Effects of an exceptional point on the dynamics of a single particle in a time-dependent harmonic trap

Raam Uzdin,[1,*] Emanuele G. Dalla Torre,[1,2] Ronnie Kosloff,[1,†] and Nimrod Moiseyev[1,‡]

[1]*Institute for Theoretical Atomic, Molecular and Optical Physics, Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 01238, USA*

[2]*Department of Physics, Harvard University, Cambridge, Massachusetts 01238, USA*

(Received 12 December 2012; revised manuscript received 22 March 2013; published 12 August 2013)

The time evolution of a single particle in a harmonic trap with time-dependent frequency $\omega(t)$ has been well studied. Nevertheless, here we show that when the harmonic trap is opened (or closed) as a function of time while keeping the adiabatic parameter $\mu = [d\omega(t)/dt]/\omega^2(t)$ fixed, a sharp transition from an oscillatory to a monotonic exponential dynamics occurs at $\mu = 2$. At this transition point, the time evolution has an exceptional point (EP) *at all instants*. This situation, where an EP of a time-dependent Hermitian Hamiltonian is obtained at any given time, is very different from other known cases. In the present case, we show that the order of the EP depends on the set of observables used to describe the dynamics. Our finding is relevant to the dynamics of a single ion in a magnetic, optical, or rf trap, and of diluted gases of ultracold atoms in optical traps.

DOI: 10.1103/PhysRevA.88.022505 PACS number(s): 31.70.Hq, 42.65.Sf

Introduction. Exceptional points (EPs) are degeneracies of non-Hermitian Hamiltonians [1,2] associated with the coalescence of two or more eigenstates. The number of studies of EPs has substantially grown since the pioneering works of Carl Bender and his co-workers on $\mathcal{PT}$-symmetric Hamiltonians [3]. These Hamiltonians have a real spectrum, which becomes complex at the EP. However, $\mathcal{PT}$ symmetry is not required to obtain an EP point, as in the case of a coalescence between two resonant states, leading to self-orthogonal states [4–6].

The physical effects of EPs have already been demonstrated in different types of experiments; see, for example, the effect of EPs on cold-atom experiments [7], on the cross sections of electron scattering from hydrogen molecules [8], and on the linewidth of unstable lasers [9]. More direct realizations of EPs in microwave experiments are given in Refs. [10–12] and in optical experiments in Ref. [13]. For theoretical studies that are relevant to these experiments, see, for example, Refs. [8,14–19]. In addition, theoretical studies predict significant effects of second-order EPs on the photoionization of atoms [20–22] and the photodissociation of molecules [23–25].

The above-mentioned studies on the effects of EPs are related to non-Hermitian time-independent Hamiltonians. Note that non-Hermitian Hamiltonians can be obtained from Hermitian Hamiltonians by imposing outgoing boundary conditions on the eigenfunctions or including complex absorbing potentials [5]. This approach allows the description of resonance phenomena in systems with finite-lifetime metastable states. Other studies considered time-periodic Hamiltonians where the EPs are associated with the quasienergies of the Floquet operator, which can be represented by a time-independent non-Hermitian matrix (see, for example, one of the first studies of EPs in atomic physics in Ref. [20]). EPs were also studied in nonperiodic systems in the context of Landau-Zener-Majorana transitions, where the EP was obtained only after analytic continuation of the actual Hamiltonian [26]. Finally, time-dependent EPs have been used to control the quantum evolution of non-Hermitian systems [27].

In this paper, we show the appearance of an EP in a system governed by a Hermitian time-dependent Hamiltonian. In contrast to the previous proposals, our model does not involve absorbing boundary conditions (i.e., resonances) and is not periodic in time. The EP is revealed by an appropriate rescaling of the time coordinate, which allows us to map the original time-dependent problem to an effective time-independent nonunitary evolution. As we will see, the order of the predicted EP depends on the physical observables used to probe the system.

