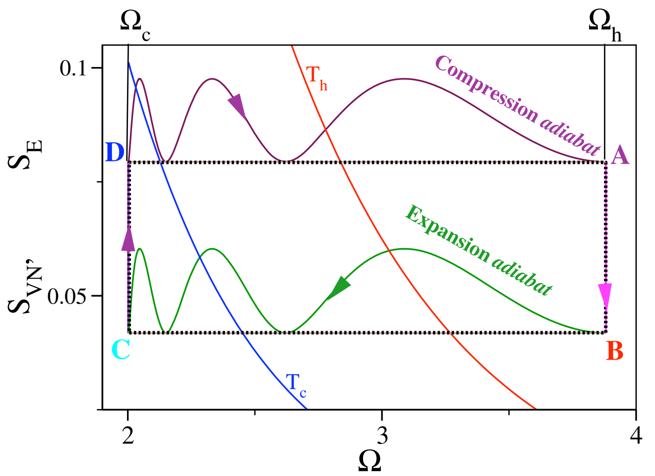

Fig. 1: (Colour on-line) Refrigerator cycle in the frequency entropy plane. The von Neumann entropy $\mathcal{S}_{vn} = -\mathrm{tr}\{\hat{\boldsymbol{\rho}} \log \hat{\boldsymbol{\rho}}\}$ (ABCD dashed rectangle) as well as the energy entropy $\mathcal{S}_E = -\sum p_i \log p_i$ are shown (p_i is the population of energy level i). The hot and cold isotherms are indicated. Both the expansion *adiabat* and the compression *adiabats* revolve exactly three periods.

The dynamics of the quantum thermodynamical observables are described by completely positive maps within the formulation of quantum open systems [11–13]. The dynamics is generated by the Liouville superoperator, $\mathcal{L}$, studied in the Heisenberg picture,

$$\frac{\mathrm{d}\hat{\mathbf{A}}}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{\mathbf{H}}, \hat{\mathbf{A}}] + \mathcal{L}_D(\hat{\mathbf{A}}) + \frac{\partial \hat{\mathbf{A}}}{\partial t}, \tag{3}$$

where $\mathcal{L}_D$ is a generator of a completely positive Liouville superoperator.

The cycle studied is composed of two segments, termed *isochores*, where the working medium is in contact with the cold/hot baths and the external control field ω is constant. In addition, there are two segments, termed *adiabats*, where the external field $\omega(t)$ varies and with it the energy level structure of the working medium. This cycle is a quantum analogue of the Otto cycle [14]. Each segment is characterized by a quantum propagator $\mathcal{U}_s$. The propagator maps the initial state of the working medium to the final state on the relevant segment. The four strokes of the cycle (see fig. 1) are

- *Isochore (isomagnetic)* $A \to B$: the field is maintained constant $\omega = \omega_h$ the working medium is in contact with the hot bath of temperature T_h. $\mathcal{L}_D$ leads to equilibrium with heat conductance Γ_h, for a period τ_h. The segment dynamics is described by the propagator $\mathcal{U}_h$.
- *Expansion adiabat (demagnetization)* $B \to C$: the field changes from ω_h to ω_c in a time period τ_{hc}. $\mathcal{L}_D = \mathcal{L}_N$ represents external noise in the controls. The propagator becomes $\mathcal{U}_{hc}$ which is the main subject of study.
- *Isochore (isomagnetic)* $C \to D$: the field is maintained constant $\omega = \omega_c$ the working medium is in contact with the cold bath of temperature T_c. $\mathcal{L}_D$ leads to equilibrium with heat conductance Γ_c, for a period τ_c. The segment dynamics is described by the propagator $\mathcal{U}_c$.
- *Compression adiabat (magnetization)* $D \to A$: the field changes from ω_c to ω_h in a time period τ_{ch}, $\mathcal{L}_D = \mathcal{L}_N$ represents external noise in the controls. The propagator becomes $\mathcal{U}_{ch}$.

