Nonlocal quantum correlations are a key resource in quantum information processing [1]. The exclusive quantum part of this correlation has been termed entanglement. For a pure bipartite state the extent of entanglement is well defined by the Schmidt rank of the state [2,3], counting the number of nonvanishing terms in the product states decomposition [4]. For mixed quantum states the notion of entanglement is more involved and various measures have been suggested [4], each focusing on particular aspects of this phenomenon. In order to take advantage of the insights learned from different measures, it is advantageous to seek for classes of states where different entanglement measures agree [5]. Such a class is the bipartite Schmidt-correlated states class [6–8], where it has been shown that the distillable entanglement [9] and relative entropy of entanglement [10] both coincide and can be calculated by a simple formula [6,7,10,11]. The Schmidt-correlated (SC) states are defined as mixtures of pure states, sharing the same Schmidt bases [6–8]. Such states naturally appear in a bipartite system dynamics with additive integrals of motion (see below and Ref. [12]).

Hence, these states form an important class of mixed states from a quantum dynamical perspective.

The present study establishes a remarkable property of the SC states: the handy, albeit obscure, negativity [13] measure of entanglement admits a clear geometrical interpretation. It is found that the negativity equals half the sum of the absolute values of the off-diagonal elements of the density matrix, which is a distance of the SC state from a pertinent separable state (see Fig. 1). As a consequence, unlike a general mixed state, a SC state is separable [16] if and only if its negativity vanishes, which implies that the Peres-Horodecki criterion [14,15] of separability is both necessary and sufficient for SC states. It should be noted, that the matrix norm used to define the distance, permits an estimation of the negativity “at a glance,” which has a strong intuitive appeal. Quantum-dynamical considerations motivate a generalization of the results to particular mixtures of SC states. It is shown that the negativity of such a mixture is less or equal to the distance of the state from a pertinent separable state.

We start from a formal definition of a SC state.

**Definition.** A mixed bipartite state \( \hat{\rho} = \sum_p p | \psi_p \rangle \langle \psi_p | \) is called Schmidt-correlated if \( | \psi_p \rangle = \sum_m c_{im} | m \rangle_1 \otimes | m \rangle_2 \) for every \( i \), i.e., all pure states in the mixture share the same Schmidt bases \( \Gamma_1 = \{ | m \rangle_1 \}_{m=1}^N \) and \( \Gamma_2 = \{ | m \rangle_2 \}_{m=1}^N \).

As a simple example of SC state consider the state \( \hat{\rho} = \sum_{i=1}^2 p_i | \psi_i \rangle \langle \psi_i | \) of a composite system of two qubits, where \( | \psi_1 \rangle = c_1 | 11 \rangle + c_2 | 00 \rangle \) and \( | \psi_2 \rangle = c_3 | 10 \rangle + c_4 | 01 \rangle \) will not be SC, unless both states are maximally entangled, in which case a common Schmidt bases exist [8].

**Theorem 1.** Let a bipartite state \( \hat{\rho} \) be a SC state with respect to Schmidt bases \( \Gamma_1 \) and \( \Gamma_2 \). Then the negativity of the state \( \hat{\rho} \) equals a distance of \( \hat{\rho} \) from a separable state \( \hat{\rho}' \) diagonal in the tensor product basis \( \Gamma = \Gamma_1 \otimes \Gamma_2 \). \( \mathcal{N}(\hat{\rho}) = \frac{1}{2} d(\hat{\rho}, \hat{\rho}') \), where \( (\hat{\rho}')_i = \delta_{i,j} (\hat{\rho})_i \), \( i,j \in \Gamma \) and the distance \( d(\hat{x}, \hat{y}) = \| \hat{x} - \hat{y} \|_a \) is induced by the matrix norm \( \| \hat{x} \|_a = \sum_{i,j} | (\hat{x})_{ij} |^2 \) [17].