The harmonic-oscillator system with changing frequency in the Heisenberg picture. The model under present study is the one-dimensional (1D) harmonic-oscillator with changing frequency, defined as

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2(t)\hat{x}^2, \tag{1}$$

where m is the mass of the particle, and $\hat{p}$ and $\hat{x}$ are, respectively, the momentum and position operators. This model has been used in the past as an example of and benchmark for many basic concepts in physics. These include parametric resonances (see, for example, [28]), when the mass is periodic in time, dynamical invariants [29–31], and coherent states [32–34], for a generic time dependence of the mass. We will show that the dynamics generated by this Hamiltonian can display an exceptional point.

We study the model (1) in the framework of Refs. [35,36], where it is shown that due to the closed commutation relations between the operators $\hat{p}^2, \hat{x}^2, \hat{p}\hat{x} + \hat{x}\hat{p}$, the model forms a su(1,1) algebra. As a basis set for this

*Present address: Department of Physics, The Hebrew University, Jerusalem, Israel.

†Permanent address: Fritz Haber Institute of Molecular Dynamics, The Hebrew University, Jerusalem, Israel.

‡Permanent address: Schulich Faculty of Chemistry and Faculty of Physics, Technion - Israel Institute of Technology, Haifa 32000, Israel.

algebra, we choose

$$\hat{\boldsymbol{O}} \equiv (\hat{O}_1,\hat{O}_2,\hat{O}_3) \equiv (\hat{H},\hat{L},\hat{D}), \tag{2}$$

where the Hamiltonian $\hat{H}$ is defined in (1), $\hat{L} = \hat{H} - m\omega^2(t)\hat{x}^2$ is the Lagrangian, and $\hat{D} = \omega(t)(\hat{x}\hat{p} + \hat{p}\hat{x})/2$. Any commutator between operators in the algebra can be expressed as a linear combination of these operators:

$$[\hat{O}_i,\hat{O}_j] = \sum_{k=1}^{3} C_{ij}^k \hat{O}_k, \tag{3}$$

where C_{jk}^l is the structure factor of the su(1,1) algebra [37]. The Heisenberg picture for the dynamics that is associated with the operators $\hat{O}_j$ is described as

$$\frac{d\hat{O}_j}{dt} = i[\hat{H},\hat{O}_j] + \frac{\partial \hat{O}_j}{\partial t}, \tag{4}$$

where $j = 1,2,3$ and we work in units in which $\hbar = 1$. These equations are explicitly given by

$$\begin{aligned} \frac{d}{dt}\hat{H} &= \frac{\partial \hat{H}}{\partial t} = m\Big(\frac{d\omega}{dt}\Big)\omega\hat{x}^2 = \omega\mu(\hat{H}-\hat{L}),\\ \frac{d}{dt}\hat{L} &= i[\hat{H},\hat{L}] + \frac{\partial \hat{L}}{\partial t} = -2\omega\hat{D} - \omega\mu(\hat{H}-\hat{L}),\\ \frac{d}{dt}\hat{D} &= i[\hat{H},\hat{D}] + \frac{\partial \hat{D}}{\partial t} = 2\omega\hat{L} + \omega\mu\hat{D}. \end{aligned} \tag{5}$$

Here we defined the dimensionless "adiabatic parameter"

$$\mu = \left[\frac{1}{\omega^2(t)}\right]\frac{d\omega}{dt}. \tag{6}$$

The equations of motion (5) conserve the "Casimir" operator [38] $\hat{C}(t) = [\hat{H}^2(t) - \hat{L}^2(t) - \hat{D}^2(t)]/\omega^2(t)$ by satisfying $d\hat{C}/dt = 0$.

In what follows, we will focus on the specific case of $\mu =$ const, corresponding to the frequency profile

$$\omega(t) = \frac{\omega(0)}{1-\mu\omega(0)t}. \tag{7}$$

In experiments, the harmonic trap is varied between two extreme values, $\omega_{\rm open}$ and $\omega_{\rm closed}$. The compression factor is given by $\omega_{\rm closed}/\omega_{\rm open} > 1$. For positive values of the adiabatic parameter, $\omega(0) = \omega_{\rm open}$ and $\omega(t_f) = \omega_{\rm closed}$. For negative values, $\omega(0) = \omega_{\rm closed}$ and $\omega(t_f) = \omega_{\rm open}$. In both cases, $t_f = |\mu|(\omega_{\rm open}^{-1} - \omega_{\rm closed}^{-1})$.

