Temperature dependence of interaction-induced entanglement

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Both direct and indirect weak nonresonant interactions are shown to produce entanglement between two initially disentangled systems prepared as a tensor product of thermal states, provided the initial temperature is sufficiently low. Entanglement is determined by the Peres-Horodecki criterion, which establishes that a composite state is entangled if its partial transpose is not positive. If the initial temperature of the thermal states is higher than an upper critical value $T_{u}$, the minimal eigenvalue of the partially transposed density matrix of the composite state remains positive in the course of the evolution. If the initial temperature of the thermal states is lower than a lower critical value $T_{b}$, the minimal eigenvalue of the partially transposed density matrix of the composite state becomes negative, which means that entanglement develops. We calculate the lower bound $T_{lb}$ for $T_{b}$ and show that the negativity of the composite state is negligibly small in the interval $T_{lb} < T < T_{uc}$. Therefore the lower-bound temperature $T_{lb}$ can be considered as the critical temperature for the generation of entanglement. It is conjectured that above this critical temperature a composite quantum system could be simulated using classical computers.

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I. INTRODUCTION

Efficient simulation of quantum dynamics on classical computers is hampered by the problem of scaling: the complexity of computation in quantum dynamics scales exponentially with the number of degrees of freedom [1]. The reason for this exponential growth is the entanglement of the degrees of freedom generated during the evolution. This problem is of a fundamental character: entanglement is viewed as one of the main peculiarities of the quantum dynamics as compared to its classical counterpart [2,3]. Asking under what conditions entanglement is generated along the evolution of the quantum system is closely associated with the question of the quantum-classical transition [4,5].

It is customary in quantum-dynamical simulations to assume that the initial state of the composite system is factorized in the relevant local basis [6]. An important question is whether the product form is conserved along the evolution [7,8]. The answer was generally found to be negative both for the pure- [8,9] and for the mixed-state [9] dynamics.

A pure composite state is entangled if and only if it is not factorized in the local basis. For mixed states the situation is more complex [10]. For a bipartite composite system separability [11] is defined as a decomposition of the density matrix of the composite system in the form

$$\hat{\rho}_{12} = \sum p_{i} \hat{\rho}_{1}^{i} \otimes \hat{\rho}_{2}^{i},$$

(1)

where $0 \leq p_{i} \leq 1$, $\Sigma p_{i} = 1$, and $\hat{\rho}_{1}$ and $\hat{\rho}_{2}$ are density matrices on Hilbert spaces of the first and the second subsystem, respectively. Separable states exhibit only classical correlations. States that cannot be represented in the form (1) exhibit correlations that cannot be explained within any classical theory and are said to be entangled. There are two qualitatively different kinds of the mixed-state entanglement [12]: free entanglement and bound entanglement. Free entanglement can be brought into a form useful for quantum-information processing and bound entanglement is “useless” in this sense.

Separable states are not of the product form generally. Thus the important question remains, under what conditions does the mixed state of the composite system evolving from the initial product (or generally separable) state develop entanglement along the evolution. If quantum correlations in the composite system do not develop during the evolution one may speculate that the dynamics of the composite system has classical character. A possible practical implication is that this “separable dynamics” could be simulated efficiently on classical computers.

The dynamics of entanglement was investigated recently in various systems: the quantum Brownian particle [13], harmonic chain [14], two-qubit system interacting with the common harmonic bath [15], Jaynes-Cummings model [16], NMR [17], various spin systems [18–20], Morse oscillator coupled to the spin bath [21], and bipartite Gaussian states in quantum optics [22] to mention just some cases. The purpose of the present paper is to explore the temperature dependence of entanglement generation in the course of evolution of a bipartite state in the limit of weak coupling and nonresonant interaction between the parts. Under these limitations nondegenerate perturbation theory was applied to the calculation of the bipartite entanglement in the evolving composite system. We have considered two cases of interaction: (1) direct interaction, when two initially disentangled systems are brought into contact (cf. Fig. 1), and (2) indirect interaction, when two noninteracting and initially disentangled systems are brought into contact with the third party (cf. Fig. 5). In both cases the initial state of the composite system was taken to be the product of the thermal states of the parts.

To establish quantum entanglement the Peres-Horodecki criterion is employed [23,24]. The Peres-Horodecki criterion states that the bipartite system is entangled when the partially transposed density matrix of the system possesses a negative eigenvalue. The converse statement is generally not true: there exist inseparable states whose partial transposes are...
positive [25]. It is proved in Ref. [12] that states whose partial transposes are positive (PPT states in what follows) do not exhibit free entanglement. Therefore PPT states are either separable or bound entangled and as a consequence are not useful in quantum-information processing. In the context of simulating a quantum composite system with classical computers, we are interested in the possibility of maintaining a separable [cf. Eq. (1)] or approximately separable form during the evolution. We conjecture on the basis of Ref. [26], where it is proved that PPT density matrices of a low rank are separable, that sufficiently cold (or, perhaps, sufficiently pure, in general) initial states that remain PPT during the evolution are at least approximately separable.

It is a well-known fact that in a sufficiently small neighborhood of a maximally mixed state, which can be viewed as a thermal state at infinite temperature, all states are separable [28–31]. Since the interactions studied in the present work are weak and nonresonant, the dynamics keeps the evolving state in the vicinity of the initial state. By assumption initial states are thermal product states. It is expected then that starting somewhere in the separable ball around the maximally mixed state, the evolving state will remain separable in the course of the evolution. Given that entanglement is generated at sufficiently low initial temperature, a finite cross-over temperature, depending on the details of the interaction, should exist: below this temperature the interaction generates entanglement in the course of the evolution of the composite system and above it the evolving state remains separable.

Since all separable states are PPT states the analogous critical temperature should exist for the PPT character of the evolution. In principle, there may exist several critical temperatures for the PPT quality for the given interaction. Therefore we define the lower and the upper critical temperatures $T_{lc}$ and $T_{uc}$ such that if the composite system evolves from the initial thermal state at temperature $T < T_{lb}$, the minimal eigenvalue of the partial transpose of the state becomes negative in the course of the evolution and if $T > T_{uc}$ the PPT character of the state is preserved. In the present work a lower bound $T_{lb}$ of the $T_{lc}$ is calculated in cases of both direct (1) and indirect (2) interactions between the parts.

Applying the Peres-Horodecki criterion to the case (1) of a direct interaction we show that for sufficiently low initial temperature of the subsystems the interaction does induce entanglement unless the ground state of either one of the subsystem is invariant under the interaction. The lower bound $T_{lb}$ of the lower critical temperatures $T_{lc}$ was calculated in the limit of weak intersystem coupling and shown to be tight: the negativity of the composite state [27], which is a quantitative counterpart of the Peres-Horodecki criterion and a measure of entanglement, is shown to be generally negligible for temperatures in the interval $T_{lb} < T < T_{uc}$.

In the case (2) of indirect two scenarios with time-scale separation are studied: (a) two “slow” noninteracting systems coupled to a “fast” third party and (b) two “fast” noninteracting systems coupled to a “slow” third party. Under some technical assumptions about the form of the interaction we find in both cases that for sufficiently low initial temperature of the noninteracting systems entanglement is induced by the interaction with the third party. We calculate the lower-bound temperature $T_{lb}$ in both cases of the timescale separation.