The product of the four propagators, $\mathcal{U}_s$ is the cycle propagator at point A:

$$\mathcal{U}_{cyc} = \mathcal{U}_{ch}\mathcal{U}_c\mathcal{U}_{hc}\mathcal{U}_h. \tag{4}$$

Eventually, independent of initial condition, after a few cycles, the working medium will reach a limit cycle characterized as an invariant eigenvector of $\mathcal{U}_{cyc}$ with eigenvalue **1** (one) [9]. The characteristics of the refrigerator are therefore extracted from the limit cycle.

The dynamics of the expansion adiabat. – The key to low temperatures is the expansion *adiabat*. In magnetic-salt–based refrigerators this segment is termed the adiabatic demagnetization stage [10,15,16]. A necessary condition for cooling is that the energy of the working medium at contact point C is lower than the equilibrium energy at temperature T_c. What are the starting conditions at the beginning of the expansion segment point B? Considering that the efficiency is limited by the Carnot cycle $\eta_{otto} \leqslant \eta_{carnot}$ for a refrigerator this leads to the reversed condition $\frac{\Omega_c}{\Omega_h} \leqslant \frac{T_c}{T_h}$. Now $\Omega_c \geqslant J$ therefore using $\Omega_c(\min) = J$:

$$T_c \geqslant J\frac{T_h}{\Omega_h}. \tag{5}$$

This condition relates the hot-end frequency ω_h to the temperature T_c [10]. To force T_c to zero $\omega_h \to \infty$ and with it Ω_h. Under these conditions at equilibrium all population is in the ground state and $\langle\hat{\mathbf{H}}\rangle_h = -\hbar\Omega_h$. This is the optimal starting point for the expansion *adiabat*. On the cold side the necessary condition for refrigeration is that the internal energy of the working medium at the end of the expansion is smaller than the equilibrium energy with the cold bath:

$$\langle\hat{\mathbf{H}}\rangle_c \leqslant \langle\hat{\mathbf{H}}\rangle_{eq}(T_c) = -\hbar\Omega_c\left(1 - 2e^{-\frac{\hbar\Omega_c}{k_b T_c}}\right), \tag{6}$$

where $\langle\hat{\mathbf{H}}\rangle_{eq}(T_c)$ is approximated by the low-temperature limit $\hbar\Omega_c \gg k_B T_c$. Such a condition is fulfilled if the populations follows adiabatically the ground state during the expansion *adiabat*. Then $\langle\hat{\mathbf{H}}\rangle_c = -\hbar\Omega_c$ and eq. (6) is fulfilled. The expansion stage requires to reduce the external field ω from a large to a very small value maintaining adiabaticity.

The orthogonal set of time-independent operators $\hat{\mathbf{B}}_i$ is closed to the dynamics, and therefore they can supply a complete vector space to represent the propagators $\mathcal{U}_{hc}$.

A more thermodynamically oriented alternative is based on a time-dependent set. The set includes the energy $\hat{\mathbf{H}}$ and two other orthogonal operators:

$$\begin{aligned}\hat{\mathbf{H}} &= \omega(t)\hat{\mathbf{B}}_1 + \mathrm{J}\hat{\mathbf{B}}_2,\\ \hat{\mathbf{L}} &= -\mathrm{J}\hat{\mathbf{B}}_1 + \omega(t)\hat{\mathbf{B}}_2,\\ \hat{\mathbf{C}} &= \Omega(t)\hat{\mathbf{B}}_3.\end{aligned} \tag{7}$$

In general the dynamics on the expansion *adiabat* is generated by $\mathcal{L} = \mathcal{L}_H + \mathcal{L}_N$, where $\mathcal{L}_H = \frac{i}{\hbar}[\hat{\mathbf{H}}, \cdot]$ and $\hat{\mathbf{H}}(t)$ the time-dependent Hamiltonian, eq. (2). The external noise generator is $\mathcal{L}_N$ defined later. For perfect adiabatic following the propagator $\mathcal{U}$ factorizes between $\hat{\mathbf{H}}$, and the pair $\hat{\mathbf{L}}$, $\hat{\mathbf{C}}$.