The parameter μ sets the degree of adiabaticity of the process. For $\mu \to 0$, the dynamics is perfectly adiabatic and the system follows the eigenstates of the instantaneous Hamiltonian. In contrast, for $\mu \to \pm\infty$, the change of the Hamiltonian is so fast that the system does not have time to change at all. As we will show, these two limits are separated by an exceptional point. A similar effect is known to occur in the vicinity of quantum critical points (see, for example, Ref. [39]) and is here shown in time-dependent nonperiodic harmonic traps.

To reveal the appearance of an EP, we introduce the dimensionless time variable $\tau = (1/\mu)\ln[\omega(t)/\omega(0)]$, satisfying $d\tau = \omega(t)dt$, and rewrite (5) as

$$i\frac{d\hat{\boldsymbol{O}}(\tau)}{d\tau} = (i\mu\mathbf{I} + \mathcal{H}_{\rm Heis})\,\hat{\boldsymbol{O}}(\tau), \tag{8}$$

where $\mathbf{I}$ is the 3×3 unit matrix and

$$\mathcal{H}_{\rm Heis} \equiv i\begin{pmatrix} 0 & -\mu & 0\\ -\mu & 0 & -2\\ 0 & 2 & 0\end{pmatrix}. \tag{9}$$

We can further simplify Eq. (8) by defining the rescaled quantities $\hat{\boldsymbol{o}}$ according to

$$\hat{\boldsymbol{o}}(\tau(t)) = \frac{1}{\omega(t)}\hat{\boldsymbol{O}}(\tau(t)) \Rightarrow \frac{d\hat{\boldsymbol{o}}(\tau)}{d\tau} = \frac{d\hat{\boldsymbol{O}}(\tau)}{d\tau} - \mu\mathbf{I}. \tag{10}$$

The resulting equation of motion $i d\hat{\boldsymbol{o}}/d\tau = \mathcal{H}_{\rm Heis}\hat{\boldsymbol{o}}$ is equivalent to a time-dependent Schrödinger equation with a non-Hermitian time-independent Hamiltonian. The matrix $\mathcal{H}_{\rm Heis}$ is $\mathcal{PT}$ symmetric [40] and its three eigenvalues

$$E_0 = 0; \quad E_\pm = \pm\sqrt{4-\mu^2} \tag{11}$$

are real for $|\mu| \leqslant 2$. The corresponding eigenvectors are given by $v_0 = (1,0,-\mu/2)$, $v_\pm = (\mu, \pm i\sqrt{4-\mu^2}, -2)/\mu$.

In contrast to the Schrödinger equation, the population of the eigenvectors in a physical state is not arbitrary, but must satisfy several constraints. For example, for $|\mu| < 2$, the eigenvectors v_+ and v_- are complex and any physical state must populate them with an equal weight in order to keep the expectation values $\langle\hat{H}\rangle$, $\langle\hat{D}\rangle$, and $\langle\hat{L}\rangle$ real. In addition, v_+ and v_- have a zero Casimir constant $\langle\hat{C}\rangle = 0$. Due to the uncertainty relation $\langle\hat{C}\rangle \geqslant \hbar^2/4$, any physical state must necessarily populate the eigenstate v_0 with nonzero weight as well. Thus, for a generic initial state, we expect more than one eigenvector to be occupied, leading to an oscillatory behavior that we describe below.

The matrix $\mathcal{H}_{\rm Heis}$ has a third-order EP at $|\mu| = 2$. At this point, all three eigenvalues and the corresponding eigenvectors (in the HLD space) coalesce. As a consequence, $[\mathcal{H}_{\rm Heis}(\mu = \pm 2)]^3 = 0$, while $[\mathcal{H}_{\rm Heis}(\mu = \pm 2)]^2 \neq 0$, demonstrating that the present EP is of third order. Third-order EPs have been discussed in the literature for time-independent $\mathcal{PT}$-symmetric Hamiltonians (see, for example, Refs. [41–44]). The main effect of EPs (of any order) on the dynamics of $\mathcal{PT}$-symmetric systems is the sudden transition from a real spectrum to a complex energy spectrum associated with gain and loss processes [15].