In case that the directly or indirectly interacting subsystems are spins the lower-bound temperature coincides with the upper critical temperature, i.e., at $T > T_{lb}$ the spins stay disentangled. In the higher-dimensional cases it is found that the negativity of the evolving composite state is negligibly small at $T > T_{lb}$. Therefore the lower-bound temperature $T_{lb}$ may be considered as an effective crossover temperature for the PPT character of the evolution. It may be suspected that $T_{lb}$ corresponds to a critical purity of the evolving state below which the state remains separable according to the general result [28–31], cited above. To see that this is not the case we note that the radius of the largest separable ball [30] (and of the largest PPT ball [31], too) about the maximally mixed state is defined by the condition that the purity of the state is low enough, namely, that $\text{Tr} \rho^2 = 1/(d-1)$, where $d$ stands for the overall dimension of the composite state. Since, as will be shown, $T_{lb}$ is a monotonically decreasing function of the coupling strength (effective) crossover to the PPT evolution is found for initial states of arbitrarily high purity, provided the coupling strength is small enough. Therefore, the effective crossover at $T_{lb}$ cannot be attributed to the “static” proximity of the evolving state to the maximally mixed state but rather $T_{lb}$ is determined dynamically.

In both cases (1) and (2) the evolution starts from an uncorrelated initial state of the composite system represented by the tensor product of thermal states of the subsystems involved. As a consequence, initially the eigenstates of the partially transposed density matrix of the composite state are nonnegative. The evolution under the interaction perturbs the initial state. The new eigenvalues of the partially transposed density matrix are calculated by nondegenerate perturbation theory assuming the coupling is weak and the interaction is nonresonant. The time dependence of the minimal eigenvalue is not analyzed in detail. As the time evolution of the density matrix is quasiperiodic the minimal eigenvalue of the partially transposed density matrix is also a quasiperiodic function of time. The interaction is said to induce entanglement if the minimal eigenvalue becomes negative in the course of the evolution.

II. ENTANGLEMENT BETWEEN TWO DIRECTLY INTERACTING SYSTEMS

A composite system $A \otimes B$ (see Fig. 1) evolves under the following Hamiltonian:

$$\hat{H}_{total} = \hat{H} + \gamma \hat{V},$$

(2)

where $\hat{H} = \hat{H}_a \otimes \hat{1} + \hat{1} \otimes \hat{H}_b$, $\hat{V} = \hat{V}_a \otimes \hat{V}_b$, and $\gamma$ scales the magnitude of the interaction. Let the initial state be

$$\hat{\rho}(0) = \hat{\rho}_a \otimes \hat{\rho}_b,$$

(3)

where both $\hat{\rho}_a$ and $\hat{\rho}_b$ are thermal states: $\hat{\rho}_{a,b} = Z_{a,b}^{-1} \times \exp(-\hat{H}_{a,b}/T)$, where $Z_{a,b}^{-1}$ is the normalization factor. The Boltzmann constant $k_B$ is unity throughout the paper. The evolution is followed in the interaction picture. Then
where

\( |i\rangle = |j\rangle \otimes |k\rangle \) be the local orthonormal basis of the system \( A \otimes B \) composed of the eigenvectors of the unperturbed Hamiltonian \( \hat{H} = \hat{H}_a + \hat{H}_b \):

\[
\hat{H}_{a,b}(t) = E_{a,b}(t),
\]

where \( E_{a,b} \), \( i = 1, 2, \ldots \), is the unperturbed energy spectrum of the Hamiltonian \( \hat{H}_{a,b} \). The initial state is of the tensor product form [see Eq. (3)]; therefore

\[
\hat{\rho}(0)|i\rangle = \hat{\rho}(0)|i\rangle = \hat{\rho}_a \otimes \hat{\rho}_b |i\rangle = P_{a,i}|i\rangle,
\]

where \( P_{a,i} = \hat{P}_{a,i} \) and \( P_{a,i} = \hat{P}_{a,i} = \delta_{i,j} \) and \( P_{a,i} = \hat{P}_{a,i} = \delta_{i,j} \). The matrix elements of \( \hat{\rho}(t)^{T_a} \) in the chosen basis are given by

\[
(i\rangle |\hat{\rho}(t)^{T_a}|j\rangle = P_{a,i} \delta_{i,j} + M_{a,jl}.
\]

where

\[
M_{a,jl} = \int_0^t i \gamma \int_0^t \langle i\rangle \langle \hat{V}_a(t') \otimes \hat{V}_b(t') \rangle \hat{\rho}_a \otimes \hat{\rho}_b \hat{\rho}^{T_a}(j) dt'.
\]

Entanglement of \( \hat{\rho}(t) \) is established by the application of the Peres-Horodecki criterion. This is carried out by calculating the partial transpose of the state. The partial transposition \( T_a \) with respect to subsystem \( A \) of a bipartite state \( \hat{\rho}_{ab} \) expanded in a local orthonormal basis as \( \hat{\rho}_{ab} = \sum \hat{\rho}_{ijkl} |i\rangle \otimes |k\rangle \) is defined as

\[
\hat{\rho}_{ab}^{T_a} = \sum \hat{\rho}_{ijkl} |j\rangle \otimes |k\rangle.
\]

The spectrum of the partially transposed density matrix does not depend on the choice of basis or on the choice of the subsystem with respect to which the partial transposition is performed. By the Peres-Horodecki criterion the eigenvalues of a partially transposed separable bipartite state are nonnegative.

The density operator (6) under the partial transposition \( (T_a) \) becomes:

\[
\hat{\rho}(T_a) = \hat{\rho}(0)^{T_a} - i \gamma \int_0^t \langle \hat{V}_a(t') \otimes \hat{V}_b(t') \rangle \hat{\rho}(0)^{T_a} dt'.
\]
locally on the subsystems and cannot entangle them. Otherwise there are negative solutions to Eq. (13) and as a consequence the partial transpose of the density operator attains negative eigenvalues already in the first order in the coupling. Therefore, according to the Peres-Horodecki criterion, entanglement develops at zero temperature. We remark that this is a well-known result for the pure-state evolution but the present derivation will be employed also in the case of a finite temperature.

To simplify the study of the generation of entanglement at finite temperatures it is assumed that the only nonzero matrix elements of $\hat{V}_{a,b}$ are those between neighboring states, i.e., $(\hat{V}_{a,b})_{ij} \approx \delta_{i,j+1}$. Under this assumption the partially transposed density matrix $\hat{\rho}(t)^{T_a}$ obtains the following structure:

$$
\rho(t)^{T_a} = \begin{pmatrix}
P_{11} & 0 & 0 & M_{1,12} & 0 & 0 & 0 & \cdots \\
0 & P_{12} & M_{1,21} & 0 & M_{1,23} & 0 & 0 & \cdots \\
0 & M'_{1,21} & P_{21} & 0 & 0 & M_{2,32} & 0 & \cdots \\
M'_{1,12} & 0 & 0 & P_{22} & 0 & 0 & M_{2,33} & \cdots \\
0 & M'_{2,23} & 0 & 0 & P_{23} & M_{2,32} & 0 & \cdots \\
0 & 0 & M'_{2,32} & 0 & M'_{2,33} & 0 & P_{33} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix},
$$

(14)

where $P_{ij}$ are defined after Eq. (10) and $M_{kl,ij}$ by Eq. (12).