The noiseless dynamics generated only by the Hamiltonian $\hat{\mathbf{H}}(t)$ is the key to adiabaticity:

$$\frac{\mathrm{d}}{\Omega \mathrm{d}t}\begin{pmatrix}\hat{\mathbf{H}}\\ \hat{\mathbf{L}}\\ \hat{\mathbf{C}}\end{pmatrix}(t) = \begin{pmatrix}\frac{\dot{\Omega}}{\Omega^2} & -\frac{J\dot{\omega}}{\Omega^3} & 0\\ \frac{J\dot{\omega}}{\Omega^3} & \frac{\dot{\Omega}}{\Omega^2} & -1\\ 0 & 1 & \frac{\dot{\Omega}}{\Omega^2}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{H}}\\ \hat{\mathbf{L}}\\ \hat{\mathbf{C}}\end{pmatrix}. \tag{8}$$

The ability of the working medium to follow the energy spectrum is defined by the adiabatic measure $\mu = \frac{J\dot{\omega}}{\Omega^3}$. If $\mu = 0$ the propagator factorizes. Constant μ minimizes the non-adiabatic deviations during the expansion [17]. In addition constant μ leads to a closed-form solution for the propagator $\mathcal{U}_{hc}$ forcing a particular scheduling of the external field $\omega(t)$ with time: $\omega(t) = Jf/\sqrt{1-f^2}$, where f is a linear function of time: $f = \frac{t}{\tau_{hc}}(\frac{\omega_c}{\Omega_c} - \frac{\omega_h}{\Omega_h}) + \frac{\omega_h}{\Omega_h}$. The adiabatic parameter μ and the time allocated to the *adiabat* τ_{hc}, obey the reciprocal relation $\mu = \frac{K_{hc}}{\tau_{hc}}$, where $K_{hc} = \frac{1}{J}(\frac{\omega_c}{\Omega_c} - \frac{\omega_h}{\Omega_h})$.

Equation (8) is integrated by defining a new time variable: $\mathrm{d}\theta = \Omega \mathrm{d}t$. The final values of Θ_{hc} becomes $\Theta_{hc} = \tau_{hc}\frac{1}{K_{hc}}\Phi_{hc}$, where $\Phi_{hc} = (\arcsin(\frac{\omega_c}{\Omega_c}) - \arcsin(\frac{\omega_h}{\Omega_h}))$ and $0 \geqslant \Phi \geqslant -\frac{\pi}{2}$.

Equation (8) is solved by noticing that the diagonal is a unit matrix multiplied by a time-dependent scalar. Therefore we seek a solution of the type $\mathcal{U}_{hc} = \mathcal{U}_1\mathcal{U}_2$, where $[\mathcal{U}_1, \mathcal{U}_2] = 0$. The integral of the diagonal part of eq. (8) becomes

$$\mathcal{U}_1 = e^{\int_0^{\tau_{hc}} \frac{\dot{\Omega}}{\Omega}\mathrm{d}t}\mathcal{I} = \frac{\Omega_c}{\Omega_h}\mathcal{I}, \tag{9}$$

which can be interpreted as the scaling of the energy levels with the variation in Ω.

To integrate $\mathcal{U}_2$, the non-diagonal parts of eq. (8) are diagonalized, leading to the eigenvalues $0, -i\sqrt{q}, i\sqrt{q}$, where $q = \sqrt{1+\mu^2}$, and the propagator

$$\mathcal{U}_2 = \begin{pmatrix}\frac{1+\mu^2 c}{q^2} & -\frac{\mu s}{q} & \frac{\mu(1-c)}{q^2}\\ \frac{\mu s}{q} & c & -\frac{s}{q}\\ \frac{\mu(1-c)}{q^2} & \frac{s}{q} & \frac{\mu^2+c}{q^2}\end{pmatrix}, \tag{10}$$

where $s = \sin(q\Theta)$ and $c = \cos(q\Theta)$.