Physical consequences. We now discuss the consequences of the EP on physical observables. Figure 1(a) shows the time dependence of the rescaled quantity $\langle\hat{o}_1 - \hat{o}_2\rangle = \langle\hat{H} - \hat{L}\rangle/\omega(t) = m\omega(t)\langle x^2\rangle$. For $\mu < 2$, this quantity shows periodic oscillations, which become exponentially growing at $\mu > 2$, highlighting the existence of an exceptional point. Similar results can be obtained for any other linear combination of the rescaled operators $\hat{\boldsymbol{o}}$, defined in Eq. (10). In Fig. 1(a), we used as time coordinate the time-dependent compression factor $\omega(t)/\omega(0)$, whose logarithm corresponds to the new time variable τ (multiplied by μ). In this scale, the rescaled observables show periodic oscillations, with period

$$T_\tau = \frac{2\pi}{\frac{1}{2}(E_+ - E_-)} = \frac{2\pi}{\sqrt{4-\mu^2}}. \tag{12}$$

This time scale diverges at the EP, as can be noticed, for example, in Fig. 1(a).

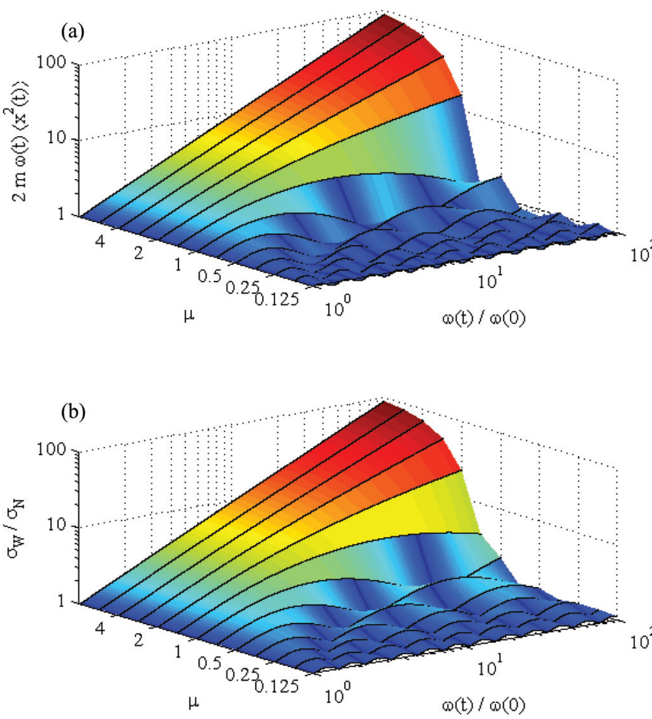

FIG. 1. (Color online) (a) Variance of the position operator $\langle x^2\rangle$, normalized by the width of the instantaneous potential $\sqrt{1/2m\omega(t)}$, as a function of the time-dependent compression factor $\omega(t)/\omega(0)$ for different values of the adiabatic parameter $\mu \geqslant 0$. The initial state is the ground state of the Hamiltonian (1) at $t = 0$, where $\langle x^2\rangle = 1/[2m\omega(0)]$. For $\mu < 0$, the same plot is obtained where now the compression factor is taken as $\omega(0)/\omega(t)$. At the exceptional point (EP) $\mu = 2$, the dynamics changes from oscillatory to monotonous. (b) Same plot for the ratio between the wide and narrow axes $\rho = \sigma_W/\sigma_N$. The curve is independent on the initial state. (The time evolution is represented by the black solid curves and the colors are a guide for the eye.)

Dealing with a time-dependent problem, the dynamics depends on the specific choice of the initial state as well. In Fig. 1(a), we chose as initial state the ground state of the Hamiltonian at $t = 0$, where $m\omega(0)\langle\hat{x}^2\rangle = 1/2$. For a generic initial state, the rescaled observables $\langle\boldsymbol{o}\rangle$ display additional trivial oscillations that persist even for $\omega = \text{const}$ and are not related to the EP.