There are two kinds of matrix elements $M_{kl,ij}$: $M_{kl,(k+1)\rightarrow(i+1)}$ and $M_{kl,(k+1)\rightarrow(i-1)}$ (other elements are their counterparts under the transposition). Matrix elements $M_{ki,(k+1)\rightarrow(i+1)}$ couple the unperturbed eigenvalues $P_{ki}$ and $P_{(k+1)\rightarrow(i+1)}$. For small coupling strength $\gamma_{kk} M_{ki,(k+1)\rightarrow(i+1)} \ll P_{ki}$ and the contribution of $M_{ki,(k+1)\rightarrow(i+1)}$ to the correction to $P_{ki}$ is negligible and cannot make the eigenvalue negative. On the other hand, the ratio $|M_{ki,(k+1)\rightarrow(i+1)}|/P_{(k+1)\rightarrow(i+1)} \ll (P_{(k+1)\rightarrow(i+1)})/P_{ki}$ can in general be arbitrarily large for low temperatures but for sufficiently high temperatures it tends to zero and as a consequence the contribution of the coupling element $M_{ki,(k+1)\rightarrow(i+1)}$ to the correction to $P_{ki}$ is negligible. It can be checked along the same lines that the ratio of the coupling matrix elements $M_{ki,(k\rightarrow(i-1))}$ to the unperturbed eigenvalues $P_{ki}$ and $P_{(k+1)\rightarrow(i-1)}$ of the partially transposed density matrix (14) vanish for sufficiently high temperature. Therefore, at least for composite systems with finite Hilbert space dimensions, there exists a finite upper critical temperature $T_{uc}$. Above $T_{uc}$ the spectrum of the partially transposed density matrix remains positive. In close vicinity of $T_{uc}$ from below the minimal eigenvalue becomes negative in the course of the evolution. These conclusions stay in accord with a general result that finite-dimensional composite systems in sufficiently small neighborhood of the maximally mixed state (i.e., thermal states at infinite temperature) are separable [28–31].

At sufficiently low initial temperature the minimal eigenvalue of the partially transposed density matrix becomes negative in the course of the evolution. This means that there exists a finite lower critical temperature $T_{lc}$. Below $T_{lc}$ the composite system $A \otimes B$ develops entanglement. In a sufficiently close vicinity of $T_{lc}$ from above the state remains PPT in the course of evolution. It is possible that $T_{lc}=T_{uc}$. This equality is confirmed in all numerical tests. A lower bound $T_{lb}$ for the lower critical temperature $T_{lc}$ can be calculated using perturbation analysis. It is shown that this bound is tight since the free entanglement in the interval $T_{lb}<T<T_{uc}$ is negligibly small under the weak-coupling assumption. Therefore, from the practical point of view the lower bound $T_{lb}$ for $T_{lc}$ can be considered as the critical temperature for entanglement. For simplicity the lower bound for the lower critical temperature is termed “the lower-bound temperature” throughout the paper.

At low temperatures the leading-order contribution to the negative eigenvalue of the partially transposed density matrix comes from the matrix elements $M_{11,22}, M_{12,21}$ (and their complex conjugates) that do not vanish at $T=0$. Therefore, to the leading order in $\gamma$, the nonvanishing eigenvalues of the partially transposed density matrix Eq. (14) are the eigenvalues of the following effective partially transposed density matrix $\hat{\rho}(t)^{T_a eff}$:

$$
\hat{\rho}(t)^{T_a eff} = \begin{pmatrix}
P_{11} & 0 & 0 & M_{11,22} \\
0 & P_{12} & M_{12,21} & 0 \\
0 & M'_{12,21} & P_{21} & 0 \\
M'_{11,22} & 0 & 0 & P_{22} 
\end{pmatrix},
$$

(15)

The critical temperature, calculated for the effective $4 \times 4$ matrix (15), is a lower bound for the lower critical temperature $T_{lc}$ of the bipartite system $A \otimes B$. The eigenvalues of Eq. (15) are eigenvalues of two $2 \times 2$ matrices:
The form strictly positive at and above the temperature has been evaluated. The other two eigenvalues are found to be strictly positive where

\[
\frac{P_{12}}{M_{12,21}} + \frac{M_{12,21}}{P_{12}} \quad (16)
\]

and

\[
\frac{P_{11}}{M_{11,22}} + \frac{M_{11,22}}{P_{22}} \quad (17)
\]

The eigenvalues of the matrix (16) are

\[
\lambda_\pm = \frac{P_{12} + P_{21} \pm \sqrt{(P_{12} + P_{21})^2 - 4(P_{12}P_{21} - |M_{12,21}|^2)}}{2},
\]

(18)

where from Eq. (12)

\[
M_{12,21} = \gamma(2|\tilde{\mathbf{V}}_a|1)(2|\tilde{\mathbf{V}}_b|1)e^{\pm i\Delta E_{11}} - 1 \quad (19)
\]

where we define \( \Delta E_{11} = E^a_a - E^b_b + E^a_b - E^b_a \), which is the lowest joint excitation energy of the composite system.

From Eq. (18), \( \lambda_\pm \) will be negative whenever \( P_{12}P_{21} < |M_{12,21}|^2 \) and positive if \( P_{12}P_{21} > |M_{12,21}|^2 \). The lower-bound temperature \( T_{ib} \) is evaluated from the condition \( P_{12}P_{21} = |M_{12,21}|^2 \). Since \( |M_{12,21}| \) is an oscillating function of time [see Eq. (19)] the amplitude of \( |M_{12,21}| \) is taken to be equal to \( \sqrt{P_{12}P_{21}} \):

\[
\frac{2\gamma}{\Delta E_{11}}(2|\tilde{\mathbf{V}}_a|1)(2|\tilde{\mathbf{V}}_b|1)(P_{11} - P_{22}) = \sqrt{P_{12}P_{21}}. \quad (20)
\]

Assuming that \( T_{ib} \) is low \( P_{11} - P_{22} = P_{11} \) and then

\[
\frac{2\gamma}{\Delta E_{11}}(2|\tilde{\mathbf{V}}_a|1)(2|\tilde{\mathbf{V}}_b|1) = \sqrt{P_{22}P_{11}^3} = e^{-\Delta E_{11}/2T_{ib}}. \quad (21)
\]

Since \( e^{-\Delta E_{11}/2T} \) is a monotonic function of the temperature, at \( T>T_{ib}\lambda_\pm > 0 \) and at \( T<T_{ib}\lambda_\pm < 0 \). Finally, the expression for the lower-bound temperature \( T_{ib} \) becomes

\[
T_{ib} = -\frac{\Delta E_{11}}{2 \ln[(2\gamma/\Delta E_{11})(2|\tilde{\mathbf{V}}_a|1)(2|\tilde{\mathbf{V}}_b|1)]} \quad (22)
\]

So far only two of the eigenvalues of the matrix (15) have been evaluated. The other two eigenvalues are found to be strictly positive at and above the temperature \( T_{ib} \). Therefore, the expression (22) defines the critical temperature for the partially transposed effective density matrix (15) and the lower bound temperature of the partially transposed density matrix (14).

Equation (22) can be generalized to an interaction term of the form \( \sum_i \gamma_i \mathbf{V}_i \otimes \mathbf{V}_i \):

\[
T_{ib} = -\frac{\Delta E_{11}}{2 \ln[(2\gamma/\Delta E_{11})\sum_i \gamma_i(2|\tilde{\mathbf{V}}_i|1)(2|\tilde{\mathbf{V}}_i|1)]}, \quad (23)
\]

provided \( \sum_i \gamma_i(2|\tilde{\mathbf{V}}_i|1)(2|\tilde{\mathbf{V}}_i|1) \neq 0 \). When this term vanishes there is no entanglement in the first order in the coupling strength \( \gamma \).