The adiabatic limit is described by $\mu \to 0$. Then eq. (10) factorizes. These are the perfect-adiabatic-following conditions. In general eq. (10) describes a periodic motion of $\hat{\mathbf{H}}$, $\hat{\mathbf{L}}$ and $\hat{\mathbf{C}}$. Each period is defined by

$$q\Theta = 2\pi l \quad l = 0, 1, 2\ldots, \tag{11}$$

where l is the winding number. At the end of each period $\mathcal{U}_2$ restores to the identity matrix. These are the frictionless conditions of adiabatic following. For intermediate times $\langle\hat{\mathbf{H}}\rangle$ is always larger than the frictionless value. The amplitude of this periodic dynamics decreases when μ becomes smaller, Cf. $\mathcal{U}_2(1,1)$ in eq. (10).

The frictionless conditions define a quantization condition for the adiabatic parameter μ:

$$\mu = \left(\left(\frac{2\pi l}{\Phi_{hc}}\right)^2 - 1\right)^{-\frac{1}{2}}. \tag{12}$$

Examining eq. (12) we find that there is no solution for $l = 0$. The first frictionless solution $l \geqslant \sqrt{\frac{\Phi_{hc}}{2\pi}}$ leads to a minimum expansion time for frictionless solutions:

$$\tau_{hc}(\mathrm{min}) = K_{hc}\sqrt{\left(\frac{2\pi}{\Phi_{hc}}\right)^2 - 1}. \tag{13}$$

The family of all frictionless solutions leads to refrigeration cycles which obey eq. (6) for any $T_c > 0$. Such frictionless refrigerators have no minimum temperature therefore they are connected to $T_c = 0$.

The effective minimal temperature. – Any realistic refrigerator is subject to noise on the external controls. Perfect adiabaticity requires precise control of the scheduling of the external field $\omega(t)$. Any deviation from perfect adiabatic following, maintaining the ground state on the expansion *adiabat* will lead to a minimum temperature. If $\langle\hat{\mathbf{H}}\rangle_c = -\hbar\Omega_c(1-\delta)$, where δ is the deviation from perfect adiabatic following then, from eq. (6) $\delta \leqslant 2e^{-\frac{\hbar\Omega_c}{k_b T_c}}$, leading to

$$T_c \geqslant \frac{\hbar\Omega_c}{-k_b\log(\delta/2)} \geqslant \frac{\hbar J}{-k_b\log(\delta/2)}. \tag{14}$$

From eq. (10), $\delta = \mu^2(1-c)/(1+\mu^2)$ which is zero for the periodic solutions. δ also vanishes when $\mu \to 0$. We will now show that even an insignificant amount of noise will lead to $\delta > 0$ and $T_c(\mathrm{min}) > 0$.

First we consider a piecewise process controlling the scheduling of ω in time. At every time interval, ω is updated to its new value. Then random errors are expected in the duration of these time intervals described by the Liouville operator $\mathcal{L}_N$. This process is mathematically equivalent to a dephasing process on the expansion *adiabat* [18]. This stochastic dynamics can be modeled by a Gaussian semigroup with the generator [13,19]

$$\mathcal{L}_{N_p}(\hat{\mathbf{A}}) = -\frac{\gamma_p}{\hbar^2}[\hat{\mathbf{H}}, [\hat{\mathbf{H}}, \hat{\mathbf{A}}]], \tag{15}$$

which is termed phase noise. The modified equations of motion on the *adiabats* become

$$\frac{\mathrm{d}}{\Omega \mathrm{d}t}\begin{pmatrix}\hat{\mathbf{H}}\\ \hat{\mathbf{L}}\\ \hat{\mathbf{C}}\end{pmatrix}(t)=\begin{pmatrix}\frac{\dot{\Omega}}{\Omega^2} & -\frac{J\dot{\omega}}{\Omega^3} & 0\\ \frac{J\dot{\omega}}{\Omega^3} & \frac{\dot{\Omega}}{\Omega^2}-\gamma_p\Omega & -1\\ 0 & 1 & \frac{\dot{\Omega}}{\Omega^2}-\gamma_p\Omega\end{pmatrix}\begin{pmatrix}\hat{\mathbf{H}}\\ \hat{\mathbf{L}}\\ \hat{\mathbf{C}}\end{pmatrix}. \tag{16}$$