**Remark.** The separable state \( \hat{\rho}' \), from which the distance induced by the norm \( \| \|_a \) is measured, is simply represented by the matrix of \( \hat{\rho} \) with off-diagonal elements stripped off. If \( \hat{\rho} \) is a pure state (which is trivially SC), then \( \hat{\rho}' \) is also the state that minimizes the relative entropy \( S(\hat{\rho} | \hat{\rho}') \) over all disentangled states \( \hat{\sigma} \). In that case, the entanglement of relative entropy is \( E_{\text{RE}}(\hat{\rho}) = \min_{\text{separable}} S(\hat{\rho} | \hat{\sigma}) = S(\hat{\rho} | \hat{\rho}') \) [18,19].

**Proof.** Let \( \hat{\rho} = \sum_{i,j} p_{ij} | \psi_{ij} \rangle \langle \psi_{ij} | \), where \( | \psi_{ij} \rangle = \sum_{m=1}^N c_{ijm} | m \rangle_1 \otimes | m \rangle_2 \).

Then \( \hat{\rho} = \sum_{ij} p_{ij} | \psi_{ij} \rangle \langle \psi_{ij} | \) is a SC state \( \{ | \psi_{ij} \rangle \}_{ij} \). By definition \( d(\hat{\rho}, \hat{\rho}') = \sum_{m=1}^N | (\hat{\rho} - \hat{\rho}')_m |^2 = \sum_{m=1}^N | \rho_{mn} - \rho_{mn}' |^2 = \sum_{m,n} | \rho_{mn} - \rho_{mn}' |^2 \).

The negativity of a state is defined as the absolute value of the sum of the negative eigenvalues of the partially transposed density operator corresponding to the state [13]. In what follows we show that \( \{ | \rho_{mn} |, m < n \} \) is the set of all the negative eigenvalues of the partially transposed \( \hat{\rho} \). This completes the proof.

The partially transposed density matrix is given by

\[
\hat{\rho}^{PT} = \sum_{mn} \rho_{mn} | m \rangle \langle n | \otimes | n \rangle \langle m | = \sum_{mn} \rho_{mn} | m \rangle \otimes | n \rangle \langle m |.
\]

Consider \( N(N-1) \) vectors

\[
| \psi_{kl} \rangle = \sum_l | \psi_{kl} | l \rangle \pm | \rho_{kl} | l \rangle | k \rangle, \quad k < l, k = 1, 2, \ldots, N - 1
\]

and \( N \) vectors

\[
| \psi_{kk} \rangle = | k \rangle | k \rangle, \quad k = 1, 2, \ldots, N.
\]
\[ \hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \]

Distance \( d(\hat{\rho}, \hat{\rho}') = \sum_{i \neq j} |\rho_{ij}| \) \( N'(\hat{\rho}) = \frac{1}{2} d(\hat{\rho}, \hat{\rho}') \)

\[ \hat{\rho}^{PT}|\psi_{kk}\rangle_x = \sum_{mn} \rho_{mn}|m\rangle\langle m| |n\rangle\langle n| (|\rho_{kl}|l\rangle \pm |\rho_{kl}|l\rangle |k\rangle) = -\sum_{mn} \rho_{mn} \delta_{mk} \delta_{kn} |m\rangle\langle m| \pm \sum_{mn} \rho_{mn} |k\rangle\langle k| \pm \rho_{kl} |m\rangle |k\rangle \times |k\rangle = \mp |\rho_{kl} (|k\rangle |l\rangle \pm |l\rangle |k\rangle) = \mp |\rho_{kl} |\langle k|\psi_{kk}\rangle_x. \]

(4)

Analogously we obtain

\[ \hat{\rho}^{PT}|\psi_{kk}\rangle = |\rho_{kk}\rangle |\psi_{kk}\rangle. \] (5)

Thus, the partially transposed matrix \( \hat{\rho}^{PT} \) has been diagonalized, its \( N = \frac{N(N-1)}{2} \) negative eigenvalues \( -|\rho_{kl}|, k < l \) have been found, which completes the proof. Corollary 1. Let a bipartite state \( \hat{\rho} \) be SC. Then \( \hat{\rho} \) is disentangled if and only if its negativity vanishes.