For the system of two interacting spins the lower bound \( T_{ib} \) given by Eq. (22) coincides with the upper critical temperature \( T_{uc} \); therefore in this case the critical temperature exists in the strict sense. Figure 2 shows results of the numerical calculation of the critical temperature as a function of coupling strength \( T_{ib} \). The composite system of two spins evolves from the initial product of thermal states under the Hamiltonian \( H = (1/2)(\gamma \hat{a}^\dagger \hat{a} + (\gamma/2 - 1) \hat{b}^\dagger \hat{b} - \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}) \). The evolution is calculated numerically for \( \omega = 1 \). The border of the shaded area represents \( T_{uc} \) calculated numerically. The dashed line represents \( T_{ib} \) according to Eq. (22) (dashed line) corresponds well to the numerical values of \( T_{uc} \). It is interesting to note that for large values of coupling the critical temperature asymptomatically tends to a finite constant value of the same order of magnitude as the energy difference between the first excited and the ground state of the unperturbed composite system.

At \( T < T_{ib} \) the minimal eigenvalue of the partially transposed state (14) is negative. We want to show that above \( T_{ib} \) the negative eigenvalues of the matrix form (14) are of higher order in \( \gamma \) and therefore are negligibly small when the coupling is weak.

Let us consider corrections to the eigenvalues \( P_{(i+1)} \) and \( P_{(i+1)} \) of the composite state (14). The order-of-magnitude estimate of the smallest one of the corrected eigenvalues is

\[
\lambda_\pm^{(i)} = (P_{(i+1)}P_{(i+1)}) - \gamma P_{(i)}/\Delta E_{ij}, \quad \Delta E_{ij} = E_{ij} + E_{ij}^{(i)} - E_{ij}^{(i)},
\]

where \( E_{ij}^{(i)} = E_{ij}^{(i+1)} - E_{ij}^{(i+1)} - E_{ij} \). For simplicity we assume \( P_{(i+1)} = P_{(i+1)} \). Then \( \lambda_\pm^{(i)} = O(P_{(i+1)} - \gamma P_{(i)}/\Delta E_{ij}) \). Below \( T_{ib} \), the minimal eigenvalue of the state (14) is \( \lambda_\pm = O(-\gamma/\Delta E_{11}) \). We shall estimate the ratio \( r_{ij} = \lambda_\pm^{(i)}/\lambda_\pm \) and show that it is negligible when the coupling is weak. We shall assume without loss of generality that the ground-state energy is zero: \( E_{ij}^{(i)} = 0 \). Then the partition function \( Z \) of the composite system is larger than unity. It follows that

\[
\begin{align*}
\text{FIG. 2.} \quad \text{(Color online)} \quad \text{The shaded area in the parameter space of the inverse initial temperature } T \text{ of two spins and the logarithm of the inverse coupling strength } \gamma \text{ represents values of } T \text{ and } \gamma \text{ where entanglement does not develop in the course of the evolution.}
\end{align*}
\]
We remark that the minimum is a monotonically slowly increasing function of $x_{ij}$.

In those cases when $x_{ij} = \Delta E_{ij}/\Delta E_{11} \gg 1$ the upper bound for $r$ scales as $O(1)$, and the corresponding negative eigenvalues of Eq. (14) are negligible. In this case we expect that $T_{lb}$ is negligible under the weak-coupling assumption.

Typically the spectrum becomes denser with increasing energy. In that case $x_{ij} = \Delta E_{ij}/\Delta E_{11} \ll 1$. Values of $\lambda_{ij}$ corresponding to $x_{ij} \ll 1$ need not be taken into account, because $T_{c} < T_{lb}$ in this case and as a consequence $\lambda_{ij} > 0$ at $T \gg T_{lb}$. At $x_{ij} = O(1)$ the upper bound for $r$ scales as $O(\gamma)$ and therefore corresponding negative eigenvalues of Eq. (14) are negligible. In this case we expect that $T_{lb} \approx T_{uc}$.

In those cases when $x_{ij} = \Delta E_{ij}/\Delta E_{11} \ll 1$ the upper bound for $r$ scales as $O(1/x_{ij})$, and the corresponding negative eigenvalues of Eq. (14) can be neglected, too.

When $x_{ij}$ is moderately larger than unity the upper bound Eq. (27) for $r$ has a local maximum. The position of the maximum weakly depends on $\gamma$; numerical calculations show $x_{ij} \approx 2 - 10$ in the range of $10^{-4} \leq \gamma \leq 10^{-1}$. The value of the minimum is a monotonically slowly increasing function of $\gamma$. In the range $10^{-4} \leq \gamma \leq 10^{-1}$ numerical estimation of $r$ gives

$$
\frac{\lambda_{ij}}{\lambda_{-}} = \frac{(\gamma/\Delta E_{ij})P_{ij} - P_{ij(j+1)}}{\gamma/\Delta E_{11}} \frac{(\gamma/\Delta E_{ij})ZP_{ij} - ZP_{ij(j+1)}}{\gamma/\Delta E_{11}}
= \frac{(\gamma/\Delta E_{ij})e^{E_{ij}/T} - e^{E_{ij(j+1)}/T}}{\gamma/\Delta E_{11}}.
$$

We are looking for the maximal value of $r_{ij}$ in the interval $0 < T < T_{c}$ corresponding to the condition $P_{ij(j+1)} < \gamma P_{ij}/\Delta E_{ij}$, i.e., to the negative values of $\lambda_{ij}$. $T_{c}$ is determined by the condition $\lambda_{ij} = 0$. The ratio $r_{ij}$ is positive in the interval $0 < T < T_{c}$ and vanishes on its borders. Therefore $r_{ij}$ has a maximum $r_{ij}^{m}$ at $0 < T_{m} < T_{c}$, which is found from the condition $\partial r_{ij}/\partial T_{m} = 0$. The calculation gives

$$
\exp[- \Delta E_{ij}/(2T_{ij}^{m})] = (\gamma/\Delta E_{ij})(E_{ij}/E_{ij(j+1)}) < (\gamma/\Delta E_{ij})
= \exp[- \Delta E_{ij}/(2T_{ij}^{m})],
$$

which proves that there is one maximum $r_{ij}^{m}$ at $0 < T_{m} < T_{c}$. We remark that $T_{c}$, corresponding to the largest $\Delta E_{ij}$ over all $i$ and $j$, is of the order of the upper critical temperature, $T_{m} = O(T_{m})$. The maximal value of $r_{ij}$ is given by

$$
r_{ij}^{m} = \frac{\Delta E_{11}}{2E_{ij} + \Delta E_{ij}} \left( 2E_{ij} / \Delta E_{ij} \right)^{2E_{ij}/\Delta E_{ij}} \left( \gamma / \Delta E_{ij} \right)^{2E_{ij}/\Delta E_{ij}},
$$

where the inequality follows from the fact that $1/e < [2E_{ij}/(2E_{ij} + \Delta E_{ij})]^{2E_{ij}/\Delta E_{ij}} < 1$ in general. As a next step we notice that $\Delta E_{11} \approx 2E_{ij}$; therefore

$$
r_{ij} < r_{ij}^{m} < \frac{\Delta E_{11}}{2E_{ij} + \Delta E_{ij}} \left( \gamma / \Delta E_{ij} \right)^{2E_{ij}/\Delta E_{ij}}
\leq \frac{\Delta E_{11}}{\Delta E_{11} + \Delta E_{ij}} \left( \gamma / \Delta E_{ij} \right)^{\Delta E_{11}/\Delta E_{ij}}.
$$

Introducing the definition $x_{ij} = \Delta E_{ij}/\Delta E_{11}$ and taking $\Delta E_{11} = 1$, which corresponds to a rescaling of the coupling strength $\gamma$, leads to

$$
\frac{\lambda_{ij}}{\lambda_{-}} = r_{ij} < \frac{1}{x_{ij}} \left( \frac{\gamma}{x_{ij}} \right)^{1/x_{ij}}.
$$

Eq. (27) shows values $0.04 - 0.1$ for the local maximum. It is clear that the upper bound Eq. (27) for $r_{ij}$ is far from being tight. In fact, numerical calculations show that $r_{ij}$ is generally much smaller. As a consequence, the corresponding negative eigenvalues of Eq. (14) can be neglected.