To isolate the effects of the EP that do not depend on the choice of the initial state, we introduce the rescaled covariance matrix

$$\begin{pmatrix} o_1 - o_2 & o_3 \\ o_3 & o_1 + o_2 \end{pmatrix} = m \begin{pmatrix} \omega(t)\langle x^2\rangle & \langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle \\ \langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle & \langle p^2(t)\rangle/\omega(t) \end{pmatrix}. \tag{13}$$

The small and large eigenvalues of this matrix, denoted by σ_N^2 and σ_W^2, respectively correspond to the variances of the narrow and wide axes of the Wigner distribution. For a harmonic oscillator, one can show that

$$\sigma_{W,N}^2 = \frac{\langle\hat{H} \pm \sqrt{\hat{L}^2 + \hat{D}^2}\rangle}{\omega}. \tag{14}$$

Note that by virtue of the Casimir constant, any quantum state satisfies $\sigma_N\sigma_W \geqslant 1/2$. For $\omega = \text{const}$ ($\mu = 0$), the evolution simply mixes $\hat{L}$ and $\hat{D}$, leaving $\langle\hat{L}^2 + \hat{D}^2\rangle$ constant. Hence, even though for $\omega = \text{const}$ the distribution rotates in phase space and changes the variance of position and momentum, the width of the narrow and wide axes of the distribution remains fixed.

For $0 < |\mu| < 2$, the expectation values of σ_N^2 and σ_W^2 oscillate in time. A convenient measure to capture this oscillatory dynamics is given by the ratio

$$\rho = \frac{\sigma_W}{\sigma_N}, \tag{15}$$

where, by definition, $\rho \geqslant 1$. For any initial state satisfying $\mu\langle\hat{D}\rangle + \langle\hat{L}\rangle^2/(4\langle\hat{H}\rangle) > 0$ [45], it is possible to show that the visibility of the fringes pattern is given by the simple expression

$$V = \frac{\rho_{\max} - \rho_{\min}}{\rho_{\max} + \rho_{\min}} = \frac{|\mu|}{2}. \tag{16}$$

At $|\mu| = 2$, the visibility becomes one, and for $|\mu| \geqslant 2$, the oscillations disappear.

Semiclassical analysis. We now present a different approach which clarifies the relation between the present problem and energy-dissipative systems. Our approach is based on the formal equivalence between quantum and classical evolution of *quadratic* Hamiltonians. To reproduce the quantum-mechanical results, one needs to complement the classical equations of motion by stochastic initial distributions, given by the Wigner transform of the initial state. Note that generic *quantum* initial conditions lead to Wigner distribution functions with negative values corresponding to "nonclassical" states.

In our case, the relevant equation of motion is Newton's law,

$$\left[\frac{d^2}{dt^2} + \omega^2(t)\right]x(t) = 0. \tag{17}$$

For a study of PT symmetry in parametric oscillators see Ref. [46]. By applying the transformation $d\tau = \omega(t)dt$, or $\frac{d}{dt} = \omega(t)\frac{d}{d\tau}$, we obtain

$$\frac{d^2}{dt^2}x = \frac{d}{dt}\left[\omega(t)\frac{d}{d\tau}x\right] = \omega'(t)\frac{d}{d\tau}x + \omega^2(t)\frac{d}{d\tau}x. \tag{18}$$

In the specific case $\mu = \text{const}$, the equation of motion becomes

$$\left[\frac{d^2}{d\tau^2} + \mu\frac{d}{d\tau} + 1\right]x(\tau) = 0. \tag{19}$$

Here we obtain the well-known equation of motion of a damped harmonic oscillator. Note that the original model, given by Eq. (17), does not involve dissipation and *a priori* one would not expect the appearance of an EP. The rescaling of the time coordinate allows us to identify an EP at $|\mu| = 2$, corresponding to the transition between an underdamped and overdamped oscillator. Equation (19) is a second-order differential equation and, as such, leads to a *second-order* EP.