We seek a product form solution: $\mathcal{U}_{hc}=\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3$, where $\mathcal{U}_3$ the noise propagator, given by $\frac{\mathrm{d}}{\Omega\mathrm{d}t}\mathcal{U}_3(t)=\mathcal{W}(t)\mathcal{U}_3(t)$, where

$$\mathcal{W}(t)=\mathcal{U}_2(-t)\begin{pmatrix}0 & 0 & 0\\ 0 & -\gamma_p\Omega & 0\\ 0 & 0 & -\gamma_p\Omega\end{pmatrix}\mathcal{U}_2(t). \tag{17}$$

We seek an approximate solution for $\mathcal{U}_3$ in the limit when $\mu\to 0$, then $\mathcal{U}_2=\mathcal{I}$ since this is the frictionless limit. Expanding eq. (17) to first order in μ leads to

$$\mathcal{W}(t)\approx -\gamma_p\Omega(t)\begin{pmatrix}0 & \mu s & -\mu(1-c)\\ \mu s & 1 & 0\\ -\mu(1-c) & 0 & 1\end{pmatrix}. \tag{18}$$

$\mathcal{U}_3(\tau_{hc})$ is solved in two steps. First evaluating the propagator for one period of Θ, for which $\Omega(t)$ is almost constant, and then the global propagator becomes the product of the one-period propagators for l periods: $\mathcal{U}_3(\tau_{hc})\approx\mathcal{U}_3(\Theta=2\pi)^l$. The Magnus expansion to second order is employed to obtain the one-period propagator $\mathcal{U}_3(2\pi)$:

$$\mathcal{U}_3(\Theta=2\pi)\approx e^{\mathcal{M}_1+\mathcal{M}_2+\cdots}, \tag{19}$$

where $\mathcal{M}_1=\int_0^{2\pi}\mathrm{d}\Theta\mathcal{W}(\Theta)$ and $\mathcal{M}_2=\frac{1}{2}\int_0^{2\pi}\int_0^{\Theta}\mathrm{d}\Theta\mathrm{d}\Theta'$ $[\mathcal{W}(\Theta),\mathcal{W}(\Theta')]+\ldots)$. The first-order Magnus term leads to

$$\mathcal{U}_3(\Theta=2\pi)_{M_1}\approx \begin{pmatrix}1 & 0 & \mu(1-e^{-2\pi\gamma_p\Omega})\\ 0 & e^{-2\pi\gamma_p\Omega} & 0\\ \mu(1-e^{-2\pi\gamma_p\Omega}) & 0 & e^{-2\pi\gamma_p\Omega}\end{pmatrix}, \tag{20}$$

for which to first order in μ, δ, the deviation from perfect adiabatic following, is zero. The second-order Magnus approximation leads to

$$\mathcal{U}_3(\Theta=2\pi)_{M_2}\approx\begin{pmatrix}C & -S & 0\\ S & C & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{21}$$

where $S=\sin\alpha$ and $C=\cos\alpha$. $\alpha=\pi\gamma_p\Omega\mu\sqrt{9\mu^2+4}$ and as $\mu\to 0$, $\alpha=2\pi\gamma_p\Omega\mu$. The second-order propagator $\mathcal{U}_3(\tau_{hc})$, for l revolutions is also a rotation matrix identical to eq. (21), with a new angle $\alpha_l=2\pi\gamma_p\mu\int_0^{2\pi l}\Omega(\Theta)\mathrm{d}\Theta=\pi\gamma_p J\ln\Big[\frac{(\Omega_h+\omega_h)(\Omega_c-\omega_c)}{(\Omega_h-\omega_h)(\Omega_c+\omega_c)}\Big]$. The deviation of $\mathcal{U}_3$ from the identity operator defines δ. Asymptotically for $\mu\to 0$, $\delta_{min}=1-\cos(\alpha_l)\approx\pi^2\gamma_p^2J^2\ln[\omega_h/J]$. Any time variation in μ will lead to $\delta>\delta_{min}$. In addition the deviation from adiabaticity δ_{min} is quadratically increasing with the dephasing parameter γ_p. Under different conditions when μ is large this tendency can be reversed.