It can be argued that although each one of the negative eigenvalues of Eq. (14) is negligible at $T \geq T_{lb}$, the (free) entanglement of the state cannot be neglected. In fact, the minimal negative eigenvalue of the partially transposed matrix is not a measure of entanglement. Various measures of entanglement have been defined. In the present context we will employ a quantitative counterpart of the Peres-Horodecki criterion, the negativity [27]

$$
N(\hat{\rho}(t)) = \frac{\|\hat{\rho}(t) \|_{1} - 1}{2}.
$$

where $\|\hat{X}\|_{1} = \text{Tr} \sqrt{\hat{X} \hat{X}^\dagger}$ is the trace norm of an operator $\hat{X}$. The negativity of the state equals the absolute value of the sum of the negative eigenvalues of the partially transposed state. When the negativity of a composite bipartite state vanishes there is no free entanglement in the state. It can be shown by an order-of-magnitude analysis similar to the analysis above that values of the negativity of the composite state, corresponding to the partial transpose (14), are generally dominated by the minimal negative eigenvalue. As a consequence, the negativity of the state, evolving from the initial thermal product state at the temperature $T \geq T_{lb}$ is negligible under the weak-coupling assumption.

Figures 3 and 4 display results of numerical calculations of the time averaged negativity of the composite state (6) as a function of initial temperature for two different kinds of unperturbed spectra of the composite system $A \otimes B$. Both $A$ and $B$ are four-level systems. The composite system evolves...
from the initial product of thermal states of $A$ and $B$ under the Hamiltonian (2).

Figure 3 presents the results of calculations for the following choice of the unperturbed spectra of $\hat{H}_a$ and $\hat{H}_b$: $E^{1,2,3,4}_a=\{\omega,5\omega,8\omega,10\omega\}$ and $E_{b}=\sqrt{\omega E_{a}}$ (we set $\omega=1$). Care was taken to avoid resonances and the spectra were chosen to become denser with increasing energy. The interaction terms in the Hamiltonian were restricted to $(\hat{V}_{ab})_{ij}=\delta_{ij}\Gamma$ and the coupling strength $\gamma=0.05$. We see that $T_{uc}$ and the time-averaged negativity $N(\hat{\rho}(t))$ is negligible in the interval $T_{lb}<T<T_{uc}$ as expected.

Figure 4 displays the time-averaged negativity $N(\hat{\rho}(t))$ as a function of initial temperature of the composite state of two interacting four-level subsystems $A$ and $B$ with the unperturbed energy spectra $E^{1,2,3,4}_a=\{\omega,3\omega,7\omega,13\omega\}$ and $E_{b}=\sqrt{\omega E_{a}}$. The composite state evolves from the initial product of two thermal states under the Hamiltonian (2), where $(\hat{V}_{ab})_{ij}=\delta_{ij}\Gamma$ and the coupling strength $\gamma=0.05$. In choosing the unperturbed spectra care was taken to avoid resonances and to ensure that the maximal value of $x_{ij}=\Delta E_{ij}/\Delta E_{11}$ equals the position of the local maximum of the upper bound (27), corresponding to $\gamma=0.05$. Figure 3 shows that the time-averaged negativity $N(\hat{\rho}(t))$ is negligible in the interval $T_{lb}<T<T_{uc}$ as expected. The value of $T_{uc}$ is given after Eq. (24), corresponding to the maximal value $\Delta E_{max}=\max_{ij}\{\Delta E_{ij}\}$ is calculated. $T_{uc}$ is in good correspondence with the value $T_{uc}$ calculated numerically.

In the following section the lower-bound temperature for the generation of entanglement in the case of indirect interaction will be calculated. It is expected on general perturbation theory grounds that the negativity is negligible in this case, too, at $T>T_{lb}$. A quantitative analysis was not carried out due to the technical complexity of the indirect interaction case. Nevertheless, all the numerical tests confirm this view. The temperature dependence of the negativity for a given coupling strength in the case of indirect interaction is essentially equivalent to the temperature dependence in the direct-interaction case plotted in Figs. 3 and 4.

### III. Entanglement Between Two Noninteracting Systems in Contact With a Common Third Party

The dynamics studied is of the composite system $A\otimes B\otimes C$ where systems $B$ and $C$ do not interact directly (see Fig. 5). The entanglement explored is of the reduced composite system $B\otimes C$.

The evolution is generated by the following Hamiltonian:

$$\hat{H}_{total} = \hat{H} + \gamma \hat{\mathbf{V}},$$

where $\hat{H}=\hat{H}_a + \hat{H}_b + \hat{H}_c$ and $\hat{\mathbf{V}}=\gamma \hat{\mathbf{V}}_b \otimes (\hat{\mathbf{V}}_b \otimes 1 + 1 \otimes \hat{\mathbf{V}}_c)$. The analysis is carried out in the interaction picture. The initial state is taken to be $\hat{\rho}(0)=\hat{\rho}_a \otimes \hat{\rho}_b \otimes \hat{\rho}_c$, where $\hat{\rho}_a$, $\hat{\rho}_b$, and $\hat{\rho}_c$ are thermal states. Since $B$ and $C$ are noninteracting entanglement will appear only in the second order in the coupling. Up to second order in $\gamma$ the state of the composite system $A\otimes B\otimes C$ becomes

$$\hat{\rho}(t) = \hat{\rho}(0) - i \gamma \int_0^t [\hat{\mathbf{V}}(t'),\hat{\rho}(0')] dt'$$

$$- \gamma^2 \int_0^t \int_0^{t'} [\hat{\mathbf{V}}(t'),[\hat{\mathbf{V}}(t''),\hat{\rho}(0)]] dt'' dt',$$

where

$$\hat{\rho}' = e^{-iHt} \hat{\rho} e^{iHt}.$$
\[ \hat{\mathbf{V}}(t) = e^{i\mathbf{H}_t t} \hat{\mathcal{V}} e^{-i\mathbf{H}_t t}. \]  

(31)

In what follows the prime on the \( \hat{\rho}(t) \) is omitted.