To understand the relation between this semiclassical finding and the previous quantum analysis, it is useful to define the velocity $v = dx/dt$ as an independent variable and compute the classical equations of motion of $x^2(t)$, $v^2(t)$, and $x(t)v(t)$. The resulting equations of motion coincide with

Eq. (9) and have three eigenvectors that coalesce at $|\mu| = 2$, indicating a third-order EP. This observation leads to the interesting conclusion that in our system, the order of the EP depends on the specific set of observables. Average quantities ($\langle x\rangle$ and $\langle p\rangle$) and variances ($\langle x^2\rangle$, $\langle p^2\rangle$, and $\langle xp + px\rangle$) are characterized, respectively, by second- and third-order EPs. At present, it is not clear whether this feature is unique to the harmonic case or generic to EPs of time-dependent Hamiltonians.

Experimental realization. Although the Hamiltonian (1) can be realized in any controllable harmonic trap (optics, plasma, etc.), we will consider here the case of either a single ion [47] or a dilute atomic cloud [48–52] in time-dependent confining traps. The realization with atomic clouds allows the measurement of expectation values in a single-shot experiment. Complications of the dynamics due to the atom-atom interactions can be avoided (minimized) by setting the atomic scattering length to zero in the vicinity of a Feshbach resonance [53].

The effects of the EP can be directly detected by preparing the atom in the ground state of the trap and measuring the spatial fluctuations $\langle x^2(t)\rangle$ (for example, using the techniques described in Refs. [50,54–58]), while the trap frequency is varied according to Eq. (7). At the EP, the graph of the rescaled quantity $\omega(t)\langle x^2(t)\rangle$ as a function of the compression factor $\omega(t)/\omega(0)$ displays a sharp transition from oscillatory to exponentially growing, as shown in Fig. 1(a). Alternatively, for a generic initial state, one should instead plot the ratio between the narrow and wide axis of the Wigner distribution, computed in Fig. 1(b). This quantity can be measured by fixing the frequency of the trap $\omega(t > t') = \omega(t')$ and measuring the variance of the position as a function of time. The minimum and the maximum of $\langle x^2(t)\rangle$ are, respectively, $\sigma_N^2/2m\omega(t')$ and $\sigma_W^2/2m\omega(t')$. This method is perhaps more time consuming, but guarantees the independence of the result on the initial preparation.

The two methods described above require one to observe the dynamics of the system for long times. The relevant time scale is determined by T_τ, defined in Eq. (12), and diverges at the EP. These long-time measurements are strongly affected by unavoidable interatomic interaction and time-dependent noise, preventing the observation of a sharp transition at $|\mu| = 2$.

To avoid this problem, we recall that the EP occurs at any instant in time and affects the short-time dynamics as well. In particular, the EP can be probed by plotting the derivatives of the rescaled quantity [such as $\omega(t)\langle x^2(t)\rangle$ and $\langle p^2(t)\rangle/\omega(t)$] as a function of the rescaled time τ. Because at the EP $(\mathcal{H}_{\text{Heis}})^3 = 0$, the rescaled observables $\langle \hat{\boldsymbol{o}}(\tau)\rangle = \exp(-i\mathcal{H}_{\text{Heis}}\tau)\,\langle \hat{\boldsymbol{o}}(\tau = 0)\rangle$ are second-order polynomial of τ and their third-order derivatives are identically equal to zero. Thus, plotting $d^3\langle \boldsymbol{o}(\tau)\rangle/d\tau^3$ allows one to directly probe both the position and the order of the EP from the short-time dynamics [59], as shown in Fig. 2.

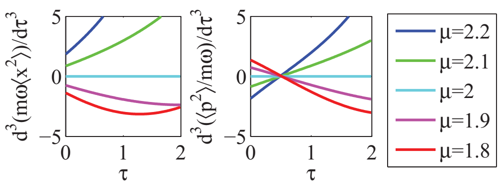

FIG. 2. (Color online) Proposed experimental method to detect the EP from the short-time dynamics. Third-order derivatives of the renormalized observables defined in Eq. (10), as a function of the renormalized time $\tau = (1/\mu)ln[\omega(t)/\omega(0)]$. At the EP $|\mu| = 2$, the third-order derivatives of the rescaled quantities $\omega\langle x^2\rangle$ and $\langle p^2\rangle/\omega$ are identically equal to zero.