Another source of external noise is due to amplitude errors in the control of the frequency $\omega(t)$. These errors are modeled by a Gaussian random process described by the Lindblad term: $\mathcal{L}_N(\hat{\mathbf{A}})=-\gamma_a\omega^2[\hat{\mathbf{B}}_1,[\hat{\mathbf{B}}_1,\hat{\mathbf{A}}]]$, where γ_a characterizes the amplitude noise. The equation of motion for the noise propagator $\mathcal{U}_3$ becomes: $\frac{\mathrm{d}}{\mathrm{d}\Theta}\mathcal{U}_3(\Theta)=\mathcal{W}(\Theta)\mathcal{U}_3(\Theta)$, where

$$\mathcal{W}(\Theta)=-\gamma_a\frac{\omega^2}{\Omega}\mathcal{U}_2(-\Theta)\begin{pmatrix}\frac{J^2}{\Omega^2} & \frac{J\omega}{\Omega^2} & 0\\ \frac{J\omega}{\Omega^2} & \frac{\omega^2}{\Omega^2} & 0\\ 0 & 0 & 1\end{pmatrix}\mathcal{U}_2(\Theta). \tag{22}$$

We seek an approximate for small μ, using a similar procedure of calculating the propagator for one period $\mathcal{U}_3(\Theta=2\pi)$. The $\mathcal{U}_3(1,1)$ element decouples from the remaining part of the propagator. As a result $1-\delta=\mathcal{U}_3(1,1)\approx e^{-\gamma_a\omega_h^2\tau_{hc}\frac{J^2}{\Omega_c^2}}$. The smallest δ is achieved for a one-period cycle, eq. (13), then $\delta_{min}\approx 4\gamma_a J\frac{\omega_h^2}{\Omega_h^2}$.

Figure 2 shows δ as a function of the propagation time for different values of μ corresponding to the quantization condition, eq. (12), calculated numerically for phase (integrating eq. (18)) and amplitude noise (integrating eq. (22)). The oscillations reflect the periodicity of eq. (10), where frictionless solutions require an integer number of periods. The amplitude of the oscillation decreases with μ. The phase and amplitude noise have a different influence on the expansion dynamics.

Figure 3 shows the minimum temperature calculated numerically as a function of expansion time τ_{hc}, for the phase and amplitude noise. The exact numerical calculation are consistent with the approximation when $\mu\to 0$. The phase noise has a monotonic decrease of $T_c(\min)$ reaching saturation as $\tau_{hc}\to\infty$, where $T_c(\min)=\frac{\hbar J}{-2k_B\log(\gamma_p J\Phi_{hc}/\sqrt{2})}$. $T_c(\min)$ of the amplitude noise is a monotonically increasing function of time which means that short expansion times lead to the minimum temperature. If both amplitude and phase noise operate simultaneously the minimum temperature will be obtained at the crossing point. This optimum will move with the ratio γ_p/γ_a.

Conclusions. – The necessary condition for the working medium to cool down to absolute zero is that $\langle\hat{\mathbf{H}}\rangle_c=-\hbar\Omega_c$. In order to start in the ground state at the hot end the working medium has to equilibrate with a very high frequency Ω_h. Perfect adiabatic following will maintain the system in its ground state which defines the frictionless solution. We found a family of additional frictionless solution obeying a quantization rule for μ, the adiabatic parameter.