Next the system is reduced to \( B \otimes C \) by taking the partial trace of \( \hat{\rho}(t) \) over the system \( A \) degrees of freedom. The partial transposition of the reduced density operator with respect to the subsystem \( B \) gives:

\[ \hat{\rho}_{bc}^{T_B}(t) = \text{Tr}_A \hat{\rho}(t) \]  

(32)

where \( \hat{\rho}_{bc}(t) = \text{Tr}_A \hat{\rho}(t) \)

\[ \hat{M} = -i \gamma \int_0^t \text{Tr}_a[\hat{\mathbf{V}}(t'), \hat{\rho}(0)]^T dt' \]  

(33)

\[ - \gamma^2 \int_0^t \int_0^{t'} \text{Tr}_a[\hat{\mathbf{V}}(t'), [\hat{\mathbf{V}}(t'), \hat{\rho}(0)]]^T dt'' dt'. \]

Let \( |ik\rangle = |i\rangle \otimes |k\rangle \) be the local orthonormal basis of the system \( B \otimes C \) composed of the eigenstates of the Hamiltonian \( \hat{\mathbf{H}}_b + \hat{\mathbf{H}}_c \):

\[ \hat{\mathbf{H}}_{bc,i}(|i\rangle) = \mathcal{E}_{bc,i}|i\rangle, \]  

(34)

where \( \mathcal{E}_{bc,i} \), \( i = 1, 2, \ldots \) is the unperturbed energy spectrum of the Hamiltonian \( \hat{\mathbf{H}}_{bc} \). Since \( \hat{\rho}_{bc}(0) = \hat{\rho}_b \otimes \hat{\rho}_c \),

\[ \lambda_s = -\frac{P_{12} + M_{12,12} + P_{21} + M_{21,21}}{2} \pm \sqrt{\left(P_{12} + M_{12,12} + P_{21} + M_{21,21}\right)^2 - 4\left(P_{12} + M_{12,12}(P_{21} + M_{21,21}) - |M_{12,21}|^2\right)} \]  

(38)

and the eigenvalue \( \lambda_- \) becomes negative when \( P_{12} + M_{12,12}(P_{21} + M_{21,21}) < |M_{12,21}|^2 \).

To calculate \( M_{12,12} \), \( M_{21,21} \), and \( M_{12,21} \) we first note that the integrand in the first order term in Eq. (33) is

\[ \text{Tr}_a[\hat{\mathbf{V}}(t'), [\hat{\mathbf{V}}(t'), \hat{\rho}(0)]]^T = \langle \hat{\mathbf{V}}_a | [\hat{\mathbf{V}}_b(t'), \hat{\rho}_b(0)]^T \hat{\mathbf{V}}_c \rangle \]  

(39)

where \( \langle \hat{\mathbf{V}}_a \rangle \) means the thermal average of the operator \( \hat{\mathbf{V}}_a \) and the notation \( \hat{\mathbf{V}} = \hat{\mathbf{V}}_a \otimes \hat{\mathbf{I}} + \hat{\mathbf{I}} \otimes \hat{\mathbf{V}}_c \) is introduced. The initial condition \( \hat{\rho}_{bc}(0) = \hat{\rho}_b \otimes \hat{\rho}_c \) was used. Since \( \hat{\rho}_{bc}(0) = \delta_{i,k}|i\rangle \) the term Eq. (39) does not contribute to the eigenvalues of the matrix (37) in the first order.

To simplify the calculation of the second order corrections it is assumed that the thermal average of the system \( A \) coupling operator \( \langle \hat{\mathbf{V}}_a \rangle \) vanishes. This assumption is not crucial for the qualitative picture of temperature dependence of the entanglement. Moreover, it is in line with common models of coupling, for example, the Caldeira-Leggett model [33], diagonal pole interaction with the electromagnetic field [34], etc. The integrand in the second-order term in Eq. (33) is

\[ \text{Tr}_a[[\hat{\mathbf{V}}(t'), [\hat{\mathbf{V}}(t'), \hat{\rho}(0)]]^T = \langle \hat{\mathbf{V}}_a(t') \hat{\mathbf{V}}_a(t') \rangle \hat{\rho}_{bc} + \hat{\rho}_b \otimes \hat{\rho}_c \]  

(40)

Expanding the thermal averages in the orthonormal basis \( \{m\} \) of the Hamiltonian \( \hat{\mathbf{H}}_a \) leads to

\[ \text{Tr}_a[[\hat{\mathbf{V}}(t'), [\hat{\mathbf{V}}(t'), \hat{\rho}(0)]]^T = \sum_{m,n} p_{a,m} \langle m | \hat{\mathbf{V}}_a \rangle |n\rangle |^2 \langle \omega_{mn}^a (t' - t') \rangle \]  

(41)

where \( \langle \omega_{mn}^a \rangle \) is the energy difference between the states \( |n\rangle \) and \( |m\rangle \) of the Hamiltonian \( \hat{\mathbf{H}}_a \), the \( \{\hat{\mathbf{X}}, \hat{\mathbf{Y}}\} \) designates the anti-commutator of operators \( \hat{\mathbf{X}} \) and \( \hat{\mathbf{Y}} \) and \( p_{a,m} = \langle \hat{\rho}_{a,m} \rangle \).
For simplicity the notation \( \hat{C}(t',t'') \) is used for the operator (41). Expressing the operator \( \hat{V}_{bc} \) in terms of \( \hat{V}_b \) and \( \hat{V}_c \) we put the matrix elements of \( \hat{C}(t',t'') \) into the following form:

\[
\langle 12 | \hat{C}(t',t'') | 21 \rangle = -2 P_{12} \sum_{m,n} p_{a,m,n}(m|\hat{V}_a|n)\bar{p}(1|\hat{V}_c|2)^2 \times \cos[\omega_{mn}(t'-t'')],
\]

\[
\langle 12 | \hat{C}(t',t'') | 21 \rangle = -2 P_{12} \sum_{m,n} p_{a,m,n}(m|\hat{V}_a|n)\bar{p}(1|\hat{V}_b|2)^2 \times \cos[\omega_{mn}(t'-t'')],
\]

\[
\langle 12 | \hat{C}(t',t'') | 21 \rangle = P_{12} \sum_{m,n} p_{a,m,n}(m|\hat{V}_a|n)^2(2|\hat{V}_b|1)(2|\hat{V}_c|1) \times e^{i\omega_{mn}(t'-t')} (e^{-i(\omega_a+\omega_c)t} + e^{-i(\omega_a+\omega_c)t'}),
\]

(42)

where \( \omega_{b,c} \) stands for the energy difference between the first excited and the ground states of the unperturbed subsystem \( B (C) \). The matrix elements \( M_{12,12}, M_{21,21}, \) and \( M_{12,21} \) are given by

\[
M_{12,12} = -\gamma^2 \int_0^t \int_0^{t'} \langle 12 | \hat{C}(t',t'') | 12 \rangle dt'\, dt'',
\]

\[
M_{21,21} = -\gamma^2 \int_0^t \int_0^{t'} \langle 21 | \hat{C}(t',t''') | 21 \rangle dt'\, dt''',
\]

\[
M_{12,21} = -\gamma^2 \int_0^t \int_0^{t'} \langle 12 | \hat{C}(t',t'') | 21 \rangle dt'\, dt''.
\]

(43)

The integration is straightforward but the final expressions are cumbersome. Two cases are considered explicitly: (a) \( \omega_{b,c} \gg \omega_{b,c} \) and (b) \( \omega_{b,c} \gg \omega_{mn} \). In both cases it is shown that at sufficiently low initial temperature of the system \( B \otimes C \) one of the eigenvalues of the matrix (37) is negative and the lower-bound temperature \( T_{lb} \) is calculated.