Proof: If \( \hat{\rho} \) is disentangled its negativity \( N(\hat{\rho}) \) vanishes by Peres-Horodecki criterion [14,15]. If \( N(\hat{\rho}) = 0 \) then by theorem 1 \( \hat{\rho} \) is separable.

The following two corollaries of theorem 1 permit an estimation of entanglement of a SC state by simply "looking at" the occupied entries of the corresponding density matrix.

Corollary 2. The negativity of the SC (with respect to Schmidt bases \( \Gamma_{1,2} \)) state \( \hat{\rho} \) equals half the sum of the off-diagonal elements of the corresponding density matrix in the Schmidt basis: \( N(\hat{\rho}) = \sum_{i \neq j} |\rho_{ij}|, i,j \in \Gamma_1 \oplus \Gamma_2 \).

Corollary 3. Let a SC state \( \hat{\rho} \) be quasidiagonal, i.e., \( \hat{\rho} = \sum_{mn} \rho_{mn} |m\rangle\langle m| |n\rangle\langle n| \), where \( mn = 1,2,\ldots,N \) and \( |m\rangle - |N\rangle \leq \Delta < N - 1 \). Then \( N(\hat{\rho}) < \Delta. \)

Proof: By corollary 2 it suffices to show that \( \frac{1}{2} \sum_{m \neq n} |\rho_{mn}| < \Delta. \) The sum of the absolute values of the off-diagonal elements can be estimated as follows:

\[ \sum_{mn} |\rho_{mn}| = \sum_{mn} |\rho_{mn}| - 1 = \sum_{mn} |\rho_{mn}| - 1 = \sum_{mn} |\rho_{mn}| - \sum_{mn} |\rho_{mn}| + \sum_{mn} |\rho_{mn}| < \Delta. \]

(6)

where the first inequality follows from the positivity of the density operator and the second is the inequality of geometric and arithmetic means. This concludes the proof.

The SC correlated states naturally emerge in certain quantum dynamical settings (see Ref. [12]). Assume a (generally non-unitary) evolution of a bipartite composite system admitting an additive integral of motion \( \hat{A} = \hat{A}_1 \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{A}_2 \), where \( \hat{A}_{1,2} \) are local observables, i.e.,

\[ \frac{d}{dt} \hat{\rho} = \mathcal{L} \hat{\rho} \]

(7)

and

\[ \frac{d}{dt} \hat{A} = \hat{A}_1 \hat{A}_2 = 0. \]

(8)

Consider local bases of eigenstates of operators \( \hat{A}_1 \) and \( \hat{A}_2 \):

\[ \Gamma_i = \{|m\rangle_m = 1,2,\ldots,N_i, \hat{A}_i |m\rangle = \lambda^i_m |m\rangle \}, \quad i = 1,2. \]

(9)

We denote as \( \mathcal{H} \) a subspace of the composite system Hilbert space \( \mathcal{H} \) spanned by the eigenstates of \( \hat{A} \), corresponding to an eigenvalue \( \lambda \):

\[ \mathcal{H} = \text{Sp}(|m\rangle_1 \otimes |n\rangle_2, \lambda^1_m + \lambda^2_n = \lambda \). \]

(10)

Let us assume that the spectra of \( \hat{A}_i, i = 1,2 \) are non-degenerate, i.e., \( \lambda^1_m = \lambda^2_m = \lambda \). Then the equation \( \lambda^1_m + \lambda^2_n = \lambda \) has fixed \( m \) and \( \lambda \) possesses a unique solution for \( n: n = \lambda^1(m) \). Therefore the map \( \lambda^1 : \Gamma_1 \rightarrow \Gamma_2 \) provides a one-to-one correspondence between a state \( |m\rangle_1 \in \Gamma_1 \) and a state \( |n\rangle_2 \in \Gamma_2 \), i.e., it defines a unique common set of Schmidt bases for the Schmidt decomposition of all \( \{|\psi\rangle \in \mathcal{H} \}

\[ \{|\psi\rangle = \sum_{\lambda^1, \lambda^2} c_{\lambda^1,\lambda^2} |m\rangle_1 |n\rangle_2. \]

(11)

Since all pure states \( \{|\psi\rangle \in \mathcal{H} \) share the same Schmidt bases their mixture is a SC state.