Concluding remarks. The dramatic effect of EPs of non-Hermitian time-independent Hamiltonian systems on the dynamics is the focus of recent theoretical and experimental studies in various fields of physics (for example, in optical or microwave experiments where the material has a complex index of refraction). Here we show that the dynamics of a system described by a *time-dependent Hermitian Hamiltonian* can be strongly affected by the EP of an effective time-independent Hamiltonian. The fact that the dynamics of the Hermitian time-dependent harmonic oscillator can be explained by the existence of an EP *at all instants* shows the richness of the dynamics of one of the most basic model Hamiltonians, which constitutes a cornerstone in a large variety of fields in physics. Our finding is both interesting for fundamental theoretical reasons and relevant to experiments with single ions and diluted BECs in time-dependent traps.

Acknowledgments. We would like to acknowledge ITAMP for their support. This work was initialized and has been carried out as a result of the friendly and scientific atmosphere at ITAMP, which encourages informal discussions on a daily basis among the permanent and visiting members of ITAMP. We would also like to sincerely thank the anonymous referee for his/her useful comments.

[1] T. Kato, *Perturbation Theory of Linear Operators* (Springer, Berlin, 1966).
[2] W. D. Heiss and H. L. Harney, Eur. Phys. J. D **17**, 149 (2001).
[3] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007), and references therein.
[4] N. Moiseyev and S. Friedland, Phys. Rev. A **22**, 618 (1980).
[5] N. Moiseyev, *Non-Hermitian Quantum Mechanics* (Cambridge University Press, Cambridge, 2011).
[6] E. Narevicius, P. Serra, and N. Moiseyev, Europhys. Lett. **62**, 789 (2003).
[7] M. V. Berry and D. H. O'Dell, J. Phys. A **32**, 2093 (1998).
[8] E. Narevicius and N. Moiseyev, Phys. Rev. Lett. **84**, 1681 (2000).
[9] M. V. Berry, J. Mod. Opt. **50**, 63 (2003).
[10] C. Dembowski, H. D. Graf, H. L. Harney, A. Heine, W. D. Heiss, H. Rehfeld, and A. Richter, Phys. Rev. Lett. **86**, 787 (2001).
[11] T. Stehmann, W. D. Heiss, and F. G. Scholtz, J. Phys. A **37**, 7813 (2004).
[12] B. Dietz, H. L. Harney, O. N. Kirillov, M. Miski-Oglu, A. Richter, and F. Schafer, Phys. Rev. Lett. **106**, 150403 (2011), and references therein.