A. Two slow systems interacting with a fast common third party

Performing the integrations in Eq. (43) and taking the leading terms in \( \omega_{b,c} \), \( \omega_{mn} \) brings us to

\[
M_{12,12} = 4 \gamma^2 \sum_{m,n} p_{a,m,n}(m|\hat{V}_a|n)^2(1|\hat{V}_c|2)^2 \frac{\sin[\omega_{mn}(\omega_a+\omega_c)t/2]}{(\omega_{mn})^2},
\]

\[
M_{21,21} = 4 \gamma^2 \sum_{m,n} p_{a,m,n}(m|\hat{V}_a|n)^2(1|\hat{V}_b|2)^2 \frac{\sin[\omega_{mn}(\omega_a+\omega_c)t/2]}{(\omega_{mn})^2},
\]

\[
M_{12,21} = 2 \gamma^2 \sum_{m,n} p_{a,m,n}(m|\hat{V}_a|n)^2(2|\hat{V}_b|1) \times (2|\hat{V}_c|1) \frac{1-e^{-i(\omega_a+\omega_c)t}}{\omega_{mn}(\omega_a+\omega_c)}. \]

(44)

At \( T=0 \) the minimal eigenvalue of Eq. (38) is given by \( \lambda_- = -\frac{|M_{12,12}M_{21,21} - |M_{12,21}|^2}{2|P_{12}|^2} \), which to the leading order in \( \omega_{b,c} \) gives \( \lambda_- = -|M_{12,21}|^2 \). This proves that the system \( B \otimes C \) becomes entangled at sufficiently low temperature. We note that this result holds at any finite temperature of the system \( A \). At infinite temperature of the system \( A \) \( M_{12,21} = 0 \) and no free entanglement is generated in the system \( B \otimes C \).

At finite initial temperature of \( B \otimes C \) the condition \( \lambda_- < 0 \) translates to \( P_{12}^2 P_{21}^2 < |M_{12,21}|^2 P_{11}^2 \) to the leading order in \( \omega_{b,c} \). The lower-bound temperature \( T_{lb} \) is found from the condition \( P_{12}^2 P_{21}^2 = |M_{12,21}|^2 P_{11}^2 \). Since \( M_{12,21} \) is an oscillating function of time the amplitude of \( |M_{12,21}| \) must be substituted for \( |M_{12,21}| \) in this equality, which leads to the following equation defining the lower bound temperature:

\[
4 \gamma^2 \langle 2|\hat{V}_b|1\rangle \langle 2|\hat{V}_c|1\rangle \sum_{m,n} p_{a,m,n}(m|\hat{V}_a|n)^2 \frac{\omega_{mn}}{(\omega_{mn})^2} \times \frac{P_{12}^2 P_{21}^2}{P_{11}^2} = \exp\left(\frac{-\omega_b + \omega_c}{2T_{lb}}\right),
\]

(45)

finally leading to

\[
T_{lb} = \frac{-\omega_b + \omega_c}{2 \ln\left(4 \gamma^2 \langle 2|\hat{V}_b|1\rangle \langle 2|\hat{V}_c|1\rangle \sum_{m,n} p_{a,m,n}(m|\hat{V}_a|n)^2 \frac{\omega_{mn}}{(\omega_{mn})^2} \right)}.
\]

(46)

A generalization of the formula to the case of interaction of the form \( \Sigma \gamma \hat{V}_a^i \otimes (\hat{V}_b^i \otimes \hat{1} + \hat{1} \otimes \hat{V}_c^i) \) can be carried out along the same lines.

The entanglement in the reduced system of two noninteracting slow spins interacting with the fast four-level bath was explored numerically and the results are plotted in Fig. 6. The shaded area in the parametric space of the logarithm of inverse coupling strength and the inverse initial temperature of the spins represents parametric values for which no entanglement develops in the course of the evolution. The border of the shaded area corresponds to the critical temperature for various coupling magnitudes. The Hamiltonian of the composite system is

\[
\hat{H} = \hat{H}_a \otimes \hat{1}_b \otimes \hat{1}_c + \frac{1}{2} \omega_b [\hat{1}_a \otimes (\hat{\sigma}_z^b \otimes \hat{\sigma}_c^b + \sqrt{2} \hat{1}_b \otimes \hat{\sigma}_z^c)]
\]

\[
+ \gamma \hat{V}_a \otimes (\hat{\sigma}_z^b \otimes \hat{1}_c + \hat{1}_b \otimes \hat{\sigma}_z^c),
\]

(47)

where \( (\hat{H}_a)_{ij} = \delta_{ij} E_{a}^{i} \), \( E_{a}^{0,1,2,3,4} = \{0, 10\omega, 20\omega, 30\omega\} \) and \( (\hat{V}_a)_{ij} = \delta_{ij} \). The temperature of the thermal initial state of the bath is \( T = 5\omega \). The value of \( \omega \) chosen for the numerical calculation is unity. The correspondence of Eq. (46) (the dashed line) to the numerical values is very good up to a coupling strength of the order of unity. We note that for large values of the coupling strength \( \gamma \) the critical temperature asymptotically tends to a finite constant value.
B. Two fast systems interacting with a slow common third party

The case $\omega_{bc} \gg \omega_{bm}$ is more complex. To demonstrate entanglement at zero temperature of the system $B \otimes C$ two simplifying assumptions were added. The first is that the temperature of the system $A$ is also zero. The second is that the matrix elements of $V_{ab}$ couple only the neighboring states: $\langle n | \hat{V}_a | m \rangle \propto \delta_{n, m+1}$. Under these two assumptions the expressions for $M_{12,12}$, $M_{21,21}$, and $M_{12,12}M_{21,21} - |M_{12,21}|^2$ become

$$M_{12,12} = P_{11} \left( \frac{2 \gamma(2|\hat{V}_a|1)(1|\hat{V}_b|2)\sin((\omega_a + \omega_c)t/2)}{\omega_c} \right)^2,$$

$$M_{21,21} = P_{11} \left( \frac{2 \gamma(2|\hat{V}_a|1)(1|\hat{V}_b|2)\sin((\omega_a + \omega_b)t/2)}{\omega_b} \right)^2,$$

$$M_{12,12}M_{21,21} - |M_{12,21}|^2 = P_{11}^2 \left( \frac{2 \gamma^2(2|\hat{V}_a|1)(1|\hat{V}_b|2)(1|\hat{V}_b|2)}{\omega_b \omega_c} \right)^2 S(t),$$

where

$$S(t) = \sin(\omega_a t)\{\sin(\omega_b t) + \sin(\omega_c t) - \sin[(\omega_a + \omega_b + \omega_c)t]\}.$$

To estimate $S(t)$ new variables $x = \sin(\omega_a t)$, $y = \sin(\omega_b t)$, and $z = \sin(\omega_c t)$ are introduced. Ignoring the zero-measure set of commensurable frequencies we can treat the function $S(t)$ as a function of three independent variables $x, y$, and $z$. The range of $S(t)$ in the cube, defined by $-1 \leq x, y, z \leq 1$, can be explored numerically and is found to be $s = S(t) < 3$, where $s \approx -1.6834$. Therefore, from Eq. (48) $M_{12,12}M_{21,21} - |M_{12,21}|^2 < 0$, which proves that at zero temperature $\lambda_\gamma < 0$ [see Eq. (38)] and the systems $B$ and $C$ are entangled by the interaction with the system $A$.