If the initial state of the composite system is a mixture of eigenstates of \( \hat{A} \), corresponding to the same eigenvalue \( \lambda \), i.e., \( \hat{\rho}(0) = \sum p_i |\psi_i^0\rangle \langle \psi_i^0| \) with \( \{|\psi_i^0\rangle \in \mathcal{H} \}

then the conservation of \( \hat{A} \) would imply that \( \hat{\rho}(t) = \sum p_i (t) |\psi_i(t)\rangle \langle \psi_i(t)| \) \( \{|\psi_i(t)\rangle \in \mathcal{H} \) at any \( t > 0 \). Therefore, the evolving state \( \hat{\rho}(t) \) remains SC and the negativity of the state can be calculated using theorem 1.

For illustration, the evolution of negativity of a composite state of a two noninteracting quantum systems coupled to a local purely dephasing baths is calculated. The composite system evolves according to the Liouville equation

\[ \frac{d}{dt} \hat{\rho} = -[\hat{A}_1, [\hat{A}_1, \hat{\rho}]] - [\hat{A}_2, [\hat{A}_2, \hat{\rho}]]. \]

(12)

The local bases of eigenstates of operators \( \hat{A}_1 \) and \( \hat{A}_2 \) is \( \Gamma_i = \{|m\rangle_i, m = 1,2,\ldots,N_i, \hat{A}_i |m\rangle = \lambda^i_m |m\rangle \}, \quad i = 1,2 \). The initial state of the composite system is a pure entangled state.
\[ |\phi(0)\rangle = \sum_m c_m |m_1\rangle |m_2\rangle. \] 

At \( t > 0 \) the solution of Eq. (12) with initial state (13) is given by

\[ \hat{\rho}(t) = \sum_{m,n} c_m^* c_n e^{-\frac{i}{\hbar} (\lambda_m - \lambda_n)^2 + \epsilon (\lambda_m - \lambda_n^2)^2} |m_1\rangle |m_2\rangle |n_1\rangle |n_2\rangle. \]

(14)

Since \( \hat{\rho}(t) \) is a SC state the theorem 1 applies and

\[ N(\hat{\rho}(t)) = \frac{1}{2} d(\hat{\rho}(t), \hat{\rho}(t)) = \frac{1}{2} \left( \sum_{i,j} |(\hat{\rho}^t)_{i,j}| - 1 \right) \]

\[ = \frac{1}{2} \left( \sum_{i,j} c_m^* c_n e^{-\frac{i}{\hbar} (\lambda_m - \lambda_n)^2 + \epsilon (\lambda_m - \lambda_n^2)^2} - 1 \right). \]

(15)

As a next step, the initial state for the evolution (7) is generalized from a state \( \hat{\rho}(0) = \sum p_i \hat{\rho}_i \) with \( |\psi_i\rangle \in \mathcal{H}_k \) to a mixture of such states, corresponding to different eigenvalues \( \lambda \) of the conserved operator \( \hat{A} \):

\[ \hat{\rho}(0) = \sum_{\lambda} p_{\lambda} \hat{\rho}^\lambda, \]

(16)

where \( \hat{\rho}^\lambda = \sum p_i \hat{\rho}_i |\psi_i\rangle \langle \psi_i| \) with \( |\psi_i\rangle \in \mathcal{H}_k \). By conservation of \( \hat{A} \) we have \( \hat{\rho}(t) = \sum p_{\lambda} \hat{A}_\lambda(t) \hat{\rho}^\lambda(t) \). An estimation of the negativity of \( \hat{\rho}(t) \) is possible using a generalization of the theorem 1 (see theorem 2 below). The result is

\[ N(\hat{\rho}(t)) \leq \frac{1}{2} d(\hat{\rho}(t), \hat{\rho}(t)), \]

(17)

where \( d(\hat{\rho}, \hat{\rho}) \) is the distance defined in Eq. (9) and the distance \( d(\hat{x}, \hat{y}) = \| \hat{x} - \hat{y} \|_a \) is induced by the norm \( \| \cdot \|_a \). 