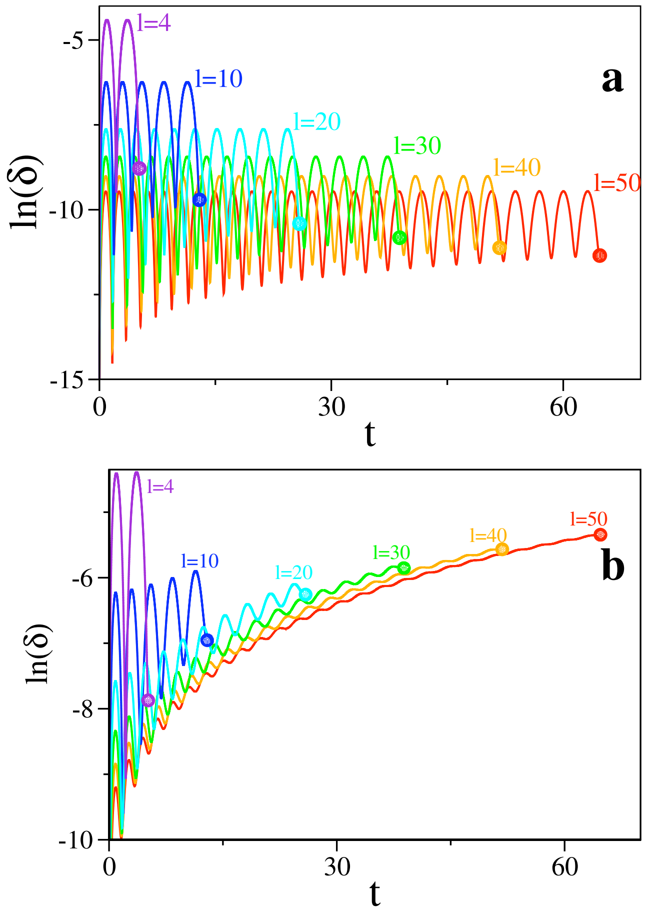

Fig. 2: (Colour on-line) The deviation δ from perfect adiabatic following as a function of the expansion time for different μ corresponding to complete revolutions l for, a) phase noise and b) amplitude noise. The end points of each expansion are marked by a filled circle corresponding to the value of δ at the end of the expansion *adiabat* $t = \tau_{hc}$. For the amplitude noise δ increases with τ_{hc}. For the phase noise the final δ decreases with τ_{hc}, *i.e.* when $l \to \infty$ and $\mu \to 0$, δ reaches its asymptotic value $\delta_{min} > 0$.

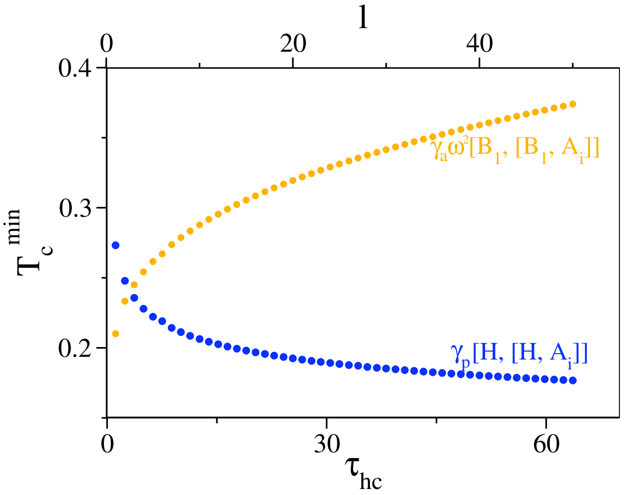

Fig. 3: (Colour on-line) The minimum temperature as a function of the expansion *adiabat* time allocation τ_{hc} (bottom scale) and l the winding number (upper scale), for the two noise models. $\Omega_h = 3.8$, $\Omega_c = 2.01$, $J = 2$ $\gamma_p = -0.001$ and $\gamma_a = -0.0001$.