[13] C. E. Rüter *et al.*, Nat. Phys. **6**, 192 (2010).
[14] S. Rotter, F. Libisch, J. Burgdorfer, U. Kuhl, and H. J. Stockmann, Phys. Rev. E **69**, 046208 (2004).
[15] S. Klaiman, U. Günther, and N. Moiseyev, Phys. Rev. Lett. **101**, 080402 (2008).
[16] K. G. Makris *et al.*, Int. J. Theor. Phys. **50**, 1019 (2011); K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Phys. Rev. A **81**, 063807 (2010).
[17] R. Uzdin, A. A. Mailybaev, and N. Moiseyev, J. Phys. A **44**, 435302 (2011).
[18] M. V. Berry and R. Uzdin, J. Phys. A **44**, 435303 (2011).
[19] R. Uzdin and N. Moiseyev, Phys. Rev. A **85**, 031804(R) (2012).
[20] O. Latinne, N. J. Kylstra, M. Dorr, J. Purvis, M. Terao-Dunseath, C. J. Joachain, P. G. Burke, and C. J. Noble, Phys. Rev. Lett. **74**, 46 (1995).
[21] H. Cartarius, J. Main, and G. Wunner, Phys. Rev. Lett. **99**, 173003 (2007).
[22] H. Cartarius and N. Moiseyev, Phys. Rev. A **84**, 013419 (2011).
[23] R. Lefebvre, O. Atabek, M. Sindelka, and N. Moiseyev, Phys. Rev. Lett. **103**, 123003 (2009).
[24] O. Atabek, R. Lefebvre, M. Lepers, A. Jaouadi, O. Dulieu, and V. Kokoouline, Phys. Rev. Lett. **106**, 173002 (2011).
[25] I. Gilary and N. Moiseyev, J. Phys. B **45**, 051002 (2012).
[26] R. Uzdin and N. Moiseyev, J. Phys. A **45**, 444033 (2012).
[27] R. Uzdin *et al.*, J. Phys. A **45**, 415304 (2012).
[28] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, New York, 1965).
[29] H. R. Lewis, Jr., Phys. Rev. Lett. **18**, 510 (1967).
[30] H. R. Lewis, Jr., J. Math. Phys. **9**, 1976 (1968).
[31] H. J. Korsch, Phys. Lett. A **74**, 294 (1979).
[32] J. G. Hartley and J. R. Ray, Phys. Rev. D **25**, 382 (1982).
[33] A. K. Rajagopal and John T. Marshall, Phys. Rev. A **26**, 2977 (1982).
[34] X. C. Gao, J. B. Xu, and T. Z. Qian, Ann. Phys. **204**, 235 (1990).
[35] Y. Rezek and R. Kosloff, New J. Phys. **8**, 83 (2006).
[36] Y. Rezek, P. Salomon, K. H. Hoffman, and R. Kosloff, Euro. Phys. Lett. **85**, 30008 (2009).
[37] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (*Dover Books on Mathematics*) (Dover, New York, 2006).
[38] The Casimir constant is simply the "interval" in special relativity. See U. Gunther and B. F. Samsonov, Phys. Rev. A **78**, 042115 (2008).
[39] A. Polkovnikov *et al.*, Rev. Mod. Phys. **83**, 863 (2011).
[40] The $\mathcal{P}$ operator for this Hamiltonian matrix is simply $\mathcal{P} =$ diagonalmatrix([1, − 1,1]), where $\mathcal{P}^2 = \mathbf{I}$ and $\mathcal{P}[\mathcal{H}_{\text{Heis}}]^* - \mathcal{H}_{\text{Heis}}\mathcal{P} = 0$.
[41] U. Günther and F. Stefani, Czech. J. Phys. **55**, 1099 (2005).
[42] W. D. Heiss, J. Phys. A **41**, 244010 (2008).
[43] E. M. Graefe *et al.*, J. Phys. A **41**, 255206 (2008).
[44] G. Demange and E.-M. Graefe, J. Phys. A **45**, 025303 (2012).
[45] This condition is, for example, satisfied by any eigenstate of the initial Hamiltonian.
[46] R. El-Ganainy, K. G. Makris, and D. N. Christodoulides, Phys. Rev. A **86**, 033813 (2012).
[47] O. Abah, J. Rossnagel, G. Jacob, S. Deffner, F. Schmidt-Kaler, K. Singer, and E. Lutz, Phys. Rev. Lett. **109**, 203006 (2012).
[48] D. S. Jin, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **77**, 420 (1996).
[49] M. O. Mewes, M. R. Andrews, N. J. vanDruten, D. M. Kurn, D. S. Durfee, C. G. Townsend, and W. Ketterle, Phys. Rev. Lett. **77**, 988 (1996).
[50] Y. Castin and R. Dum, Phys. Rev. Lett. **77**, 5315 (1996).
[51] Yu. Kagan, E. L. Surkov, and G. V. Shlyapnikov, Phys. Rev. A **54**, R1753 (1996).
[52] P. O. Fedichev and U. R. Fischer, Phys. Rev. A **69**, 033602 (2004).
[53] C. Chin *et al.*, Rev. Mod. Phys. **82**, 1225 (2010).
[54] M. R. Andrews *et al.*, Science **273**, 84 (1996).
[55] C. C. Bradley, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. **78**, 985 (1997).
[56] S. Levy, E. Lahoud, I. Shomroni, and J. Steinhuser, Nature (London) **449**, 579 (2007).
[57] O. Lahav, A. Itah, A. Blumkin, C. Gordon, S. Rinott, A. Zayats, and J. Steinhauer, Phys. Rev. Lett. **105**, 240401 (2010).
[58] I. Shammass, S. Rinott, A. Berkovitz, R. Schley, and J. Steinhauer, Phys. Rev. Lett. **109**, 195301 (2012).
[59] For practical purposes, the order of numerical differentiation can be reduced by applying Eq. (8).