Theorem 2 generalizes theorem 1 to particular mixtures of SC states. Consider a composite Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) of bipartite quantum system. Let \( \Gamma_1 = \{ |m_1\rangle, m_1 = 1, \ldots, N_1 \} \) be an orthonormal basis of the local Hilbert space \( \mathcal{H}_1 \) and \( \Gamma_2 = \{ |m_2\rangle, m_2 = 1, \ldots, N_2 \} \) be an orthonormal basis of the local Hilbert space \( \mathcal{H}_2 \). Consider a one-to-one correspondence \( f^\lambda \) between a subset \( S^\lambda_1 \subset \Gamma_1 \) and a subset \( S^\lambda_2 \subset \Gamma_2 \). The map \( f^\lambda \) defines a subspace \( \mathcal{H}^\lambda \subset \mathcal{H} \) spanned by the states of the form \( |m_1\rangle \otimes |f^\lambda(m_2)\rangle \):

\[ \mathcal{H}^\lambda = \text{Sp} \{ |m_1\rangle \otimes |f^\lambda(m_2)\rangle \} \subset \mathcal{H}^\lambda. \]

(18)

All pure states \( |\psi^\lambda\rangle \in \mathcal{H}^\lambda \) share the same Schmidt bases by construction. Therefore, a mixture \( \hat{\rho}^\lambda = \sum p_{\lambda} |\phi_{\lambda}\rangle \langle \phi_{\lambda}| \) is a SC state by definition and will be called \( \lambda \)-SC state for brevity in what follows.

Consider a family of such maps \( f^\lambda \), indexed by \( \lambda \), with the following property:

\[ f^\lambda \circ |m_1\rangle = f^{\lambda^2} \circ |m_1\rangle \Leftrightarrow \lambda_1 = \lambda_2. \]

(19)

Then the following result can be proven.

**Theorem 2.** Let a bipartite state \( \hat{\rho} \) be a mixture of \( \lambda \)-SC states: \( \hat{\rho} = \sum p_{\lambda} \hat{\rho}_{\lambda} \) with respect to local bases \( \Gamma_1, \Gamma_2 \) and a family of injective maps \( f^\lambda \): \( S^\lambda_1 \subset \Gamma_1 \rightarrow \Gamma_2 \) with the property (19).

Then the negativity of the state \( \hat{\rho} \) is less or equals a distance of \( \hat{\rho} \) from a separable state \( \hat{\rho} \) diagonal in the basis \( \Gamma_1 \otimes \Gamma_2 \): \( N(\hat{\rho}) \leq \frac{1}{2} d(\hat{\rho}, \hat{\rho}) \), where \( \hat{\rho}_{ij} = \delta_{ij} \hat{\rho}_{ij} \) and the distance \( d(\hat{x}, \hat{y}) = \| \hat{x} - \hat{y} \|_a \) where \( \| \cdot \|_a = \sum_{ij} |\langle \hat{x}_{ij}| \hat{y}_{ij} \rangle| \).

**Proof.** Since the negativity is entanglement monotone [13] the following holds:

\[ N\left( \sum_{\lambda} p_{\lambda} \hat{\rho}^\lambda \right) \leq \sum_{\lambda} p_{\lambda} N(\hat{\rho}^\lambda). \]

(20)

By theorem 1:

\[ \sum_{\lambda} p_{\lambda} N(\hat{\rho}^\lambda) = \sum_{\lambda} p_{\lambda} \frac{1}{2} d(\hat{\rho}^\lambda, \hat{\rho}^\lambda) = \sum_{\lambda} p_{\lambda} \frac{1}{2} \| \hat{\rho}^\lambda - \hat{\rho}^\lambda \|_a. \]

\[ = \sum_{\lambda} p_{\lambda} \frac{1}{2} \sum_{i,j} |(\hat{\rho}^\lambda)_{i,j} - 1 | \]

\[ = \sum_{\lambda} p_{\lambda} \frac{1}{2} \sum_{i,j} |(\hat{\rho}^\lambda)_{i,j} - 1 | \]

\[ = \frac{1}{2} \sum_{\lambda} \sum_{i,j} |(\hat{\rho}^\lambda)_{i,j} - 1 |. \]