The main result of this study is that any noise in the controls of $\omega(t)$ will eliminate the frictionless solutions leading to a minimum temperature $T_c(\min)$. This finding is consistent with experiments on the demagnetization cooling of a gas [20] which obtained a minimum temperature an order of magnitude larger than the theoretical prediction [21], which attributes the discrepancy to the noise in the controls. The logarithmic dependence on the noise parameters means that $T_c(\min)$ is of the oder of J. The devastating effect of the amplitude noise is caused by the non-commutativity of $\hat{\mathbf{H}}_{ext}(\omega)$ with the instantaneous Hamiltonian $\hat{\mathbf{H}}(t)$. In this case the friction increases with the expansion time, *i.e.* the noise accumulates. The surprise is the negative effect of phase noise. Its generator, eq. (15), can be interpreted as the result of the weak measurement of $\hat{\mathbf{H}}$, forcing collapse to a state diagonal in energy [22]. It would seem that this should improve adiabaticity and reduce friction. For an engine model we found such an effect of enhanced performance [18]. At the minimum temperature the refrigerator dephasing is harmful. Dephasing destroys the periodic $\delta = 0$ solutions. On the contrary, the noise reduces δ in the non-periodic expansions so it acts as a lubricant, nevertheless these expansions do not lead to minimum temperature. The optimal cooling conditions with minimum δ are obtained for small and constant μ. Under these conditions δ is dominated by the dephasing noise and δ_{min} reaches a non-zero minimum asymptotic value when $\tau_{hc} \to \infty$ and with it $\mu \to 0$. This noise causes $T_c(\min) > 0$, *i.e.* the absolute zero is not connected. In the thermodynamic tradition is it possible to generalize from this example? Preliminary analysis of other models which without noise show a frictionless expansion [7,23] support this ansatz.

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REFERENCES

[1] Nernst W., *Nachr. Kgl. Ges. Wiss. Goettingen*, **1** (1906) 40.

[2] Nernst W., *Ber. K. Preuss. Akad. Wiss.*, **52** (1906) 933.

[3] Landsberg P. T., *Rev. Mod. Phys.*, **28** (1956) 363.

[4] Belgiorno F., *J. Phys A: Math.Gen.*, **36** (2003) 8165.

[5] Feldmann Tova and Kosloff Ronnie, *Phys. Rev. E*, **61** (2000) 4774.

[6] Kosloff Ronnie, Geva Eitan and Gordon Jeffrey M., *Appl. Phys.*, **87** (2000) 8093.

[7] Rezek Yair, Salamon Peter, Hoffmann Karl Heinz and Kosloff Ronnie, *EPL*, **85** (2009) 30008.
[8] Feldmann Tova and Kosloff Ronnie, *Phys. Rev. E*, **68** (2003) 016101.
[9] Feldmann Tova and Kosloff Ronnie, *Phys. Rev. E*, **70** (2004) 046110.
[10] Oja A. S. and Lounasmaa O. V., *Rev. Mod. Phys.*, **69** (1997) 1.
[11] Lindblad G., *Commun. Math. Phys.*, **48** (1976) 119.
[12] Alicki R. and Lendi K., *Quantum Dynamical Semigroups and Applications* (Springer-Verlag, Berlin) 1987.
[13] Breuer H.-P. and Petruccione F., *Open Quantum Systems* (Oxford University Press) 2002.
[14] Kosloff Ronnie and Feldmann Tova, *Phys. Rev. E*, **65** (2002) 055102 1.
[15] Kurti N., *Physica*, **109** (1982) 1737.
[16] Hakonen P., Lounasmaa O. V. and Oja A., *J. Magn. & Magn. Mater.*, **100** (1991) 394.
[17] Feldmann Tova and Kosloff Ronnie, arXiv:0906.0986 (2009).
[18] Feldmann Tova and Kosloff Ronnie, *Phys. Rev. E*, **73** (2006) 025107(R).
[19] Gorini V. and Kossakowski A., *J. Math. Phys.*, **17** (1976) 1298.
[20] Fattori M., Koch T., Goetz S., Griesmair A., Hensler S., Stuhler J. and Pfau T., *Nat. Phys.*, **2** (2006) 765.
[21] Hensler S., Greiner A., Stuhler J. and Pfau T., *Europhys. Lett.*, **71** (2005) 918.
[22] Diosi L., *Weak measurements in quantum mechanics*, in *Encyclopedia of Mathematical Physics*, edited by Fransoise J.-P., Naber G. L. and Tsou S. T., Vol. **4** (Elsevier, Oxford) 2006, p. 276.
[23] Chen Xi, Ruschhaupt A., Schmidt S., del Campo A., Guery-Odelin D. and Muga J. G., arXiv:0910.0709 (2009).