The lower-bound temperature is determined by the condition $\lambda_\gamma = 0$, which translates to $(P_{12} + M_{12,12})(P_{21} + M_{21,21}) = |M_{12,21}|^2$ [see Eq. (38)]. The latter condition can be put in the form $(M_{12,12}M_{21,21} - |M_{12,21}|^2) + P_{12}P_{21} + P_{12}M_{21,21} + P_{21}M_{12,12} = 0$. Since $M_{12,12}$ and $M_{21,21}$ are nonnegative independent functions of time the minimum value of $(M_{12,12}M_{21,21} - |M_{12,21}|^2) + P_{12}P_{21} + P_{12}M_{21,21} + P_{21}M_{12,12}$ is obtained at $M_{21,21} = M_{12,12} = 0$. Then the lower-bound temperature can be calculated from the condition that the amplitude of $M_{12,12}M_{21,21} - |M_{12,21}|^2$ equals $-P_{12}P_{21}$:

$$2 \gamma^2 \sqrt{|s|} \left( \frac{2|\hat{V}_a|1(1|\hat{V}_b|2)(1|\hat{V}_b|2)}{\omega_b \omega_c} \right)^2 S(t) = \sqrt{\frac{P_{12}P_{21}}{P_{11}}},$$

finally leading to

$$T_{lb} = \frac{- \omega_b - \omega_c}{2 \ln \left( \frac{2 |\hat{V}_a|1(1|\hat{V}_b|2)(1|\hat{V}_b|2)}{\omega_b \omega_c} \right)}.$$

It is interesting to note that $T_{lb}$ in this case does not depend on the time scales of the slow system.

The entanglement in the reduced system of two noninteracting fast spins interacting with the slow four-level bath was explored numerically and the results are plotted in Fig. 7. The shaded area in the parametric space of the logarithm of inverse coupling strength and the inverse initial temperature of the spins represents parametric values for which no entanglement develops in the course of the evolution. The border of the shaded area is good. Temperature is measured in units of $\omega$. 

FIG. 6. (Color online) The shaded area in the parameter space of the inverse initial temperature $T$ of the slow spin and the logarithm of the inverse coupling strength $\gamma$ represents values of $T$ and $\gamma$ where entanglement does not develop in the course of the evolution. The composite system of two slow spins interacting with a fast four-level system evolves from the initial product of thermal states under the Hamiltonian (47). The dashed line is the plot of $T_{lb}$, Eq. (46). Up to the coupling $\gamma = 1$ its correspondence with the border of the shaded area is very good. Temperature is measured in units of $\omega$. 

FIG. 7. (Color online) The shaded area in the parameter space of the inverse initial temperature $T$ of the fast spins and the logarithm of the inverse coupling strength $\gamma$ represents values of $T$ and $\gamma$ where entanglement does not develop in the course of the evolution. The composite system of two fast spins interacting with the slow four-level system evolves from the initial product of thermal states under the Hamiltonian (52). The dashed line is the plot of $T_{lb}$, Eq. (51). Up to the coupling $\gamma = 1$ its correspondence with the border of the shaded area is good. Temperature is measured in units of $\omega$. 

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for various coupling magnitudes. The Hamiltonian is chosen to be similar to the previous example, [see Eq. (47)], but time scales of the subsystems are reversed:

$$\hat{H} = \hat{H}_a \otimes \hat{I}_b \otimes \hat{I}_c + 5\omega [\hat{I}_a \otimes (\hat{\sigma}_b^x \otimes \hat{1}_c + \sqrt{2}\hat{I}_b \otimes \hat{\sigma}_c^z)] + \gamma \hat{V}_a \otimes (\hat{\sigma}_b^y \otimes \hat{1}_c + \hat{I}_b \otimes \hat{\sigma}_c^y),$$

(52)

where $\langle \hat{H}_{a ij} \rangle = \delta_{ij} E_a$, $E_a^{[1,2,3,4]}(0, \omega, 2\omega, 3\omega)$, and $\hat{V}_{a ij} = \delta_{ij}$. The temperature of the thermal initial state of the bath was chosen as $T = 0.01\omega$, which is small compared to the energy scale of the bath chosen for the numerical calculation: $\omega = 1$. The dashed line in Fig. 7 is a plot of Eq. (51) and the correspondence with the border of the shaded area at coupling strength up to the order of unity is good.

IV. SUMMARY AND CONCLUSIONS

Entanglement is created by both direct and indirect weak interaction between two initially disentangled systems prepared in thermal states at sufficiently low temperatures. The study is restricted to the conditions where the ground states of both systems are not invariant under the interaction and the interaction is nonresonant. As a consequence, the present analysis left out some interesting models such as the Jaynes-Cummings model [35]. The generation of entanglement in cases of the weak resonant direct and indirect interactions will be considered elsewhere.

In the case of indirect interaction to show entanglement at $T = 0$ we have assumed that the thermal average of the third party coupling term in the initial state vanishes. The reason for the assumption was technical. It should be noted that many system-bath models of linear coupling satisfy this assumption [33]. The additional technical assumption was that the coupling terms of the noninteracting parties possess matrix elements only between the adjacent energy states. Here, too, the assumption is general for weak-coupling models. Two cases of time-scale separation were considered explicitly. The first is the case of two slow systems interacting via the fast third common party. The second is the case of two fast systems interacting via the slow third common party. In the first case the entanglement was shown to appear at sufficiently low initial temperature of the slow systems for any finite temperature of the third party. In the second case the entanglement develops at sufficiently low initial temperature of the fast systems. In this case we assumed that the third party was prepared at zero temperature and that the third-party coupling agent has nonvanishing matrix elements only between the adjacent energy states. This assumption is stronger than just assuming that its thermal average vanishes.

In these cases of indirect interaction and in the case of the direct interaction between the parts we have shown that if the initial temperature of the bipartite state is zero entanglement is generated by the interaction. From this result and the general theorem about the existence of a finite separable ball about the maximally mixed state [28–31] it follows that at least one crossover from the entanglement-generating to the separable evolution should exist at some finite initial temperature of the state. The same is true for the crossover from the NPPT (non-partial-positive transpose) to the PPT evolution. The lowest temperature corresponding to a crossover was termed the lower critical temperature $T_{lb}$ and the highest was termed the upper critical temperature $T_{uc}$. Numerical experiments suggest that there is one such crossover for the interactions studied and therefore $T_{lb} = T_{uc}$. In both cases of a direct and an indirect interaction between the initially disentangled systems, prepared in thermal states, we calculated the lower bound $T_{lb}$ for the lower critical temperature $T_{lc}$. When the initial temperature of both thermal states is below $T_{lb}$ the interaction generates entanglement in the course of the evolution. For temperatures above the lower bound $T_{lb}$ the negativity of the partially transposed composite state is zero in the leading order in the coupling strength and therefore negligible in the weak-coupling limit. It follows that $T_{lb}$ may be considered as the physical critical temperature for the negativity of the composite state.

It was found that $T_{lb}$ is a monotonically decreasing function of the coupling strength. As a consequence, the purity of the state evolving from the initial thermal state at $T = T_{lb}$ can be arbitrary high at weak enough coupling, but the entanglement generation is zero to the leading order in the coupling strength. This proves that far from the maximal separable (and the maximal PPT) ball [30,31] centered at the maximally mixed state there exist a quantum dynamics, which is effectively PPT.

In Ref. [26] it was proved that a PPT density operator of a sufficiently small rank is separable. We speculate that, aiming at a less stringent criterion for the approximate separability, one may be satisfied with showing that the PPT state is sufficiently pure.

A high-temperature limit is generally considered as bringing about classical features into the quantum evolution. A quantum system at high temperature can be efficiently simulated on the classical computer. The present analysis suggests that there exists a PPT (or perhaps, even separable) dynamics in a relatively cold quantum region, which hopefully can be simulated with a moderate scaling.

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The Jaynes-Cummings model of an interacting two-level system and a quantized field mode was investigated in Ref. [16]. It was found that no free entanglement is generated in the course of the evolution of the composite system if the initial temperature of both the subsystems is sufficiently high.