(21)

From the property (19) it follows that

\[ \sum_{\lambda} p_{\lambda} \sum_{i,j} |(\hat{\rho}^\lambda)_{i,j} - 1 | = \sum_{\lambda} \left( \sum_{i,j} p_{\lambda} |(\hat{\rho}^\lambda)_{i,j} - 1 | \right) \]

\[ = \sum_{\lambda} \left( \sum_{i,j} |(\hat{\rho}^\lambda)_{i,j} - 1 | \right) \]

(22)

Combining Eqs. (21) and (22) we get

\[ \sum_{\lambda} p_{\lambda} N(\hat{\rho}^\lambda) = \frac{1}{2} \sum_{\lambda} p_{\lambda} \sum_{i,j} |(\hat{\rho}^\lambda)_{i,j} - 1 | = \frac{1}{2} \sum_{\lambda} \sum_{i,j} |(\hat{\rho}^\lambda)_{i,j} - 1 | \]

\[ = \frac{1}{2} d(\hat{\rho}^\lambda, \hat{\rho}^\lambda). \]

(23)

From inequality (20) and Eq. (23) it follows that

\[ N(\hat{\rho}) \leq \frac{1}{2} d(\hat{\rho}, \hat{\rho}). \]

(24)
is basis dependent, i.e., is not invariant under unitary transformations (and, in particular, is not invariant under local unitary transformations). Nevertheless, the basis in which the correspondence of this distance to the negativity is established is the Schmidt basis, which is a preferred basis for representing SC states [21].

In a SC state the negativity equals half the sum of the absolute values of the off-diagonal elements of the density matrix in the Schmidt bases. This finding suggests the “at a glance” estimation of the entanglement of SC states: the state is “substantially” entangled if and only if the off-diagonal entries in the corresponding density matrix are “substantially” populated. In particular, if the corresponding density matrix is quasidiagonal, i.e., the off-diagonal elements populate the strip about the diagonal of width \( \Delta \), the negativity is bounded by \( \Delta \).

We have considered Schmidt-correlated states and particular mixtures of Schmidt-correlated states. These states emerge in dynamical models with conservation laws. Dynamics where the conservation laws are relaxed generate mixed states that are not SC. Simulations of open-system dynamics, similar to those in Ref. [12], have suggested a generalization of the geometrical interpretation of negativity to arbitrary mixed states. It is conjectured that the negativity of an arbitrary mixed state is bounded by half the minimal distance \( d \) of the state to a corresponding separable state, where the distance is minimized over all possible local bases:

\[
N(\hat{\rho}) \leq \frac{1}{2} \min_{\Gamma_1 \otimes \Gamma_2} \left\{ d(\hat{\rho}, \hat{\rho}') = \sum_{|\hat{\rho}|, |\hat{\rho}'| \in \Gamma_1 \otimes \Gamma_2} |(\hat{\rho} - \hat{\rho}')_{ij}|, (\hat{\rho}')_{ij} \right\} = \delta_{ij}(\hat{\rho})_{ij}.
\]

(25)

An interesting question is whether this minimal distance itself is an entanglement monotone [20]. Since \( \sum_i |(\hat{\rho} - \hat{\rho}')_{ij}| = 2\sum_i |(\hat{\rho})_{ij}| \) the conjecture (25) implies that the negativity of an arbitrary state is bounded by half the sum of the off-diagonal elements of the corresponding density matrix in any local bases, which gives an intuitive appraisal of the negativity (and entanglement) of an arbitrary mixed state.

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[21] It should be noted that finding common Schmidt bases of SC states is a difficult problem in itself, which has been addressed in Ref. [8] in the case of a mixture of two pure states.